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# The Gravity Field of a Particle. II

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# The gravity field of a particle. II

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This is a sequel to a paper of 3 years ago, which studied the orbits of 'comets' near a 'sun' regarded as a point source of gravitation according to general relativity. That paper expressed the forms of the orbits in terms of elliptic functions, but its method was not so well adapted to a study of the time in those orbits.

In the first half of the present work these orbits and their associated times are described in a simple form, the results being expressed in terms of integrals of elementary functions, which can be easily worked out either by quadratures or by approximation.

One result of the earlier paper was the proof that no orbit can have perihelion inside $r = 3m$, and in the later part of the present work a method is proposed in order to study this region, since no comet can return from it. It is supposed that flashes are emitted both from a distant observatory and from a comet, each signalling the ticks of his clock according to the time it is keeping. These are observed by the other and compared with the time on its own clock.

The method serves to describe occurrences between $r = 3m$ and the 'barrier' at $r = 2m$, and it points to some unexpected results in the matter of the comet passing the barrier, which call for explanation.

## 1. Introduction

The present paper is a sequel to one published 3 years ago (Darwin 1958). In that paper (which will be quoted here as I) a study was made of various orbits that would be described by a 'comet' going round a 'sun', when this sun was taken as a point source of gravitational attraction obeying Einstein's principles of general relativity. The solutions were given in terms of elliptic functions, and these are well suited for the description of the form of the orbit, but they are not so well adapted for the study of time, because the formulae would then involve the rather troublesome 'elliptic integrals of the third kind'. Now in the study of relativity the behaviour of time is much the most difficult feature, since in so many ways it contradicts our intuitions, and it therefore seemed that it might be worth while to develop this branch of the subject further.†

The work falls into two rather distinct parts. In the first a study is made of the time for the elliptic orbits of which the form has been already worked out—roughly speaking, simple expressions are found from which the 'equation of time' for a planet could be calculated.

It was, however, shown in I that no orbit could have its perihelion inside the distance conventionally described as $r = 3m$, whereas the gravitational field is defined down to the 'barrier' at $r = 2m$, and it seemed desirable to study this part of space also. To avoid pitfalls in the understanding of time, the principle was adopted that this should be done by making only what may be called practical experiments such as would record the actual observations that might be made by observers. The results of this part of the work have turned out to be curious and unexpected, in that it does appear that it may be possible to explore the region beyond the barrier.

† I again owe my thanks to Professor W. H. McCrea, F.R.S., for his useful criticisms on the subject.

It must of course be recognized that the present work studies results which could not arise in nature—except for the formulae given in §4 below which yield convenient approximations that could be used in practice. The reason is that even the densest of matter can never be confined in a space small enough to make possible the observation of any of the present calculations, because the orbits would always lead to direct collisions.

It is worth while to see this in a little more detail. The central equation $m = \frac{4\pi}{3} a^3 \rho G c^{-2}$ relates Einstein's measure of the mass of the sun with its actual radius and density. For our sun $m$ is $1{\cdot}5 \times 10^5$ cm whereas $a$ is $7 \times 10^{10}$ cm, roughly $5 \times 10^5$ as large. The orbits here studied are those in which $a$ is to be at least as small as $m$. This need could be met by keeping the sun's radius the same, but increasing its density a millionfold. Thus to show the effects it would be necessary to have a white dwarf star of the size of our sun, but therefore having a mass a million times as great. Alternatively, if the sun's density is to be kept constant, it is easy to calculate that its radius must be about $5 \times 10^{13}$ cm. This is nearly a thousand times the sun's radius, so that its mass would have to be increased three hundred million times.

Nevertheless, even if regarded only as a mathematical study, some of the present results seem sufficiently unexpected to merit consideration.

## 2. Elliptic orbits

The basic equations determining planetary orbits are:

$$-\frac{r}{r-2m}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 - r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 + \frac{r-2m}{r}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 = 1, \tag{2·1}$$

$$r^2\frac{\mathrm{d}\theta}{\mathrm{d}s} = p, \tag{2·2}$$

$$\frac{r-2m}{r}\frac{\mathrm{d}t}{\mathrm{d}s} = C. \tag{2·3}$$

The other angular variable $\phi$ need not be considered in the present work.

These are not only three integrals of the motion, but (2·1) also determines the magnitude of $s$ in relation to the other variables. In consequence of this $s$ measures the 'local time', that is to say the time that would be indicated by an atomic clock carried on a planet travelling round the sun. Alternatively, since an atomic clock is a rather complicated mechanism, we may suppose it replaced by a tuning-fork executing infinitesimal vibrations, and previously calibrated by comparison with a similar tuning-fork vibrating in a position fixed relative to the sun at an infinite distance. This tuning-fork will then indicate the local time.

If $u$ is written for $1/r$ there results from the integrals the well-known equation

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 = 2mu^3 - u^2 + \frac{2m}{p^2}u - \frac{1-C^2}{p^2}. \tag{2·4}$$

For an elliptic orbit the cubic on the right of (2·4) has three real positive roots, and the motion takes place between the second and third of them. It would serve no useful purpose to consider the solution of the cubic equation, but instead the orbit will be expressed in terms of these two roots. The most convenient way of doing this is in terms of the quantities often used in the elementary theory of planetary motion, $e$ the eccentricity and $l$ the *latus rectum*.†

Perihelion is at $l/(1+e)$ and aphelion at $l/(1-e)$. The three roots of the cubic are then $\frac{1}{2m}-\frac{2}{l}, \frac{1+e}{l}, \frac{1-e}{l}$, and the first of these must be greater than the second. This implies

$$l > 2m(3+e). \tag{2·5}$$

If these two roots are equal, the orbit will end by asymptotically approaching a circle of radius $2m(3+e)/(1+e)$ in the manner described in I.

The equation (2·4) may then be written as

$$\left(\frac{du}{d\theta}\right)^2 = 2m\left(u-\frac{1}{2m}+\frac{2}{l}\right)\left(u-\frac{1+e}{l}\right)\left(u-\frac{1-e}{l}\right)$$
$$= 2mu^3-u^2+2u\frac{l-m(3+e^2)}{l^2}-\frac{(l-4m)(1-e^2)}{l^3}. \tag{2·6}$$

If $m/l$ is written as $\mu$ for convenience, it follows that

$$\frac{1}{p^2}=\frac{1-\mu(3+e^2)}{lm}, \tag{2·7}$$

$$\frac{C^2}{p^2}=\frac{[1-2\mu(1+e)][1-2\mu(1-e)]}{lm}. \tag{2·8}$$

These are the only forms in which the dynamical integration constants will need to be used.

Now set

$$u = (1+e\cos\chi)/l, \tag{2·9}$$

so that $\chi = 0$ and $2\pi$ at successive perihelia, and $\pi$ at the intermediate aphelion. Then substitution in (2·6) gives

$$(d\chi/d\theta)^2 = 1-2\mu(3+e\cos\chi). \tag{2·10}$$

In the ordinary theory of planetary orbits, there are certain angles which are called *anomalies*, the word being derived from the ancient Greek study of planetary motion. These are named the *true*, the *mean*, and the *eccentric* anomalies, and to these we now add a fourth one, the angle $\chi$ which may be called the *relativistic anomaly*.

## 3. The elements of the orbits

It is easy to express all the elements of planetary orbits as integrals using $\chi$ as the independent variable. First,

$$\theta = \int d\chi[1-2\mu(3+e\cos\chi)]^{-\frac{1}{2}} \tag{3·1}$$

which gives an elliptic integral to modulus $\sqrt{[4\mu e/(1-6\mu+2\mu e)]}$.

† This letter is so widely used for the purpose that it seemed best to adopt it here, even though it clashes with the use of it in I where it signified the asymptotic distance in a hyperbolic orbit. That use will arise here only once, so it seems best not to introduce a new letter for it, but to denote it by $l_*$.

As to the time, it must be recognized that there are two times, which may be called *internal* and *external*. The internal time is that kept by a clock on the planet, and it is measured by $s$. The external time is the time kept by a clock at a great distance but at rest relative to the sun. If it is set up in an observatory far away on the line of the pole of the planet's orbital plane, the time delay of the observations will be constant throughout the orbit and, allowing for this constant delay, the external time will be given by $t$.

The appropriate measure for the time is the year as given by Kepler's Third Law. This is

$$Y = 2\pi l^{\frac{3}{2}} m^{-\frac{1}{2}} (1-e^2)^{-\frac{3}{2}}. \tag{3·2}$$

Then for the internal time,

$$\begin{aligned} s &= \int \mathrm{d}\chi \frac{1}{pu^2}\frac{\mathrm{d}\theta}{\mathrm{d}\chi} \\ &= \frac{1}{2\pi} Y (1-e^2)^{\frac{3}{2}} [1-\mu(3+e^2)]^{\frac{1}{2}} \int \mathrm{d}\chi (1+e\cos\chi)^{-2} [1-2\mu(3+e\cos\chi)]^{-\frac{1}{2}}. \end{aligned} \tag{3·3}$$

For the external time,

$$\begin{aligned} t &= \int \mathrm{d}\chi \frac{C}{1-2mu} \frac{1}{pu^2}\frac{\mathrm{d}\theta}{\mathrm{d}\chi} \\ &= \frac{1}{2\pi} Y (1-e^2)^{\frac{3}{2}} [1-2\mu(1+e)]^{\frac{1}{2}} [1-2\mu(1-e)]^{\frac{1}{2}} \\ &\quad \times \int \mathrm{d}\chi (1+e\cos\chi)^{-2} [1-2\mu(1+e\cos\chi)]^{-1} [1-2\mu(3+e\cos\chi)]^{-\frac{1}{2}}. \end{aligned} \tag{3·4}$$

It may be noted that if the two *equations of time* are needed in detail, quadratures must be used, and for a very eccentric orbit the factor $(1+e\cos\chi)^{-2}$ produces integrands which are unbalanced, since aphelion gives much the largest contributions. This can be avoided by the use of the eccentric anomaly $\psi$, which is here related to $\chi$ by the equation

$$(1+e\cos\chi)(1-e\cos\psi) = 1-e^2. \tag{3·5}$$

## 4. Approximate solutions

The greatest density that ordinary matter can have ensures that unless $l$ is very much greater than $m$ the planet will collide with the material of the sun. It is therefore justifiable to expand the integrals in powers of $\mu$ and retain only the first power. If for any reason higher accuracy is needed, the calculations of the coefficients of $\mu^2$, etc., are perfectly straightforward, but they will not be given here.

Then to the first order, all quantities being measured from perihelion,

$$\theta = \chi + \mu(3\chi + e\sin\chi). \tag{4·1}$$

The times are more simply expressed in terms of the eccentric anomaly as defined by (3·5):

$$s = \frac{1}{2\pi} Y[(\psi - e\sin\psi) + \tfrac{1}{2}\mu(1-e^2)(3\psi - e\sin\psi)], \tag{4·2}$$

$$t = \frac{1}{2\pi} Y[(\psi - e\sin\psi) + 3\mu(1-e^2)\psi]. \tag{4·3}$$

The year's totals are thus $\Theta = 2\pi(1+3\mu)$, (4·4)

$$S = Y[1+\tfrac{3}{2}\mu(1-e^2)], \quad (4\cdot5)$$

$$T = Y[1+3\mu(1-e^2)]. \quad (4\cdot6)$$

The years here are the 'anomalistic years' taken from perihelion to perihelion. The 'sidereal years' measured from the first point of Aries will vary in length according to the position of perihelion, but their averages will be

$$S_{\text{sid.}} = Y[1-\tfrac{3}{2}\mu(1+e^2)] \quad \text{and} \quad T_{\text{sid.}} = Y[1-3\mu e^2].$$

These results were given in I for circular orbits; $S_{\text{sid.}}$ corresponds to the formula there given as $\sqrt{(1-3m/R)}$, while the fact that $T_{\text{sid.}} = Y$ corresponds to the fact that the angular velocity in a circular orbit is exactly that given by Kepler's Third Law.

It may also be noted that the clock-rate for the planetary clock at any place is

$$\mathrm{d}s/\mathrm{d}t = 1-\tfrac{1}{2}\mu[3+e^2+4e\cos\chi],$$

which reduces to $1-\frac{1}{2}\mu(3+e)(1+e)$ at perihelion and $1-\frac{1}{2}\mu(3-e)(1-e)$ at aphelion.

## 5. Example of an eccentric orbit

It seemed that it would be interesting to examine a rather extreme case by numerical quadrature. If $l = 8m$, the parabolic orbit with $e = 1$ terminates on a circle of radius $4m$, and so the orbit with $l = 8m$, $e = 0{\cdot}9$ was studied. This has perihelion at $r = 4{\cdot}21m$ and aphelion at $r = 80m$; and the planetary clock-rates at perihelion and aphelion are 0·531 and 0·986, respectively. Though selected by pure chance it proved a rather convenient choice for computation, because with $\mu = \frac{1}{8}$ the expression (2·10) has a factor $1-e\cos\chi$, and this can be read off without further computation from another part of the table.

No great precision was aimed at. The integrands of (3·1), (3·3) and (3·4) were calculated at every 10° from 0° to 180° and added up. In the cases of $S$ and $T$ the factor $(1+e\cos\chi)^{-2}$ becomes large near 180°, and this may have introduced some error, but it was partly checked by the use of the eccentric anomaly, which showed that the integrals were substantially correct.

By dividing the summations into the two parts from 0° to 90°, and 90° to 180° it was easy to calculate the separate contributions of the parts of the orbit where $r < l$ and $r > l$. This may illustrate to a small extent the characteristics of the two equations of time. These two parts are denoted by $(p)$ and $(a)$ below. Then

$$\Theta(p) = 2\pi\times 1{\cdot}854, \quad \Theta(a) = 2\pi\times 0{\cdot}810;$$

$$S(p) = Y\times 0{\cdot}044, \quad S(a) = Y\times 1{\cdot}050;$$

$$T(p) = Y\times 0{\cdot}072, \quad T(a) = Y\times 1{\cdot}087.$$

These may be compared with the values in a Keplerian orbit of the same elements $l$ and $e$. For this

$$\Theta(p) = 2\pi\times 0{\cdot}50, \quad \Theta(a) = 2\pi\times 0{\cdot}50;$$

$$T(p) = Y\times 0{\cdot}018, \quad T(a) = Y\times 0{\cdot}982.$$

It will be noticed that most of the effect on $\Theta$ arises near perihelion, where in this rather extreme case the planet goes nearly twice round before escaping. On the



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other hand, very little of the time, whether internal or external, is spent near perihelion where the velocity becomes great, and therefore the year's length is much less affected. Also, the slowing of the planet's clock in this part of the orbit has not very much influence in the comparison of the total internal and external years.

## 6. Hyperbolic orbits

Most of the preceding work also applies *mutatis mutandis* to hyperbolic orbits. Perihelion is at $l/(e+1)$, but we now need to know the three quantities $\theta_1$ the final value of $\theta$, $V$ the velocity at infinity and $l_*$ the perpendicular distance of the final asymptote from the sun.

The final value of $\chi$ is given by (2·9) as $\chi_1 = \pi - \cos^{-1}(1/e)$, and from this (3·1) gives the final value of $\theta_1$, in the form of an elliptic integral. In the case when $\mu^2$ is negligible this is

$$\theta_1 = (1+3\mu)[\pi - \cos^{-1}(1/e)] + \mu\sqrt{(e^2-1)}. \tag{6·1}$$

The velocity at infinity is given by $C = 1/\sqrt{(1-V^2)}$. The equations (2·7) and (2·8) still hold, and it is easy to derive

$$V^2 = \frac{\mu(1-4\mu)(e^2-1)}{[1-2\mu(e+1)][1+2\mu(e-1)]}. \tag{6·2}$$

For small $\mu$ this gives $V^2 = \mu(e^2-1)$ with the next term only involving $\mu^3$.

Again, $l_* = p\sqrt{(1-V^2)}/V$ which yields

$$l_* = l/\sqrt{[(e^2-1)(1-4\mu)]} \tag{6·3}$$

or, with neglect of $\mu^2$,

$$l_* = l(1+2\mu)/\sqrt{(e^2-1)}.$$

## 7. Associate orbits

The orbits hitherto discussed are those which describe motions between the second and third roots of (2·6). The same equation, however, admits of a solution where $u$ is greater than the first root, and this will be briefly considered, with a view to matters arising in connexion with the field near the sun. This orbit may be called the *associate* orbit to the elliptic or hyperbolic type of orbit so far discussed.

It starts at aphelion at the first root of (2·6) and ends in the sun. Its dynamical constants are again given by (2·7) and (2·8) in terms of $l$ and $e$. For such an orbit aphelion is at some value less than $2m(3+e)/(1+e)$, that is to say it lies between $6m$ and $2m$, and so is always outside the barrier.

To solve this case the substitution replacing (2·9) is

$$u = \left(\frac{1}{2m} - \frac{2}{l}\right) + \left(\frac{1}{2m} - \frac{3+e}{l}\right)\tan^2 \tfrac{1}{2}\xi \tag{7·1}$$

and from this (2·6) gives

$$(\mathrm{d}\xi/\mathrm{d}\theta)^2 = 1 - 2\mu(3 - e\cos\xi). \tag{7·2}$$

It is then easy to write down integrals like (3·1), (3·3) and (3·4) for $\theta$, $s$ and $t$. These hold down to $r = 2m$ without question, but at this place $t$ becomes infinite. This is not so for $\theta$ and $s$, which remain finite right down to $r = 0$, where $\xi = \pi$.

This point was touched on in I, § 10, and the general questions of the last part of the orbit will be discussed here later, but the matter may be illustrated by a simple example. Take $l = 8m$, $e = 0$. This has aphelion at $4m$, and (7·2) shows that $\theta = 2\xi$.

In polar co-ordinates the orbit is $r = 8m(1+\cos\frac{1}{2}\theta)/(3+\cos\frac{1}{2}\theta)$. It crosses $r = 2m$ where $\cos\xi = -\frac{1}{3}$, or $\theta = 220°$. As estimated by the external time $t$ it takes an infinite time to reach this place, but its internal time has been slowed down to such an extent that it is still finite there, and if it is justifiable to regard the motion as continuing, it will end at $\theta = 2\pi$. The internal time down to $\theta = 2\pi$ is

$$\sqrt{\tfrac{5}{8}}\,Y[1-(5/4\sqrt{2})].$$

This may be compared with the sidereal internal year for the circular orbit of radius $8m$ which is $\sqrt{\frac{5}{8}}\,Y$.

## 8. General consideration of the region below $3m$

In what follows it will be convenient to take $m = 1$, since unlike the previous work there will be no orbital element, such as $l$, with which it has to be compared. For the case of our sun this means that the unit of length will be 1·5 km, and of time 5 $\mu$s.

It was shown in I that no orbit could have perihelion inside $r = 3$. Any comet entering this region must inevitably end in the sun, and so the region between 3 and the barrier at 2 requires different methods for its study from those hitherto used. It is natural to impose the condition that the experiments designed to test the field in this region should, at least in idea, be practicable ones.

One such method is the use of *radar*. An observatory $A$ is set up at a great distance from the sun. At intervals it emits flashes of light, which are reflected back to it by the comet $B$, and the interval between emission and return is timed. In ordinary radar the velocity of light is assumed to be constant, but here in general the path of the ray will be curved and also its velocity will vary. This would make a rather formidable problem, which I have not attempted to solve. It seemed sufficient to take the extreme case of a comet travelling directly from $A$ towards the sun, for then it becomes fairly simple, because though the velocity of the light-rays will still vary, they will travel in the direct line between $A$ and $B$.

A more elaborate experiment will also be considered, in which there is an observer in the comet $B$, as well as the one at $A$. It would seem legitimate to suppose that the passenger on $B$ could escape to the outside world with his records by means of a rocket fired just before the comet's final crash. Once again it will be supposed that the comet's motion is on the straight line starting from $A$ and ending in the sun. $A$ and $B$ are both equipped with standard clocks, so that $A$'s clock will tick in $t$-time and $B$'s in $s$-time. Each clock emits a flash of light at each of its ticks, and these flashes are seen by the other, counted and compared with the number of ticks of its own clock, both counts starting from the instant when $B$ left $A$.

It is to be noted that instead of counting ticks and flashes, each observer might be supposed to be watching the dial of the other's clock through a telescope and comparing it with the reading of the dial of his own clock. In that case he could compare the two clock-readings with infinite accuracy and not merely as a pair of integers. Nevertheless, I shall use the expressions *tick* and *flash* as the easiest way to describe the experiments, but it justifies the use of fractions of a tick or of a flash to any degree of accuracy.

These results will be calculated below, but it may be well first to consider what a crude consideration of the subject would suggest. We know that a clock in a planet


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$B$ going in a circle at radius $r$ runs at a rate $\sqrt{(1-3/r)}$ compared to $A$, so that by counting $B$'s flashes $A$ will say that $B$'s clock is going slow. Conversely $B$ will say that $A$'s clock is going fast. We also know from the case of special relativity that a moving clock appears to go slower the higher its speed. These facts would suggest that inside $r = 3$ $B$'s clock would go still slower as he approaches the sun, and that the comet's death would be indicated by his clock-rate going to zero as observed by $A$. Conversely, though we know that for free motions in outer space each clock reckons that the other's is going slow, we might conjecture that this effect would later be offset by the slowing down for the inner orbits, so that $B$ would observe that $A$'s flashes arrived at a rate increasing all the time, and the comet's imminent death would be indicated to $B$ by the flashes arriving from $A$ at a rate tending to infinity. We shall verify the first of these suppositions, but the second will be shown to be completely wrong.

## 9. The use of radar

For the travel of light-rays it will be convenient to use the symbol $\tau$ instead of $t$, so that $t$ may be reserved for the description of the comet's motion. For any light-ray $\mathrm{d}s = 0$, so that its motion is here described by the equation

$$-\frac{r}{r-2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2+\frac{r-2}{r} = 0, \tag{9·1}$$

or

$$\frac{\mathrm{d}\tau}{\mathrm{d}r} = \pm\frac{r}{r-2}. \tag{9·2}$$

If the ray starts from $R$ and ends nearer the sun at $r$, the negative sign must be taken and this gives a formula conveniently written as $\tau(R)-\tau(r)$, (9·3)

where

$$\tau(r) = r+2\ln(r-2). \tag{9·4}$$

For the returning ray the positive sign is to be taken in (9·2), and the value is again given by (9·3).

If $r$ is not far from $R$ the logarithmic terms are negligible, and the distance is measured by halving the total time interval as $R-r$, which determines the distance between $A$ and $B$ at the half-time between the emission and return of the ray, just as it does in the ordinary use of radar.

But as $r$ approaches 2 the last term becomes large, and finally infinite when $r = 2$. Now if the comet is fired from $A$ at a high speed, it should evidently take only a short time to arrive at $r = 2$, and yet we have the curious result that $A$ requires an infinite time to verify this. The use of radar would appear not to be a very powerful method.

## 10. The exchange of flashes

The equations governing the motion of $B$ are now

$$-\frac{r}{r-2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2+\frac{r-2}{r}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 = 1, \tag{10·1}$$

$$\frac{r-2}{r}\frac{\mathrm{d}t}{\mathrm{d}s} = C. \tag{10·2}$$

Write $u$ for $1/r$, and $D^2$ for $C^2-1+2u$, (10·3)
and then, since $B$ is moving towards the sun, these yield

$$\frac{\mathrm{d}s}{\mathrm{d}r} = -\frac{1}{D}, \tag{10·4}$$

$$\frac{\mathrm{d}t}{\mathrm{d}r} = -\frac{C}{D(1-2u)}. \tag{10·5}$$

These are easily integrated, and formulae will be given later, but they are not algebraically very simple, and some of the results can be obtained in a general form without working the integrals out.

Define the functions $s(r)$ as $\int_{r_0}^{r}\frac{\mathrm{d}r}{D}$, and $t(r)$ as $\int_{r_0}^{r}\frac{C\mathrm{d}r}{D(1-2u)}$, where $r_0$ may have any convenient value whatever. Then at the time when $B$ has reached $r$, his clock will have given $s(R)-s(r)$ ticks, and at this time $A$ will have given $t(R)-t(r)$ ticks. $A$ will have emitted this number of flashes, but a number $\tau(R)-\tau(r)$ of them will be still travelling in space towards $B$ so that $B$, when he is at $r$, will only have counted $[t(R)-t(r)]-[\tau(R)-\tau(r)]$ of them. On the other hand, by the time $B$ reaches $r$ he will have emitted $s(R)-s(r)$ flashes, and the last of these will arrive at $A$ at a time later by the amount $\tau(R)-\tau(r)$, by which time his clock will have given

$$[t(R)-t(r)]+[\tau(R)-\tau(r)]$$

ticks.

Briefly, the experiment of $B$ will consist in comparing $t-\tau$ flashes with $s$ ticks, which will record his observation when he is at $r$. On the other hand, the experiment of $A$ will compare $s$ flashes with $t+\tau$ ticks of his clock, but this observation will refer not to the instant at which it is made, but to the past time when $B$ was at $r$.

## 11. Flash-rates

We can immediately set down what may be called the two flash-rates, that is to say the number of flashes per tick at either station. In the case of $B$ this rate will refer to the time when he is at $r$, whereas in the case of $A$ it will refer to the past time when $B$ was at $r$, though the observation will only be made later.

Then recalling (9·2), (10·4) and (10·5), we have

$$F_A = \frac{\mathrm{d}s}{\mathrm{d}(t+\tau)} = C-D \tag{11·1}$$

and

$$F_B = \frac{\mathrm{d}(t-\tau)}{\mathrm{d}s} = \frac{1}{C+D}. \tag{11·2}$$

Take $C>1$ so that $B$ would have finite velocity $V$ at infinity. Then $C=(1-V^2)^{-\frac{1}{2}}$, and as a first application consider the case where $r$ is only a little less than $R$. The term in $u$ is then negligible and $D=V(1-V^2)^{-\frac{1}{2}}$. Then both $F_A$ and $F_B$ give the result $\sqrt{(1-V)/(1+V)}$, which is the well-known result for the Doppler effect in free space.

When $r$ gets near 2, $D$ approaches $C$ in value, and $F_A$ tends to zero. Thus the death of the comet is indicated by $F_A$ going to zero as $r$ approaches 2, in agreement with the form suggested in § 8. On the other hand, $F_B$ decreases all the time as $B$ moves from $R$ downwards, contrary to the suggestion made in § 8, but it is still finite at $r = 2$, being there $\frac{1}{2}\sqrt{(1-V^2)}$. There is so to speak nothing to tell $B$ that he is in the middle of a crisis at the place where $A$ would suggest that he is dying. This is one of the unexpected results, and it will be discussed later. For the present we will consider only values of $r$ above 2.

## 12. Integrated results

In carrying out the integrations there are minor differences according to whether $C$ is greater or less than 1. The case $C = (1-V^2)^{-\frac{1}{2}} > 1$ will be taken here, though the results are much the same if $C < 1$, provided it is sufficiently great for $B$ to start its motion from near $A$.

The most useful functions to calculate are $s$, $t-\tau$ and $t+\tau$, and choice must be made for the arbitrary constants $r_0$ at the lower limits of their integrals. Since all the useful results will depend on the differences between pairs of integrals, any values would do, but as will appear both $s$ and $t-\tau$ are finite at $r = 0$, so that is the natural starting point for them. The same is not true of $t+\tau$, but a natural choice is suggested by making this integral match the other two in form.

Table 1

| $r$ | $F_A$ | $F_B$ | $s$ | $t-\tau$ | $t+\tau$ |
|---|---|---|---|---|---|
| 1000 | 0·499 | 0·499 | 1319 | 657 | 2680 |
| 100 | 0·487 | 0·497 | 124·3 | 60·8 | 276·3 |
| 10 | 0·377 | 0·471 | 9·44 | 4·13 | 29·6 |
| 5 | 0·269 | 0·448 | 4·00 | 1·61 | 13·23 |
| 4 | 0·219 | 0·437 | 2·95 | 1·17 | 9·17 |
| 3 | 0·141 | 0·424 | 2·02 | 0·77 | 4·00 |
| 2 | 0 | 0·400 | 1·15 | 0·41 | $-\infty$ |
| 1 | — | 0·351 | 0·43 | 0·13 | — |
| 0 | — | 0 | 0 | 0 | — |

By routine processes it is then found that, if $k$ is written for $(1-V^2)/V^2$,

$$s = \sqrt{[kr(r+2k)]} - 2k^{\frac{3}{2}}J_1, \tag{12·1}$$

$$t-\tau = \frac{1}{V}\sqrt{[r(r+2k)]} - r + 2\frac{3V^2-1}{V^3}J_1 - 4J_2, \tag{12·2}$$

$$t+\tau = \frac{1}{V}\sqrt{[r(r+2k)]} + r + 2\frac{3V^2-1}{V^3}J_1 + 4J_3, \tag{12·3}$$

where

$$J_1 = \ln[\sqrt{r} + \sqrt{(r+2k)}]/\sqrt{(2k)}, \tag{12·4}$$

$$J_2 = \ln[\sqrt{r} + V\sqrt{(r+2k)}]/[V\sqrt{(2k)}], \tag{12·5}$$

$$J_3 = \ln[\sqrt{r} - V\sqrt{(r+2k)}]/[V\sqrt{(2k)}]. \tag{12·6}$$

The formulae are too complicated for their values to be easily grasped, and a numerical example will help. Take $C = \frac{5}{4}$, so that $V = \frac{3}{5}$, giving a Doppler effect of $\frac{1}{2}$ at the start of the motion. Then table 1 gives some of the values.

To illustrate the use of the table assume that $A$ is at $R = 1000$, and consider the readings when the comet is at 4. $B$ will have counted $1319 - 2·95$ ticks as against

657 − 1·17 flashes. When $A$ later gets the corresponding observation he will have counted 1319 − 2·95 flashes as against 2680 − 9·17 ticks.

It may be remarked that the numerical values for large $r$ are not given here to the same number of decimal places as the later ones, because the purpose of the table is merely for illustration, and it would have no importance to do so. With a view to the later discussion of the region below $r = 2$, the values of $F_B$, $s$ and $t-\tau$ are entered for the rows 2, 1, 0.

## 13. Speculations about the region below $r = 2m$

These results only discuss the extreme case of a comet moving at a rather high velocity straight towards the sun. It would be a difficult problem to treat of approach in a spiral, for not only would the orbits require elliptic functions of the type of I, § 10, for their description, but the flashes would travel along different paths from that of the comet. I have not attempted this problem. However, it seems a reasonable conjecture that similar results would emerge, in that the flash-rate $F_A$ would go to zero at $r = 2$, while $F_B$ would be finite there. The results of § 7 confirm this to some extent, since though there was no attempt there at evaluating flash-rates, it did appear that $t$ would become infinite when $B$ reached 2, but neither $\theta$ nor $s$ showed any discontinuity there. It is interesting then to examine the consequences of supposing that it is legitimate to accept the results for $s$ below $r = 2$.

The first point that emerges seems to imply a contradiction in the results of table 1. When $B$ reaches 2 the flash-rate he observes is finite, and there seems nothing to tell him that anything is happening there. $F_B$ only vanishes at $r = 0$, so that $B$ would say that he died not at 2 but at 0, and he would claim that his clock had ticked a total of 1319 times. On the other hand $A$ would say that $B$'s death occurred at $r = 2$, when he had emitted only 1319 − 1·15 flashes. It must, however, be noted that $A$ will have taken an infinite time to count this number of flashes, so that he has no means by which he could observe the last 1·15 ticks of $B$'s clock. This apparent contradiction may perhaps be attributed to a deficiency in the flash-tick method of experiment like that in the radar experiment.

It may also be noted that $B$ will have counted a total of 657 flashes, but this tells him nothing about the $t$-time of his death at $r = 0$, since the last flash he receives will have been emitted from $A$ at an earlier $t$-time, but there is no way for him to estimate how much earlier it was. Nor can $A$ give the value since it has taken him an infinite time even to measure the effects at $r = 2$.

A rather more elaborate experiment may be imagined which emphasizes the incompleteness of the results. Suppose that the motion of $B$ is suddenly reversed at some place, or to avoid dynamical questions about the reversal, suppose that a second comet $B'$ starts out from the sun, and travels towards $A$, passing $B$ with reversed velocity at the place $r_1$, and setting his clock by that of $B$ as he passes him.

If $r_1 > 2$ the problem is quite definite. The detailed calculations merely call for changes of sign in (10·4) and (10·5), but they need not be given here. The results are that $F'_B = 1/F_A$, and $F'_A = 1/F_B$. Furthermore, the total number of ticks of $B$ and $B'$ on $B'$'s arrival at $A$ is $2[s(R)-s(r_1)]$, while the total number of ticks of $A$ is

$$2[t(R)-t(r_1)].$$




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Take the example of table 1 with $r_1 = 3$. Then at the moment $B'$ passes $B$ the flash-rate will suddenly change from 0·424 to 1/0·141 = 7·092, and then it will decrease to 2 as $B'$ travels towards $A$. Again $A$'s flash-rate will suddenly change from 0·141 to 1/0·424 = 2·358, and then it will diminish, also ending at 2. The total number of the ticks of $B$ and $B'$ will be 2(1319 – 2·02), while the total number of $A$'s ticks will be (2680 – 4·00) + (657 – 0·77). On the other hand if the switch-over occurs for $r_1$ only infinitesimally above 2, the result is reached that when $B'$ arrives at $A$ his clock will only have given about two ticks more than this, yet he will find that $A$ has lived a nearly infinite time since the original departure of $B$.

Now consider what will be observed if the switch-over from $B$ to $B'$ occurs at $r_1 = 1$. The total number of ticks recorded by the two $B$'s will be 2(1319 – 0·43). As to flashes from $A$, $B$ will have received (657 – 0·13), but no flashes will reach $B'$ while he is going from 1 to 2. At $r = 2$ he will start receiving flashes at an infinite rate, which will have decreased to the rate 2 when he reaches $A$. But the important point to note is that there is no way for him to assign the $t$-time of his arrival there.

## 14. Conclusions about the orbit inside $r = 2m$

In the first place it must be noted that the whole time of the comet's reaching the sun must be very short. Even if the comet's acceleration is disregarded, it would in the numerical example only take about 1660 units of time for him to get there. Yet with the flash-tick method it takes an infinite time to make the observations.

It must be noted that the whole calculation starts from (2·1) to (2·3), and that the independent variable there is $s$, and not $t$. In the general study of differential equations it is usual to find that all relations derived from them hold up to the points where singularities occur through the variables reaching infinity. This rather strongly suggests that the values of $\theta$ and $s$ inside $r = 2m$ do have a physical meaning, and that it is only $t$ that fails there. On the other hand, when $B'$ passes $B$ inside $2m$, no way has been devised for deciding the $t$-time of his arrival at $A$.

It would seem that to clear the matter up, some new different experiment must be devised. Such an experiment might reveal a weakness in the flash-tick method, and it might show that in spite of the finite value of $F_B$ at $r = 2m$, nevertheless the comet did crash there. Alternatively it might show a way of assigning the $t$-time of the arrival of $B'$ at $A$, and it seems not impossible that this might give a physical meaning to the fact suggested by (2·3) that $t$-time goes backwards inside $r = 2m$.

It is to be hoped that some method may be devised which will clear up these curious results.

## Reference

Darwin, Sir Charles 1958 *Proc. Roy. Soc.* A, **249**, 180 (part I).