10 August 1998

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 245 (1998) 31–34

On the weak field approximation of the Brans–Dicke theory of gravity

A. Barros [a], C. Romero [b,1]

[a] Departamento de Física, Universidade Federal de Roraima, 69310-270, Boa Vista, RR, Brazil

[b] Departamento de Física, Universidade Federal da Paraíba, Caixa Postal 5008, 58059-970, João Pessoa, PB, Brazil

Received 19 September 1997; revised manuscript received 1 April 1998; accepted for publication 11 May 1998

Communicated by J.P. Vigier

Abstract

It is shown that in the weak field approximation solutions of Brans–Dicke equations are simply related to the solutions of general relativity equations for the same energy-momentum tensor. A method is developed which permits one to obtain Brans–Dicke solutions from Einstein solutions. To illustrate the method some examples are discussed. © 1998 Elsevier Science B.V.

It is a well known fact that most of the mathematical difficulties of general relativity theory lies in the high non-linearity of the Einstein field equations. On physical grounds this non-linearity means that the gravitational field interacts with itself, and the field contributes to its own source. However, under the special circumstance when the gravitational field is weak one can linearize the field equations thereby ignoring this feedback effect. Such a procedure, which leads to a great mathematical simplification of the gravitational field equations, has always found a wide range of applications over the years.

Certainly, the weak field approximation technique is not restricted to general relativity. It has been applied to the Brans–Dicke theory of gravity, another metric theory which also makes use of a highly non-linear set of field equations [1].

In this paper we investigate how solutions of linearized Einstein equations are related to solutions of linearized Brans–Dicke equations when both correspond to the same energy–momentum tensor.

To begin with let us recall that in the weak field approximation of general relativity we assume that the space-time metric tensor deviates only slightly from the flat space-time metric tensor. To put it more precisely we write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \tag{1}$$

where $\eta_{\mu\nu}$ denotes Minkowski metric tensor and $h_{\mu\nu}$ is to be considered a small perturbation term. The linearized equations are obtained from direct substitution of (1) into the Einstein's equations keeping only first-order terms in $h_{\mu\nu}$.

On the other hand, Brans–Dicke field equations are given by

$$G_{\mu\nu} = \frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega}{\phi^2}(\phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}) + \frac{1}{\phi}(\phi_{,\mu;\nu} - g_{\mu\nu}\Box\phi) \,, \tag{2}$$

[1] E-mail: cromero@fisica.ufpb.br.

PII S0375-9601(98)00382-X

$$\Box \phi = \frac{8\pi T}{2\omega + 3}, \tag{3}$$

where ϕ is a scalar field, ω is a dimensionless coupling constant and T denotes the trace of the energy-momentum tensor $T_{\mu\nu}$. Although (2) and (3) represent the more usual or standard form of Brans-Dicke equations we are going to consider equivalently the so-called *Einstein representation* [2] given by

$$\bar{G}_{\mu\nu} = 8\pi G_0 \times \left[\bar{T}_{\mu\nu} + \frac{2\omega + 3}{16\pi G_0 \phi^2}\left(\phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}\right)\right], \tag{4}$$

$$\bar{\Box} \ln(G_0\phi) = \frac{8\pi G_0}{2\omega + 3}\bar{T}, \tag{5}$$

which are obtained from (2) and (3) by doing the transformation

$$\bar{g}_{\mu\nu} = G_0 \phi g_{\mu\nu}, \tag{6}$$

$$\bar{T}_{\mu\nu} = G_0^{-1}\phi^{-1} T_{\mu\nu}, \tag{7}$$

where G_0 is an arbitrary constant and the bar in $\bar{G}_{\mu\nu}$, $\bar{\Box}$ and $\bar{T}$ just means that these quantities are now calculated using the unphysical metric $\bar{g}_{\mu\nu}$.

In the weak field approximation of Brans-Dicke theory in addition to (1) we must also assume that

$$\phi = \phi_0 + \epsilon, \tag{8}$$

where $\epsilon = \epsilon(x)$ is a first-order term in the energy density and $|\epsilon/\phi_0| \ll 1$.

Taking into account (8) and setting $G_0 = 1/\phi_0$ the transformation equations (6) and (7) become

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}, \tag{9}$$

$$\bar{T}_{\mu\nu} = (1 - \epsilon G_0) T_{\mu\nu} = T_{\mu\nu}, \tag{10}$$

where

$$\bar{h}_{\mu\nu} = h_{\mu\nu} + \epsilon G_0 \eta_{\mu\nu}, \tag{11}$$

and only first-order terms in the mass density have been kept.

Now, substituting (8) in the field equations (4) and having in view (9) and (10) we get

$$\bar{G}_{\mu\nu} = 8\pi G_0 T_{\mu\nu}. \tag{12}$$

On the other hand, the scalar field equation (3) becomes

$$\Box \epsilon = \frac{8\pi T}{2\omega + 3}. \tag{13}$$

It turns out then that Eqs. (12) are formally identical to the field equations of general relativity with G_0 replacing the Newtonian gravitational constant G. Therefore, if $\bar{g}_{\mu\nu}(G, x)$ is a known solution of the Einstein equations in the weak field approximation for a given $T_{\mu\nu}$, then the Brans-Dicke solution corresponding to the same $T_{\mu\nu}$ will be given in the weak field approximation just by taking the inverse of Eq. (6), i.e.,

$$g_{\mu\nu}(x) = G_0^{-1}\phi^{-1}\bar{g}_{\mu\nu}(G_0, x) = [1 - \epsilon(x) G_0]\bar{g}_{\mu\nu}(G_0, x), \tag{14}$$

or, equivalently,

$$h_{\mu\nu}(x) = \bar{h}_{\mu\nu}(G_0, x) - \epsilon(x) G_0 \eta_{\mu\nu}. \tag{15}$$

Thus, we conclude that the general problem of finding solutions of Brans-Dicke equations of gravity in the weak field approximation may be reduced to solving Einstein field equations for the same energy-momentum tensor.

It should be noted that the Einstein tensor $\bar{G}_{\mu\nu}$ which appears in the left hand side of (12) must be calculated in the weak field approximation, i.e., taking $\bar{g}_{\mu\nu}$ as given by (9).

As to the function $\epsilon(x)$, which appears in the conformal factor of the metric $\bar{g}_{\mu\nu}(G_0, x)$, it may be calculated directly from (13) and will be given as a retarded integral in the form

$$\epsilon(x) = \frac{2}{2\omega + 3}\int \frac{T(t - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, d^3x', \tag{16}$$

with $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ in Eq. (13). Let us conclude this paragraph with a remark concerning the constant ϕ_0, which comes out in the weak field approximation. Actually, in order that Brans-Dicke theory possesses a Newtonian limit this constant must be related to the Newtonian gravitational constant G by setting [1]

$$\frac{1}{\phi_0} = \left(\frac{2\omega + 3}{2\omega + 4}\right) G. \tag{17}$$

Thus we have

$$G_0 = \left(\frac{2\omega+3}{2\omega+4}\right) G. \tag{18}$$

We shall go through two solutions of Brans-Dicke equations in the weak field approximation and show how they can be directly obtained with the help of the method just outlined.

Let us start with the line element which describes the space-time generated by a static string the energy-momentum tensor of which is given by

$$T^{\mu}_{\nu} = \delta(x)\delta(y)\mathrm{diag}(\mu, 0, 0, -p),$$

where μ is the linear energy density and p is the pressure in the z direction. The solution of this problem in the context of general relativity was first worked out by Vilenkin [3]. Using the weak field approximation Vilenkin solved the field equations and obtained in cartesian coordinates

$$\bar{h}_{00} = \bar{h}_{33} = 4G(\mu + p)\ln(\rho/\rho_0), \tag{19}$$

$$\bar{h}_{11} = \bar{h}_{22} = 4G(\mu - p)\ln(\rho/\rho_0), \tag{20}$$

where $\rho = (x^2+y^2)^{1/2}$ and ρ_0 is a constant. Now, using the weak field approximation to approach this problem in Brans-Dicke theory we have to solve Eq. (13), which takes the form

$$\nabla^2\epsilon = -\frac{8\pi}{2\omega+3}(\mu - p)\delta(x)\delta(y), \tag{21}$$

whose solution is readily found to be

$$\epsilon = -\frac{4(\mu-p)}{2\omega+3}\ln\frac{\rho}{\rho_0}. \tag{22}$$

Then, from (14) it follows that the sought-for line element, which describes the space-time generated by the string in Brans-Dicke theory, is given by

$$\begin{aligned} ds^2 = &\left[1 + \frac{4(\mu-p)G_0}{2\omega+3}\ln\frac{\rho}{\rho_0}\right] \\ &\times\left[\left(1 + 4(\mu+p)G_0\ln\frac{\rho}{\rho_0}\right)dt^2\right. \\ &-\left(1 - 4(\mu-p)G_0\ln\frac{\rho}{\rho_0}\right)(dx^2+dy^2) \\ &\left.-\left(1 - 4(\mu+p)G_0\ln\frac{\rho}{\rho_0}\right)dz^2\right]. \end{aligned} \tag{23}$$

Particularly, for a vacuum string, $p = -\mu$, and turning to cylindrical coordinates (23) reduces to

$$\begin{aligned} ds^2 = &\left[1 + \frac{8\mu G_0}{2\omega+3}\ln\frac{\rho}{\rho_0}\right]\left[dt^2 - dz^2\right. \\ &\left.-\left(1 - 8\mu G_0\ln\frac{\rho}{\rho_0}\right)(d\rho^2 + \rho^2 d\theta^2)\right]. \end{aligned} \tag{24}$$

Finally, introducing a new coordinate ρ' by the transformation $\rho = \rho_0(\rho'/a)^b$, where $a = \rho_0(1 - 8\mu G_0)^{-1/2}$ and $b = (1 - 4\mu G_0)^{-1}$, and neglecting second-order terms in μG_0 we arrive at

$$\begin{aligned} ds^2 = &\left[1 + \frac{8\mu G_0}{2\omega+3}\ln\frac{\rho'}{\rho_0}\right] \\ &\times[dt^2 - dz^2 - d\rho'^2 - (1 - 8\mu G_0)\rho'^2 d\theta^2], \end{aligned} \tag{25}$$

which is the result obtained in Ref. [4].

The second example comes from Vilenkin's solution corresponding to the space-time of a static massive plane [3]. In this case, the source of the gravitational field consists of an infinite static plane wall parallel to the (y, z) plane. For a homogeneous energy surface distribution σ the energy-momentum tensor T^{μ}_{ν} is given by $T^{\mu}_{\nu} = \delta(x)\mathrm{diag}(\sigma, 0, -p, -p)$, where p is the pressure. In the weak field approximation of the linearized Einstein equations yield the solution

$$\begin{aligned} ds^2 = &[1 + 4\pi G(\sigma + 2p)|x|]dt^2 \\ &-[1 - 4\pi G(\sigma - 2p)|x|]dx^2 \\ &-[1 - 4\pi G\sigma|x|](dy^2 + dz^2). \end{aligned} \tag{26}$$

As before, in order to get the corresponding solution in Brans-Dicke theory, we must solve Eq. (13), which in this case will be given by

$$\Box\epsilon = \frac{8\pi}{2\omega+3}(\sigma - 2p)\delta(x). \tag{27}$$

Due to planar symmetry $\epsilon = \epsilon(x)$ and (27) is reduced to

$$\frac{d^2\epsilon}{dx^2} = -\frac{8\pi}{2\omega+3}(\sigma - 2p)\delta(x),$$

hence yielding the solution

$$\epsilon = -\frac{4\pi}{2\omega+3}(\sigma - 2p)|x|. \tag{28}$$

Therefore, from (14) we obtain

$$\begin{aligned} ds^2 = & \left[1 + \frac{4\pi G_0}{2\omega+3}(\sigma - 2p)|x|\right] \\ & \times \Big[(1 + 4\pi G_0(\sigma + 2p)|x|)dt^2 \\ & - (1 - 4\pi G_0(\sigma - 2p)|x|)dx^2 \\ & - (1 - 4\pi G_0\sigma|x|)(dy^2 + dz^2)\Big], \end{aligned} \tag{29}$$

which represents the space-time generated by the static massive plane in Brans-Dicke theory. For a vacuum domain wall, $p = -\sigma$ and

$$h_{00} = -h_{22} = -h_{33} = -\frac{8\pi G_0 \sigma \omega |x|}{2\omega + 3}, \tag{30}$$

$$h_{11} = \frac{24\pi G_0 \sigma(\omega + 1)|x|}{2\omega + 3} \tag{31}$$

in accordance with Ref. [4].

An interesting Brans-Dicke solution that can be easily worked out in this context corresponds to the space-time and the scalar field generated by a static point of mass [1]. Another illustrative example is provided by the global monopole [5,6].

To conclude we would like to briefly comment on the result expressed by Eq. (14). Essentially, this equation means that in the weak field approximation the metric tensor calculated from Brans-Dicke equations is quasi-conformally related to the metric tensor calculated from Einstein equations for the same energy-momentum tensor. The term quasi-conform here should be understood in the sense that in going from the Einstein solution $\bar{g}_{\mu\nu}(G, x)$ to the corresponding Brans-Dicke solution $g_{\mu\nu}(x)$ apart from the scale factor $\lambda(x) \equiv 1 - \epsilon(x)G_0$ one must replace G for G_0 in $\bar{g}_{\mu\nu}$, i.e., $g_{\mu\nu}(x) = \lambda(x)\bar{g}_{\mu\nu}(G_0, x)$.

An immediate physical consequence of that concerns the trajectories of light rays. For it is evident that null geodesics in both space-times described by $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$ are closely related: the only change involved is the replacement of the Newtonian gravitational constant G by the new ω-dependent "effective" gravitational constant $G_0 = [(2\omega + 3)/(2\omega + 4)]G$. For a value of ω consistent with solar system observations and experiments, say $\omega \sim 500$ [7], it means that massless particles travelling in the space-time described by $g_{\mu\nu}$ would "feel" a decrease in the gravitational strength as $G_0 \sim 0.999G$.

Finally, it is worth mentioning that in the weak field approximation when $\omega \to \infty$ the Brans-Dicke solution goes over the corresponding Einstein solution, although this does not always happen in the case of exact solutions [8]. Indeed, to prove this statement just note that when $\omega \to \infty$ we have, respectively, from (16) and (18) that $\epsilon(x) \to 0$ and $G_0 \to G$.

C. Romero thanks CNPq (Brazil) for financial support.

References

[1] C. Brans, R.H. Dicke, Phys. Rev. 124 (1961) 925.
[2] R.H. Dicke, Phys. Rev. 125 (1962)
[3] A.Vilenkin, Phys. Rev. D 23 (1981) 852.
[4] A.Barros, C. Romero, J. Math. Phys. 36 (1995) 5800.
[5] M. Barriola, A. Vilenkin, Phys. Rev. Lett. 63 (1989) 341.
[6] A. Barros, C. Romero, Phys. Rev. D 56 (1997) 6688.
[7] C.M. Will, Theory and Experiment in Gravitational Physics (Cambridge University Press, 1993).
[8] C. Romero, A. Barros, Phys. Lett. A 173 (1993) 243; N. Banerjee, S. Sen, Phys. Rev. D 56 (1997) 1334.