

# An accurate equation for the gravitational bending of light by a static massive object

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## ABSTRACT

An exact analytical expression for the bending angle of light due to a non-rotating massive object, considering the actual distances from source and observer to the gravitational mass, is derived. Our novel formula generalizes Darwin’s well-known equation for gravitational light bending, where both source and observer are placed at infinite distance from the lensing mass, and provides excellent results in comparison with the post-Newtonian (PPN) formalism up to first order. As a result, the discrepancy between our recent expression and the PPN approach is 6.6 mas for sun-grazing beams coming from planet Mercury, with significant differences up to 2 mas for distant starlight. Our findings suggest that these considerations should not be dismissed for both solar system objects and extragalactic sources, where non-negligible errors might be present in ultraprecise astrometry calculations.

**Key words:** gravitational lensing; weak – relativistic processes – astrometry.

## 1 INTRODUCTION

The gravitational deflection of light by a massive body has been a subject of intense research for over three centuries. According to Newton (1730), if a light ray from a distant star passes near a massive body, it would be bent a very small amount due to the object’s gravity. However, it was not until 1804 when this bending angle was first calculated by Soldner (1804), resulting in a value of 0.87 arcsec for sun-grazing starlight. A century later, Einstein (1916) reported a deflection angle of 1.75 arcsec within the framework of the general relativity (GR) theory, twice the value as obtained by Newtonian mechanics. This result was experimentally confirmed by Eddington from the May 1919 solar eclipse expeditions (Dyson, Eddington & Davidson 1920; Will 2015) and subsequent measurements via Very Long Baseline Interferometry (VLBI), a technique capable of measuring bending angles from distant radio sources with high accuracy (Shapiro 1967; Lebach et al. 1995; Li et al. 2022a, b).

Apart from providing a means to test GR, gravitational deflection of light has been widely employed to observe the properties of very distant galaxies, as well as to infer the mass of astrophysical objects, since the massive body acts as a gravitational lens with a characteristic magnifying effect (Schneider, Ehlers & Falco 1992; Frittelli 2003; Ye & Lin 2008). Accordingly, gravitational lensing is indeed a milestone of astronomy with wide-ranging applications, covering extra-solar planets (Turyshev & Toth 2022), black hole lensing (Iyer & Petters 2007; Virbhadra 2009), or string theory (He et al. 2022; Kong, Yang & Yu 2024). In order to calculate the light deflection angle for static massive objects, the prevalent GR formalism considers that the path of a light ray is a null

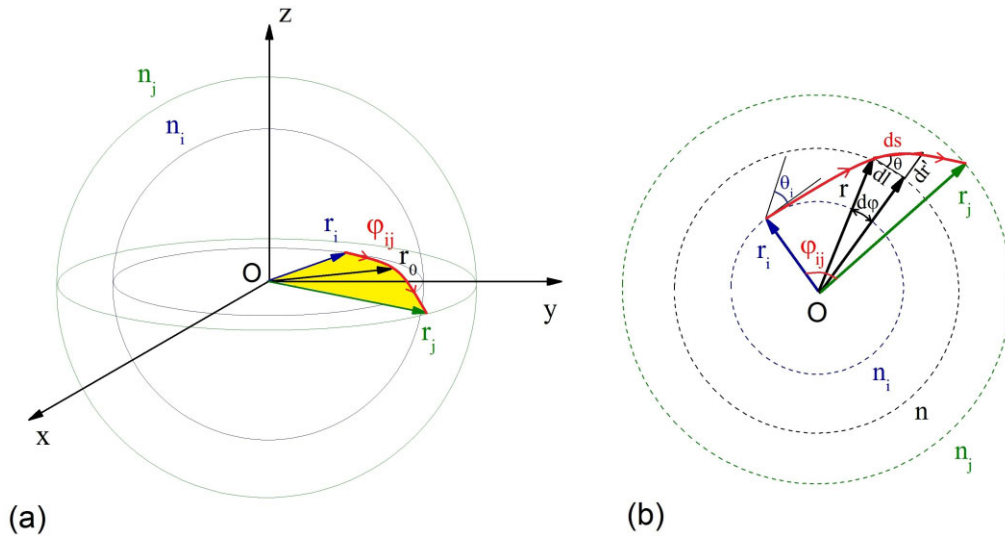
geodesic in different manifolds (Misner, Thorne & Wheeler 1973; Chandrasekhar 1983; Wald 1984; Bozza et al. 2005).

An alternative method to study the effect of gravity on light is the so-called material medium approach (hereafter MMA), based on the idea of representing the gravitational field as an optical medium with an effective refractive index. In fact, this different conception of light bending has a long history since the early days of GR (Dyson et al. 1920; Whitehead 1922). Eddington himself admitted that the gravitational deflection effect on light could be imitated by a refractive medium filling the space round the Sun, giving an appropriate velocity of light. Specifically, this refractive index at a distance  $r$  from the centre of the Sun should be  $[1 - (r_s/r)]^{-1}$ , where  $r_s$  corresponds to the Sun’s Schwarzschild radius (Dyson et al. 1920). Therefore, a light ray passing through a material medium will be deviated due to the refractive index variation of the associated media, in accordance with the well-established GR explanation.

Apart from the Eddington’s analysis on gravitational light bending, the MMA was first developed by Tamm during the 1920s (Tamm 1924, 1925). This innovative idea was used by several authors to discuss the optical phenomena for the deflection of electromagnetic waves by a gravitational field (Balazs 1958; Plebanski 1960; De Felice 1971), mainly for non-rotating masses in the Schwarzschild geometry (Fischbach & Freeman 1980; Nandi & Islam 1995; Evans, Nandi & Islam 1996a, b; Sen 2010; Feng & Huang 2019, 2020), where the medium refractive index could be expressed as an infinite power series of  $(r_s/r)$  terms (Schneider et al. 1992; Petters, Levine & Wambsganss 2002; Roy & Sen 2019; Meneghetti 2021; Hwang & Noh 2024). Moreover, the same method was also applied to estimate the light deflection angle caused by the rotation of gravitating bodies (Roy & Sen 2015) or charged massive objects (Roy & Sen 2017).

In both theoretical frameworks (i.e. the MMA and GR) some authors consider an asymptotic scenario where source and observer are placed at infinite distance from the lensing mass, which is actually

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**Figure 1.** (a) Schematic representation of our spherical-stratified medium in a non-flat spacetime (i.e. the Schwarzschild spacetime in our case). The gravitational mass  $M$  is located at point  $O$  and each concentric sphere presents the same refractive index value, according to equation (6). The light source and observer are placed at coordinates  $r_i$  and  $r_j$ , respectively, and a light ray propagates between both positions. The closest approach distance to the static mass is  $r_0$ , whereas the rotation angle is given by  $\phi_{ij}$ . Without loss of generality, we restrict ourselves to the equatorial plane where the polar angle is  $\pi/2$  (b) top view of the equatorial plane with the elements required to compute the propagation trajectory of a light ray in this graded medium.

a reasonable approach given the large distances involved. On the other hand, some researchers have studied a finite-distance scheme with specific light paths, addressing the problem numerically (Feng & Huang 2019, 2020) or via approximate deflection angle equations (Zschocke 2011; Ishihara et al. 2017; Takizawa & Asada 2023). In this regard, the formalism provides a means to analytically calculate the bending angle to any desired order of accuracy by expanding a general formula in  $M/r$  terms (Cowling 1984), where  $M$  is the mass of the gravitational body. In particular, an approximate first order PPN equation has been widely used throughout the literature to determine precise bending angle calculations (Will 1993; Ni 2017; Li et al. 2022a, b). Nevertheless, to the best of our knowledge, an exact analytical expression for the light deflection angle due to a stationary massive body, within a finite-distance scenario, has not yet been reported.

In this article, we employ the MMA formalism to derive an accurate equation for the gravitational deflection of light by a static massive object that generalizes Darwin’s well-known formula (Darwin 1961), where infinite distances from source and observer to the gravitational mass are assumed. As a result, we show that non-negligible errors in the positioning of celestial objects should be avoided if we take into consideration our recent equation. Furthermore, we also test the validity of our MMA via numerical calculation of the gravitational time delay of light (commonly named the Shapiro time delay).

The paper is organized as follows. In Section 2, we present our MMA method to deduce an exact analytical equation for the gravitational deflection angle of light in the Schwarzschild spacetime, considering the actual distances from source and observer to the gravitational body. Moreover, the Shapiro time delay is also revisited within the MMA framework. In Section 3, we describe our fundamental analytical and numerical results, where the appropriateness of our novel equation in ultraprecise astrometry is highlighted. Finally, we summarize our main results and conclusions in Section 4.

## 2 THEORETICAL BASIS

In this section, we develop our MMA to derive a general equation for the bending angle of light due to a static massive object. Then, the Shapiro time delay is addressed theoretically for completeness.

### 2.1 Material medium approach

Let us first analyse Fig. 1(a), a spherical-stratified medium in a non-flat space-time (i.e. the Schwarzschild space-time in our case). The gravitational mass  $M$  is located at point  $O$  and modifies the refractive index of the surrounding media, in accordance with the MMA. For our static body, this index of refraction  $n(r)$  is spherically symmetric and depends exclusively on the radial coordinate  $r$  and its Schwarzschild radius  $r_s = 2GM/c^2$ , where  $G$  is the gravitational constant and  $c$  the speed of light in vacuum. In this context, a light ray describes a specific path in this graded medium where  $r_i$  and  $r_j$  indicate the positions of source and observer, respectively. The parameter  $r_0$  is the closest approach distance to our lensing mass, whereas  $\phi_{ij}$  stands for the angle between both locations. For the sake of simplicity, we restrict ourselves to the equatorial plane where the polar angle is  $\pi/2$ .

A detailed description of this plane, with the elements required to compute the propagation trajectory of a light ray in a graded medium, is depicted in Fig. 1(b). First, we need to pay special attention to the relation between the radial length of the light’s infinitesimals (Misner et al. 1973)

$$dr' = dr \left(1 - \frac{r_s}{r}\right)^{-1/2} = ds \sin \theta, \quad (1)$$

where  $dr'$  corresponds to the proper distance in our curved spacetime, unlike the coordinate distance  $dr$  (not shown in this figure) applicable to a flat space. Moreover,  $ds$  denotes the length of an infinitesimal ray path and  $dl$  is the lateral length.

Considering the basic relation between the infinitesimal angle  $d\varphi$  and  $ds$  (Feng & Huang 2019)

$$rd\varphi = ds \cos \theta, \quad (2)$$

and the previous equation (1), we trivially obtain the following differential equation for light propagation

$$r \frac{d\varphi}{dr} = \left(1 - \frac{r_s}{r}\right)^{-1/2} \cot \theta, \quad (3)$$

which satisfies the renowned Bouguer's law in geometric optics

$$n_i r_i \cos \theta_i = n r \cos \theta = q. \quad (4)$$

Here, the impact parameter of the light ray  $q$  is a constant for a spherical-stratified material medium  $n = n(r)$ , as in our case. Accordingly, we can express equation (3) as

$$\frac{d\varphi}{dr} = \frac{q}{r} \frac{1}{\sqrt{n^2 r^2 - q^2}} \left(1 - \frac{r_s}{r}\right)^{-1/2}. \quad (5)$$

Let us now assume that the medium refractive index  $n(r)$  is just the positive square root of Eddington's proposal (Dyson et al. 1920)

$$n(r) = \left(1 - \frac{r_s}{r}\right)^{-1/2}. \quad (6)$$

Introducing equation (6) into equation (5), the general differential equation for light propagation in such a stratified medium can be written in the following way

$$\frac{d\varphi}{dr} = \frac{1}{r^2} \left[ \frac{r^3 - q^2(r - r_s)}{q^2 r^2 (r - r_s)} \right]^{-1/2} \left(1 - \frac{r_s}{r}\right)^{-1/2}, \quad (7)$$

which depends on the mass of our central object and the impact parameter of the light ray.

Within the scope of GR theory, a light beam in a Schwarzschild space-time obeys the following ordinary differential equation in the equatorial plane (Misner et al. 1973)

$$\frac{d\varphi}{dr} = \frac{1}{r^2} \left[ \frac{r^3 - q^2(r - r_s)}{q^2 r^3} \right]^{-1/2} \left(1 - \frac{r_s}{r}\right)^{-1/2}, \quad (8)$$

where, again,  $q$  corresponds to the impact parameter of the light ray. Please, note the similarity between equations (7) and (8). In fact, a complete equivalence is achieved for the weak-field approximation when  $r \gg r_s$ . In other words, our MMA formalism reproduces exactly the propagation of light in a gravitational field, provided that the ray paths are sufficiently distant from the Schwarzschild radius of the non-rotating body. This requisite is fulfilled in our astronomical scenarios provided that  $r_s = 2.9$  km for the Sun and  $r = 695\,700$  km for sun-grazing light beams.

Hence, the angle  $\varphi_{ij}$  between positions  $r_i$  and  $r_j$  can be accurately evaluated via the MMA (Feng & Huang 2019; please, see again Fig. 1(b))

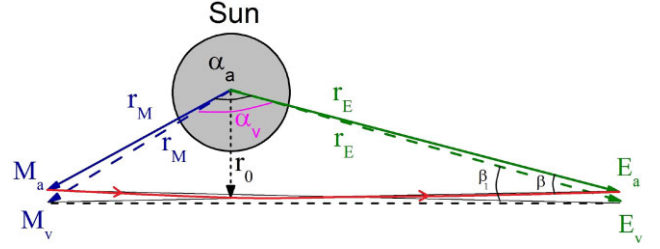
$$\varphi_{ij} = \int_{r_i}^{r_j} d\varphi = \int_{r_i}^{r_j} \frac{dl}{r} = \int_{r_i}^{r_j} \frac{dr'}{r \tan \theta}. \quad (9)$$

Substituting Bouguer's law, equation (4), into equation (9) and performing some elementary algebra, we obtain

$$\varphi_{ij} = \int_{r_i}^{r_j} dr \frac{n_0 r_0}{r \sqrt{n^2 r^2 - n_0^2 r_0^2}} \left(1 - \frac{r_s}{r}\right)^{-1/2}, \quad (10)$$

where the refractive index  $n(r)$  is given by equation (6). It is worth mentioning that equation (10) has analytical solutions in terms of incomplete elliptic integrals of first kind, as briefly addressed.

Once the basic formalism has been introduced, let us now study a specific example where our new approach should be appropriate.



**Figure 2.** Our MMA method applied to a suitable scenario in the solar system. A light ray coming from Mercury ( $M_a$ ) passes close to the Sun and reaches the Earth ( $E_a$ ). Due to the gravitational light bending, a virtual position of each planet occurs, represented by  $M_v$  and  $E_v$ . In this situation, the deflection angle  $\Delta\alpha^{(MMA)}$  is computed as the difference between the actual angle  $\alpha_a$  and the virtual angle  $\alpha_v$  via equations (10) and (11).

Our light source will be planet Mercury near its solar superior conjunction, and a light ray coming from this planet passes close to the Sun and reaches the Earth. This situation is illustrated in Fig. 2 where our light beam travels from the actual position of Mercury ( $M_a$ ) to the Earth ( $E_a$ ). It should be remarked that, due to the gravitational light bending, an observer on Earth would experience a virtual position of Mercury ( $M_v$ ). The inverse scenario is fully applicable, where now  $E_v$  stands for the virtual location of the Earth. Furthermore, the average distances from the Sun to each planet are denoted by  $r_M$  and  $r_E$ ,  $r_0$  stands for the closest approach of the ray path to the Sun, and  $\beta$  is the angle between Mercury (in the absence of gravitational bending) and the Sun as seen by an Earth's observer (Li et al. 2022a, b).

Therefore, the deflection angle  $\Delta\alpha^{(MMA)}$  is calculated as the difference between the actual angle  $\alpha_a$  and the virtual angle  $\alpha_v$  (Feng & Huang 2019)

$$\Delta\alpha^{(MMA)} = \alpha_a - \alpha_v = (\varphi_{0M} + \varphi_{0E}) - \left[ \arccos\left(\frac{r_0}{r_M}\right) + \arccos\left(\frac{r_0}{r_E}\right) \right], \quad (11)$$

where  $\varphi_{0M}$  and  $\varphi_{0E}$  correspond to the angle between  $r_0$  and each planet's actual position, evaluated via equation (10). As the difference between the angles  $\alpha_a$  and  $\alpha_v$  is very small, we can consider that  $\beta \simeq \beta_1$  in order to determine a practical expression for the observation angle  $\beta$ .

Additionally, the deflection angle under the PPN formalism up to first-order reads (Li et al. 2022a, b)

$$\Delta\alpha^{(PPN)} = (1 + \gamma) \frac{GM_s}{r_0 c^2} (\cos \beta - \cos \delta), \quad (12)$$

where  $M_s$  is the solar mass,  $\sin \beta = r_0/r_E$ ,  $\sin \delta = r_0/r_M$ , and  $\gamma$  stands for the dimensionless PPN parameter used to characterize the contribution of the space-time curvature to the gravitational deflection. In this regard, we will assume that  $\gamma = 1$  (as theoretically established in GR) since this choice does not influence significantly the results of the positions of celestial bodies in the solar system (Li et al. 2022a).

When  $r_0$  is far less than the distance from the Sun to both the Earth (observer) and Mercury (source), equation (12) transforms into the celebrated Einstein's formula up to first order (Einstein 1916; Misner et al. 1973; Mutka & Mähönen 2002)

$$\Delta\alpha^{(Ein)} = \frac{2r_s}{r_0}, \quad (13)$$

where a deflection angle of 1.7518 arcsec for starlight grazing the solar surface has been universally accepted. As shortly discussed, a

detailed comparison between the first order PPN formula, equation (12), and our exact MMA expression will be carried out.

## 2.2 Exact analytical equation for the bending angle

The analytical solution of equation (10) has been obtained via a Wolfram Mathematica software. After a careful analysis, the physically acceptable solution for the angle  $\varphi_{ij}$  is given by

$$\varphi_{ij} = 2n(r_i) \sqrt{\frac{r_0(r_i - r_s)}{r_i Q}} F(z_i, k) - 2n(r_j) \sqrt{\frac{r_0(r_j - r_s)}{r_j Q}} F(z_j, k), \quad (14)$$

where  $Q^2 = (r_0 - r_s)(r_0 + 3r_s)$  and  $F(z, k)$  is the Legendre elliptic integral of the first kind, with the following relations for the Jacobi amplitude  $z(r)$

$$\sin^2 z = \frac{2r_0 r_s + r(r_s - r_0 + Q)}{r(3r_s - r_0 + Q)}, \quad (15)$$

and the elliptic modulus  $k$

$$k^2 = \frac{3r_s - r_0 + Q}{2Q}. \quad (16)$$

So, equation (11) can be rewritten as (please, see again Fig. 2)

$$\begin{aligned} \Delta\alpha^{(\text{MMA})} = & \left[ 4n_0 \sqrt{\frac{r_0 - r_s}{Q}} F\left(\frac{\pi}{2}, k\right) \right. \\ & - 2n(r_M) \sqrt{\frac{r_0(r_M - r_s)}{r_M Q}} F(z(r_M), k) \\ & \left. - 2n(r_E) \sqrt{\frac{r_0(r_E - r_s)}{r_E Q}} F(z(r_E), k) \right] \\ & - \left[ \arccos\left(\frac{r_0}{r_M}\right) + \arccos\left(\frac{r_0}{r_E}\right) \right]. \end{aligned} \quad (17)$$

In the asymptotic case, that is, when both source and observer are placed at infinite distance from the gravitational body, equation (17) reduces to the well-known Darwin's formula (Darwin 1961; Misner et al. 1973; Mutka & Mähönen 2002)

$$\Delta\alpha^{(\text{Dar})} = 4 \sqrt{\frac{r_0}{Q}} \left[ F\left(\frac{\pi}{2}, k\right) - F(z_\infty, k) \right] - \pi, \quad (18)$$

by just considering  $r_E, r_M \rightarrow \infty$  in our generalized formula, where now

$$\sin z_\infty^2 = \frac{r_s - r_0 + Q}{3r_s - r_0 + Q}. \quad (19)$$

If only the light source distance to the lensing mass is significantly higher than the closest approach  $r_0$ , we can take the limit  $r_M \rightarrow \infty$  to obtain a modified version of equation (17)

$$\begin{aligned} \Delta\alpha^{(\text{MMA})} = & \left[ 4n_0 \sqrt{\frac{r_0 - r_s}{Q}} F\left(\frac{\pi}{2}, k\right) \right. \\ & - 2n(r_E) \sqrt{\frac{r_0(r_E - r_s)}{r_E Q}} F(z(r_E), k) \\ & \left. - 2 \sqrt{\frac{r_0}{Q}} F(z_\infty, k) \right] - \left[ \arccos\left(\frac{r_0}{r_E}\right) + \frac{\pi}{2} \right], \end{aligned} \quad (20)$$

which constitutes an excellent tool to accurately calculate the gravitational deflection angle for extragalactic emitters.

## 2.3 Shapiro time delay calculation

So far, we have analysed the gravitational bending angle on the basis of our MMA method. Let us now investigate another crucial parameter related to the effect of gravitational bodies on light propagation. We are referring to the Shapiro time delay  $\Delta t$ , the relativistic time shift in the round-trip travel time for light signals reflecting off other planets (Shapiro 1964; Shapiro et al. 1971; Reasenberg et al. 1979). According to the astronomical scenario described in Fig. 2,  $\Delta t$  can be easily expressed as

$$\Delta t = 2 \left[ (t_{0M} + t_{0E}) - \frac{1}{c} \left( \sqrt{r_M^2 - r_0^2} + \sqrt{r_E^2 - r_0^2} \right) \right], \quad (21)$$

where  $t_{0M}$  and  $t_{0E}$  stand for the light propagation time between  $r_0$  and each planet's actual position (i.e. Mercury and the Earth in our situation).

The usual way to deduce an exact expression for the time  $t_{ij}$  that a light ray takes to travel from position  $r_i$  to  $r_j$  is through GR considerations (Wald 1984)

$$t_{ij}^{(\text{GR})} = \frac{\sqrt{r_j^2 - r_i^2}}{c} + \frac{r_s}{c} \log \left( \frac{r_j + \sqrt{r_j^2 - r_i^2}}{r_i} \right) + \frac{r_s}{2c} \sqrt{\frac{r_j - r_i}{r_j + r_i}}, \quad (22)$$

nevertheless, we can also apply our MMA formalism to compute these time lapses in an alternative manner.

Looking back at Fig. 1(b), the parameter  $t_{ij}$  can be evaluated as (Feng & Huang 2019, 2020)

$$t_{ij}^{(\text{MMA})} = \int_{r_i}^{r_j} \frac{nds}{c} = \int_{r_i}^{r_j} \frac{ndr'}{c\sqrt{1 - \cos^2 \theta}}, \quad (23)$$

which transforms into the following expression, once the relation between  $dr'$  and  $dr$ , equation (1), has been considered

$$t_{ij}^{(\text{MMA})} = \frac{1}{c} \int_{r_i}^{r_j} dr \frac{n^2 r}{\sqrt{n^2 r^2 - n_0^2 r_0^2}} \left( 1 - \frac{r_s}{r} \right)^{-1/2}, \quad (24)$$

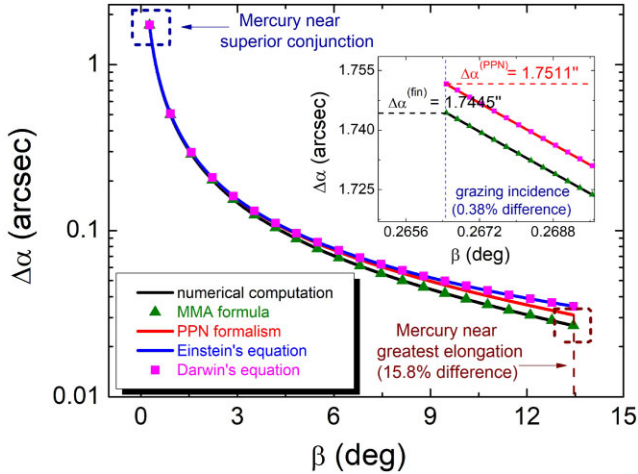
where, again, the refractive index  $n(r)$  is given by equation (6).

As in the case of the gravitational light bending, equation (24) has analytical solutions in terms of incomplete elliptic integrals of different kinds. However, due to the mathematical complexity of the final expression and its limited usefulness, we have not included this new equation in our article. As briefly discussed, we will show the equivalence between the GR formula for the Shapiro time delay, equation (22), and our numerical calculations via equation (24).

## 3 RESULTS

In this section, we present some analytical and numerical results concerning the gravitational light bending and Shapiro time delay via our MMA formalism. As a consequence, we want to emphasize the advisability of using our new analytical expressions to prevent undesired ultraprecise astrometry errors.

Hence, we represent in Fig. 3 the gravitational deflection angle for light beams coming from Mercury (please, see again Fig. 2), where  $\Delta\alpha$  is shown as a function of the angle  $\beta$ . One notices that the numerical computation of equation (11; black solid curve) and Einstein's first-order formula equation (13; blue solid curve) differ substantially for higher values of  $\beta$ . This discrepancy is reduced in the case of the first-order PPN formalism, equation (12), where the difference between both methods reaches 15.8 per cent when Mercury is located near its greatest elongation. For completeness, the deflection angle results derived via Darwin's formula, equation (18),



**Figure 3.** The gravitational deflection angle  $\Delta\alpha$  for a light beam coming from Mercury, passing near the Sun and reaching the Earth, versus the observation angle  $\beta$  (please, see again Fig. 2). The numerical computation of equation (11) and the PPN formalism up to first order, equation (12), differ for higher values of  $\beta$ . Moreover, the MMA equation (17) based on elliptical integrals fits precisely our numerical calculations. For completeness, the deflection angle results derived via Darwin's formula, equation (18), are also depicted. The inset shows the situation for a sun-grazing beam, where now the error between both schemes reduces to 0.38 per cent.

are also depicted in Fig. 3, where a full agreement with Einstein's equation is attained.

It can be noticed that our MMA formula, equation (17), fits precisely to the numerical calculation of the deflection angle equation (11). Moreover, when Mercury is at its superior conjunction (i.e.  $\beta \simeq 0.26$  deg), the difference between both methods also exists but to a lesser extent, as appreciated in the figure inset. Accordingly, a discrepancy of 0.38 per cent is achieved for solar grazing incidence if we do not consider our MMA equation. Our previous results indicate the importance of taking into consideration our exact analytical formula when calculating the gravitational deflection angle  $\Delta\alpha$ ,

especially when solar system distances are involved. In fact, non-negligible errors are also presented when extrasolar distances are considered, as explained ahead.

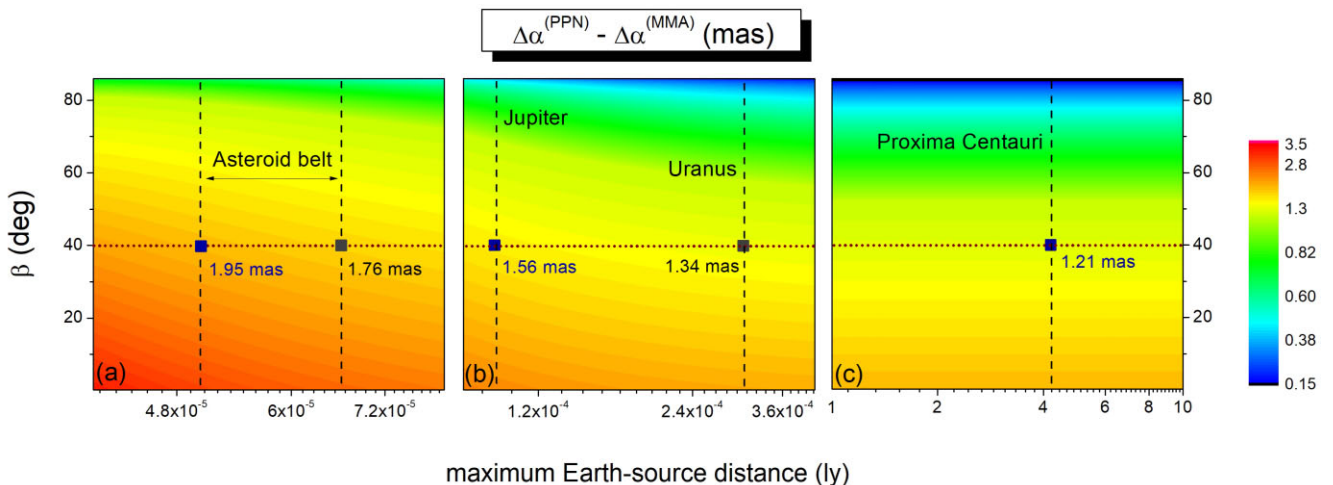
Consequently, we have studied in Fig. 4 the absolute difference between the first-order PPN formalism and our MMA method for  $\Delta\alpha$  in the case of solar system objects (left and central panels) and extrasolar sources (right panel). This parameter has been computed via equations (12) and (17) for Figs 4(a) and (b), whereas our asymptotic formula, equation (20), has been used for distant starlight calculations (right panel) as a function of the angle  $\beta$  and the maximum source-Earth distance in light years (ly). For an observation angle of 40 deg, a discrepancy between 1.76 and 1.95 mas is attained for solar system sources within the asteroid belt distance, while a lower deviation of 1.21 mas is also encountered for Proxima Centauri, as shown in Fig. 4(c). Furthermore, significant differences are also reported for sun-grazing light beams coming from solar system planets like Jupiter (2.35 mas) and Uranus (2.11 mas), dropping to 0.20 mas for extrasolar light emitters at larger observation angles ( $\beta \simeq 80$  deg).

On the other hand, the Shapiro time delay  $\Delta t$  for different source-Earth distances is depicted in Fig. 5, where the concrete examples of Mercury and Jupiter are illustrated. As in Fig. 4, we have assumed the maximum distances from emitter to observer. The solid lines represent our MMA results performed numerically via equations (21) and (24), whereas the squares indicate the  $\Delta t$  values computed via GR formalism, equations (21) and (22). A total agreement between both models is observed for different  $\beta$  angles, noting the appropriateness of our MMA to describe light propagation in the presence of static gravitational masses.

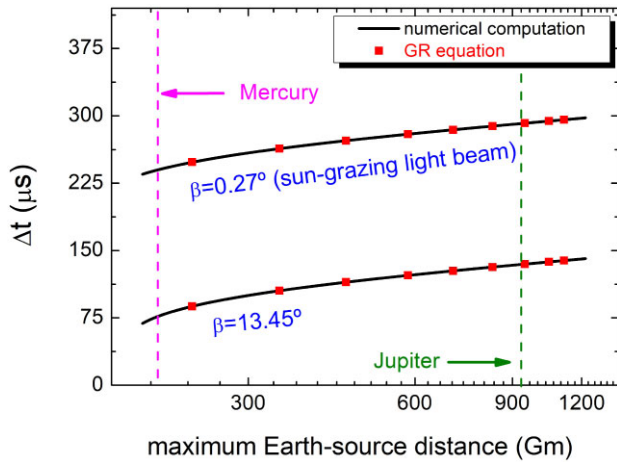
#### 4 DISCUSSION AND CONCLUSIONS

Summarizing, an MMA has been developed to determine an exact analytical expression for the bending angle of light due to a static massive body, considering the actual distances from source and observer to the gravitational mass. The validity of our new method has been checked throughout this article.

It is worth mentioning that a key conclusion of our work is the desirability of taking into account our novel accurate expressions,



**Figure 4.** Absolute difference between the first-order PPN formalism and our MMA method for  $\Delta\alpha$  in the case of solar system objects (left and central panels) and extrasolar sources (right panel). This parameter has been computed via equations (12) and (17) for Figs 4(a) and 4(b), whereas our asymptotic formula, equation (20), has been used for distant starlight calculations (right panel) as a function of the angle  $\beta$  and the maximum source-Earth distance in light years. It can be noticed that the contours in the right panel are flat due to the large distances involved (that is, the source is infinitely far away in all cases), but our results are non-zero and depend on the angle  $\beta$  because the Earth is not at infinite distance from the Sun, compared to the impact parameter of the light ray.



**Figure 5.** Shapiro time delay  $\Delta t$  versus the maximum source-Earth distance evaluated numerically via our MMA model, equations (21) and (24), for two different observation angles  $\beta$ . The red squares indicate the  $\Delta t$  values computed via GR formalism, equations (21) and (22). A full agreement between both theoretical models is observed.

equations (17) and (20), when calculating the gravitational deflection angle of light. In fact, relevant errors in the positioning of celestial objects may occur if our model is overlooked, as presented in Figs 3 and 4. For instance, the absolute difference between the MMA method and the first-order PPN formalism at an observation angle of 40 deg is 1.21 mas for starlight coming from Proxima Centauri, while the angular diameter of this star is about 1 mas (Ségransan et al. 2003). In this respect, a precise location of this star might help to accurately estimate its wide-binary orbit around  $\alpha$  Centauri A and B (Banik & Kroupa 2019).

Moreover, this bending angle inaccuracy is also greater than  $\Delta\alpha$  disagreement when modelling our gravitational mass as a static or a rotating body. Indeed, as reported by Roy and Sen within the framework of an asymptotic-based MMA in Kerr geometry (Roy & Sen 2015), the deflection angle for distant starlight grazing the Sun is 1.7520 arcsec for light ray prograde orbits, whereas 1.7519 arcsec is achieved in a retrograde scenario. Provided the bending angle value of 1.7512 arcsec for a stationary gravitational object via the first-order PPN formalism, the corresponding deviation if one neglects solar rotation is roughly 0.8 mas, in comparison with an absolute difference of 2 mas when our MMA equation is obviated.

Besides the assumption of a non-rotating central mass, it should be stated that the principal constraint of our MMA model comes from the aforementioned weak-field approximation, that is, when  $r \gg r_s$ . This means that our new approach cannot explain the strong deflection of light by a central mass, where the bending angles are not small (Bisnovaty-Kogan & Tsupko 2015). In this situation, light beams trajectories are relatively close to  $r_s$  (as in the case of a Schwarzschild black hole) and several turns near the photon sphere are completed before reaching the observer. As a consequence,  $\Delta\alpha = 2m\pi$  rad for an integer  $m$ , a physical phenomenon beyond the scope of our work.

Despite all our calculations in this article are based on light deflection by the Sun, the gravitational light bending by massive objects in the solar system, such as planet Jupiter, has recently gained a great deal of attention (Crosta & Mignard 2006; Brown 2021; Li et al. 2022a, b), due to its potential applications in microarcsecond astrometry. After a detailed comparison between our MMA equation and the first-order PPN formalism for distant

starlight grazing Jupiter's limb, we conclude that the difference between both methods is roughly  $0.002 \mu\text{as}$ , far beyond the milliarcsecond regime described in this article. However, this discrepancy should be significant in future sub-microarcsecond accuracy for the gravitational bending of light (Brown 2013).

It should be emphasized that the fundamental reason for the difference between our MMA results and previous theories is that the source and observer are in general not infinitely far away, compared to the impact parameter of the light ray at the deflecting massive body, apart from the approximate character of the PPN method discussed in this article. In essence, our exact analytical expressions might constitute useful tools to accurately calculate the gravitational deflection angle of light due to a static massive body, which should be relevant to current and future research in order to prevent undesired errors in ultraprecise astrometry.

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## DATA AVAILABILITY

Provided the theoretical nature of this paper, all data and numerical results generated or analysed during this study are included in this article (and based on the references therein). All the numerical calculations have been carried out on the basis of a Fortran 90 compiler and Mathematica software.

## REFERENCES

- Balazs N. L., 1958, *Phys. Rev.*, 110, 236
- Banik I., Kroupa P., 2019, *MNRAS*, 487, 1653
- Bisnovaty-Kogan G. S., Tsupko O. Y., 2015, *Plasma Phys. Rep.*, 41, 562
- Bozza V., De Luca F., Scarpetta G., Sereno M., 2005, *Phys. Rev. D*, 72, 083003
- Brown A. G. A., 2013, White Paper–Space-Time Structure Explorer Sub-microarcsecond astrometry for the 2030s. GAIA-CG-TN-LEI-AB-039. European Space Agency, The Netherlands
- Brown A. G. A., 2021, *ARA&A*, 59, 59
- Chandrasekhar S., 1983, *The Mathematical Theory of Black Holes*. Clarendon Press, Oxford
- Cowling A. C., 1984, *MNRAS*, 209, 415
- Crosta M. T., Mignard F., 2006, *Class. Quantum Gravity*, 23, 4853
- Darwin C., 1961, *Proc. R. Soc. London A*, 263, 39
- De Felice F., 1971, *Gen. Relativ. Gravit.*, 2, 347
- Dyson F. W., Eddington A. S., Davidson C., 1920, *Phil. Trans. R. Soc. A*, 220, 291
- Einstein A., 1916, *Ann. der Phys.*, 354, 769
- Evans J., Nandi K. K., Islam A., 1996a, *Gen. Relativ. Gravit.*, 28, 413
- Evans J., Nandi K. K., Islam A., 1996b, *Am. J. Phys.*, 64, 1404
- Feng G., Huang J., 2019, *Optik*, 194, 163082
- Feng G., Huang J., 2020, *Optik*, 224, 165685
- Fischbach E., Freeman B. S., 1980, *Phys. Rev. D*, 22, 2950
- Frittelli S., 2003, *MNRAS*, 344, L85
- He Q. et al., 2022, *MNRAS*, 511, 3042
- Hwang J.-C., Noh H., 2024, *Gen. Relativ. Gravit.*, 56, 8
- Ishihara A., Suzuki Y., Ono T., Asada H., 2017, *Phys. Rev. D*, 95, 044017

- Iyer S. V., Petters A. O., 2007, *Gen. Relativ. Gravit.*, 39, 1563
- Kong D., Yang D., Yu H.-B., 2024, *ApJ*, 965, L19
- Lebach D. E., Corey B. E., Shapiro I. I., Ratner M. I., Weber J. C., Rogers A. E. E., Davis J. L., Herring T. A., 1995, *Phys. Rev. Lett.*, 126, 041303
- Li Y. et al., 2022a, *ApJ*, 925, 47
- Li Y., Xu Y., Bian S., Lin Z., Li J., Liu D., Chaojie H., 2022b, *ApJ*, 938, 58
- Meneghetti M., 2021, *Introduction to Gravitational Lensing*. Springer, Switzerland
- Misner C. W., Thorne K. S., Wheeler J. A., 1973, *Gravitation*. Princeton Univ. Press, San Francisco
- Mutka P. T., Mähönen P. H., 2002, *ApJ*, 576, 107
- Nandi K. K., Islam A., 1995, *Am. J. Phys.*, 63, 251
- Newton I., 1730, *Opticks: Or, A Treatise of the Reflections, Refractions, Inflections and Colours of Light*. Royal Society, London
- Ni W. T., 2017, *One Hundred Years of General Relativity: From Genesis and Empirical Foundations to Gravitational Waves, Cosmology and Quantum Gravity*. World Scientific, New Jersey
- Petters A. O., Levine H., Wambsgans J., 2002, *Singularity Theory and Gravitational Lensing*. Birkhäuser, Boston
- Plebanski J., 1960, *Phys. Rev.*, 118, 1396
- Reasenberg R. D. et al., 1979, *ApJ*, 234, L219
- Roy S., Sen A. K., 2015, *Astrophys. Space Sci.*, 360, 23
- Roy S., Sen A. K., 2017, *Z. Naturforsch. A*, 72, 1113
- Roy S., Sen A. K., 2019, *J. Phys.: Conf. Ser.*, 1330, 012002
- Schneider P., Ehlers J., Falco E. E., 1992, *Gravitational Lenses*. Springer, New York
- Ségransan D., Kervella P., Forveille T., Queloz D., 2003, *A&A*, 397, L5
- Sen A. K., 2010, *Astrophysics*, 53, 560
- Shapiro I. I., 1964, *Phys. Rev. Lett.*, 13, 789
- Shapiro I. I., 1967, *Science*, 157, 806
- Shapiro I. I., Ash M. E., Ingalls R. P., Smith W. B., Campbell D. B., Dyce R. B., Jurgens R. F., Pettengill G. H., 1971, *Phys. Rev. Lett.*, 26, 1132
- Soldner J. G. v., 1804, *On the Deflection of a Light Ray from its Rectilinear Motion, by the Attraction of a Celestial Body at Which it Nearly Passes* by. Berliner Astronomisches Jahrbuch, Berlin
- Takizawa K., Asada H., 2023, *Phys. Rev. D*, 108, 104055
- Tamm I. E., 1924, *J. Russ. Phys. Chem. Soc.*, 56, 248
- Tamm I. E., 1925, *J. Russ. Phys. Chem. Soc.*, 57, 1
- Turyshchev S. G., Toth V. T., 2022, *MNRAS*, 515, 6122
- Virbhadra K. S., 2009, *Phys. Rev. D*, 79, 083004
- Wald R. M., 1984, *General Relativity*. The University of Chicago Press, Chicago
- Whitehead A. N., 1922, *The Principle of Relativity*. Cambridge Univ. Press, Cambridge
- Will C. M., 1993, *Theory and Experiment in Gravitational Physics*. Cambridge Univ. Press, Cambridge
- Will C. M., 2015, *Classical Quantum Gravity*, 32, 124001
- Ye X.-H., Lin Q., 2008, *J. Opt. A Pure Appl.*, 10, 075001
- Zschocke S. A., 2011, *Class. Quantum Gravity*, 28, 125016

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