◊ V.1

Computing the Intersection of a Line and a Cone

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◊ Introduction ◊

Computing the intersection of a line and an object is a common operation in computer graphics, for example, when ray tracing. Computation of the intersection of a line and a cylinder has been treated in previous gems (Cychosz and Waggenspack 1994, Shene 1994). This gem extends the latter work by computing the intersection of a line and a cone through geometric means.

◊ Definitions ◊

The notation and defining formulas are presented for three geometric objects:

- $\ell(\mathbf{B}, \mathbf{d})$: the line defined by base point $\mathbf{B}$ and direction vector[1] $\mathbf{d}$.
- $\mathcal{P}(\mathbf{B}, \mathbf{n})$: the plane defined by base point $\mathbf{B}$ and normal vector $\mathbf{n}$.
- $\mathcal{C}(\mathbf{V}, \mathbf{v}, \alpha)$: the cone defined by vertex $\mathbf{V}$, axis direction $\mathbf{v}$, and cone angle α.

In these definitions, bold-face roman type indicates a vector quantity. Moreover, upper (lower)-case vectors are position (direction) vectors. Position vectors are sometimes referred to as points. Therefore, $\mathbf{P}$ and P are equivalent. The normalized cross product $\mathbf{u} \otimes \mathbf{v} = \mathbf{u} \times \mathbf{v} / ||\mathbf{u} \times \mathbf{v}||$ is also employed.

◊ Problem Statement ◊

Given a test line $\ell(\mathbf{D}, \mathbf{d})$ and cone $\mathcal{C}(\mathbf{V}, \mathbf{v}, \alpha)$, determine the point of intersection by computing a t such that point $\mathbf{D} + t\mathbf{d}$ lies on $\mathcal{C}(\mathbf{V}, \mathbf{v}, \alpha)$ or show that no intersection exists.

[1] In this exposition, $\|\mathbf{d}\| = 1$ holds for any direction vector $\mathbf{d}$.

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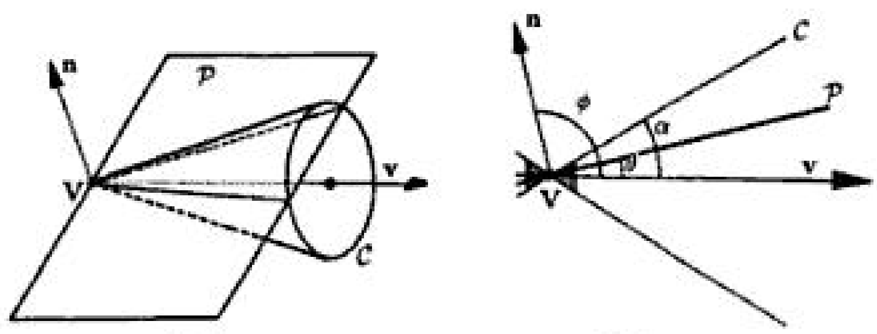

(a) (b)

Figure 1. The normal vector **n** of plane $\mathcal{P}$.

◊ The Algorithm ◊

If $\mathbf{V} \in \ell$, the intersection point is $\mathbf{V}$. Therefore, in what follows, $\mathbf{V} \notin \ell$ holds.

Consider the plane $\mathcal{P}$ determined by $\mathbf{V}$ and ℓ. Its normal vector is $\mathbf{n} = \mathbf{d} \otimes \overrightarrow{DV}$. However, if $\mathbf{v} \cdot \mathbf{n} > 0$, $\mathbf{n}$ is reversed. This ensures that $\mathcal{P}$ lies "between" $\mathbf{n}$ and $\mathbf{v}$ (Figure 1). Therefore, the desired plane is $\mathcal{P}(\mathbf{V}, \mathbf{n})$. Since $\mathcal{P}$ contains $\mathbf{V}$, $\mathcal{P} \cap \mathcal{C}$ is either a point (i.e., $\mathbf{V}$), or consists of one or two lines. In the following, the computation of $\ell \cap \mathcal{C}$ will be reduced to the computation of $\ell \cap (\mathcal{P} \cap \mathcal{C})$. In other words, the intersection lines of $\mathcal{P} \cap \mathcal{C}$ will be computed and intersected with ℓ. However, prior to the intersection computation, a disjoint test is needed.

Let θ be the angle between $\mathbf{v}$ and $\mathcal{P}$ [Figure 1(a)]. By trichotomy exactly one of the following conditions is true:

- $\theta > \alpha$: $\mathcal{P} \cap \mathcal{C}$ is $\mathbf{V}$, and $\ell \cap \mathcal{C}$ is empty.
- $\theta = \alpha$: $\mathcal{P} \cap \mathcal{C}$ is the tangent line of $\mathcal{P}$ and $\mathcal{C}$, and $\ell \cap \mathcal{C}$ consists of at most one point.
- $\theta < \alpha$: $\mathcal{P} \cap \mathcal{C}$ consists of two lines, and $\ell \cap \mathcal{C}$ consists of at most two points.

However, using θ directly is not as efficient as using $\cos\theta$, since the latter can be obtained easily as follows. Let $\phi = \theta + 90°$ be the angle between $\mathbf{n}$ and $\mathbf{v}$ [Figure 1(b)]. Therefore, $\cos\phi = \mathbf{n} \cdot \mathbf{v}$ and

$$\cos\theta = \cos(\phi - 90°) = \sin\phi = (1 - \cos^2\phi)^{1/2} = (1 - (\mathbf{n} \cdot \mathbf{v})^2)^{1/2}.$$

Since the cosine function is monotonically decreasing between 0° and 90°, $\cos(x) > \cos(y)$ if and only if $x < y$ for $0° \leq x, y \leq 90°$. Therefore, with $\cos\alpha$ and $\cos\theta$, tests $\theta > \alpha$, $\theta = \alpha$, and $\theta < \alpha$ can be replaced by $\cos\theta < \cos\alpha$, $\cos\theta = \cos\alpha$, and $\cos\theta > \cos\alpha$, respectively.

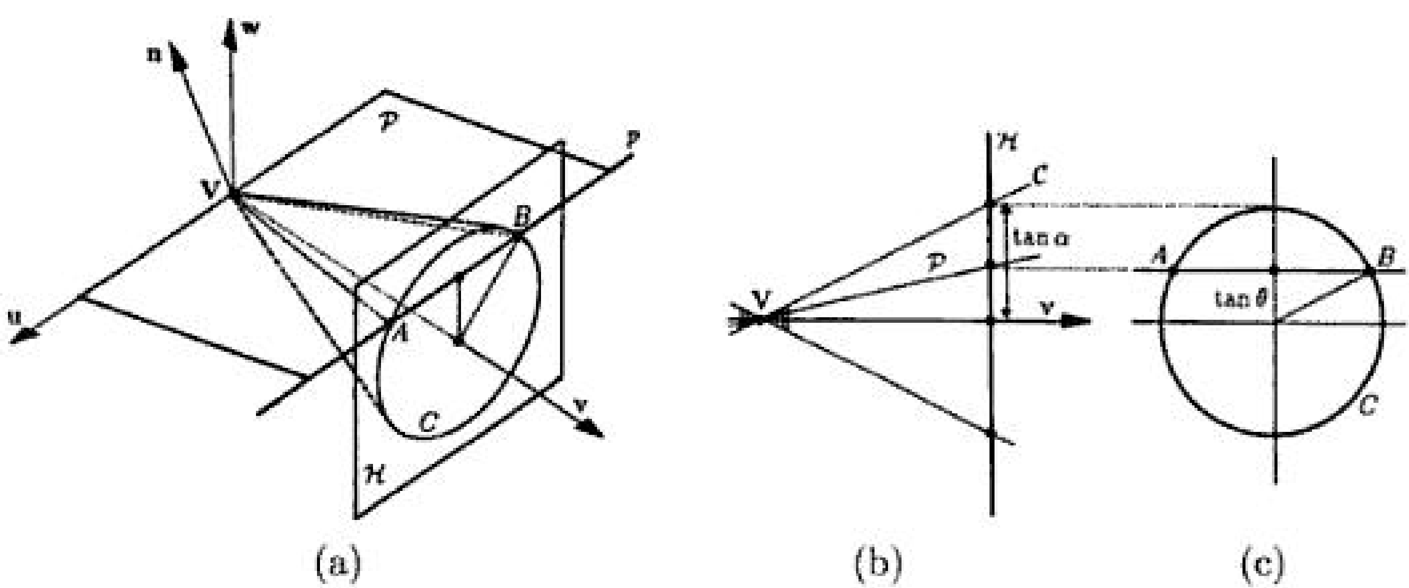

Figure 2. The *u*-*v*-*w* coordinate system and related information.

Assuming $\cos\theta \geq \cos\alpha$, two steps are required to compute $\ell \cap \mathcal{C}$: (1) computing $\mathcal{P} \cap \mathcal{C}$, and (2) computing $\ell \cap (\mathcal{P} \cap \mathcal{C})$. For the first step, a well-chosen coordinate system is vital. Since $\mathbf{n}$ and $\mathbf{v}$ are not parallel, $\mathbf{v} \times \mathbf{n}$ is well defined. Let vectors $\mathbf{u}$ and $\mathbf{w}$ be defined as follows:

$$\mathbf{u} = \mathbf{v} \otimes \mathbf{n},$$
$$\mathbf{w} = \mathbf{u} \otimes \mathbf{v} = (\mathbf{v} \otimes \mathbf{n}) \otimes \mathbf{v}.$$

Then $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are perpendicular to each other and form a right-handed u-v-w coordinate system with origin at $\mathbf{V}$ [Figure 2(a)]. Since $\mathbf{n} \perp \mathbf{u}$ and $\mathbf{V} \in \mathcal{P}$, $\mathcal{P}$ contains the u-axis and is perpendicular to the vw-plane.

Using this coordinate system, the direction vectors of $\mathcal{P} \cap \mathcal{C}$ are computed as follows. Consider a plane $\mathcal{H}$ with $v = 1$ in the u-v-w coordinate system. $\mathcal{H} \cap \mathcal{C}$ is a circle C, while $\mathcal{H} \cap \mathcal{P}$ is a line p. Let p and C intersect at A and B. Then the intersection of $\mathcal{P}$ and $\mathcal{C}$ consists of two lines, $\overleftrightarrow{VA}$ and $\overleftrightarrow{VB}$. Thus, if their direction vectors, $\delta_1 = \overrightarrow{VA}$ and $\delta_2 = \overrightarrow{VB}$, can be found, $\mathcal{P} \cap \mathcal{C}$ will be determined.

To compute A and B, first note that their w-coordinates are both equal to $\tan\theta$, and that $\frac{1}{2}\overline{AB} = (\tan^2\alpha - \tan^2\theta)^{1/2}$, where $\tan\alpha$ is the radius of circle C [Figure 2(b) and (c)]. Since $\overleftrightarrow{AB}$ is parallel to the u-axis, direction vectors $\delta_1 = \overrightarrow{VA}$ and $\delta_2 = \overrightarrow{VB}$ can be computed as follows:

$$\delta_1 = \mathbf{v} + (\tan\theta)\mathbf{w} + (\tan^2\alpha - \tan^2\theta)^{1/2}\mathbf{u},$$
$$\delta_2 = \mathbf{v} + (\tan\theta)\mathbf{w} - (\tan^2\alpha - \tan^2\theta)^{1/2}\mathbf{u}.$$

Therefore, the intersection lines of $\mathcal{P}$ and $\mathcal{C}$ are simply $\ell_1(\mathbf{V}, \delta_1)$ and $\ell_2(\mathbf{V}, \delta_2)$. Without loss of generality, assume $||\delta_1|| = ||\delta_2|| = 1$. Note that if $\mathcal{P}$ is tangent to $\mathcal{C}$, $\alpha = \theta$, and $\ell_1 = \ell_2$.

Finally, computing $\ell_1 \cap \ell$ and $\ell_2 \cap \ell$ yields the desired result. Determining the intersection point of two coplanar lines is not difficult. If δ_1 and $\mathbf{d}$ have the same or opposite direction (i.e., $\mathbf{d} \times \delta_1 = \mathbf{0}$, or equivalently $||\mathbf{d} \cdot \delta_1|| = 1$), ℓ_1 and ℓ are parallel to each other and there is no intersection point. Otherwise, there exist r and s such that $\mathbf{D} + r\mathbf{d} = \mathbf{V} + s\delta_1$. Since $\mathbf{g} \times \mathbf{g} = \mathbf{0}$ holds for any nonzero vector $\mathbf{g}$, computing the cross product with δ_1, the preceding formula gives

$$r\mathbf{d} \times \delta_1 = (\mathbf{V} - \mathbf{D}) \times \delta_1.$$

Computing the inner product with $\mathbf{d} \times \delta_1$ yields

$$r = \frac{[(\mathbf{V} - \mathbf{D}) \times \delta_1] \cdot (\mathbf{d} \times \delta_1)}{||\mathbf{d} \times \delta_1||^2}.$$

Thus, $\ell_1 \cap \ell$ is computed. Replacing δ_1 with δ_2 yields $\ell_2 \cap \ell$.

In practice, the computation for r could be simpler. Let $\pi_i(\mathbf{x})$ be the ith component of vector $\mathbf{x}$. Then

$$r = \frac{\pi_i((\mathbf{V} - \mathbf{D}) \times \delta_1)}{\pi_i(\mathbf{d} \times \delta_1)},$$

where $\pi_i(\mathbf{d} \times \delta_1)$ is a nonzero component of vector $\mathbf{d} \times \delta_1$.

Remark. Since a cylinder is a cone with its vertex at infinity, the algorithm presented here provides another way of computing the intersection of a line and a cylinder. In this case, $\mathcal{P}$ is the plane that is parallel to the cylinder axis and contains the given line, and $\mathcal{P} \cap \mathcal{C}$ degenerates to a pair of parallel lines. Consequently, the computation is reduced to computing the intersection points of this pair of lines with the given one.

◇ Acknowledgment ◇

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◇ Bibliography ◇

(Cychosz and Waggenspack 1994) J. M. Cychosz and W. N. Waggenspack, Jr. Intersecting a ray with a cylinder. In Paul Heckbert, editor, *Graphics Gems IV*, pages 356–365. AP Professional, Boston, 1994.