Addendum to 'Gravitational Geons in 1+1 Dimensions'

Dan N. Vollick

Irving K. Barber School of Arts and Sciences
University of British Columbia Okanagan
3333 University Way
Kelowna, B.C.
Canada
V1V 1V7

Abstract

In a recent paper [1] I found gravitational geons in two classes of 1+1 dimensional theories of gravity. In this paper I examine these theories, with the possibility of a cosmological constant, and find strong field gravitational geons. In the spacetimes in [1] a test particle that is reflected from the origin suffers a discontinuity in $d^2t/d\tau^2$. The geons found in this paper do not suffer from this problem.

In a recent paper [1] I examined geons in 1+1 dimensional theories of gravity with a spacetime metric given by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 \ , \tag{1}$$

where r is a radial-like coordinate with $r \geq 0$. Imposing a reflecting boundary condition at $r = 0$ leads to the problem that $d^2t/d\tau^2$ suffers a jump discontinuity upon reflection unless $f^{'}(0) = 0$. This follows from the geodesic equation

$$\frac{d^2t}{d\tau^2} = -\left(\frac{f'}{f}\right)\frac{dr}{d\tau}\frac{dt}{d\tau} \tag{2}$$

with $dr/d\tau$ changing sign upon reflection. None of the solutions in [1] satisfied this condition, except (13) with $B = 0$. In this paper I find additional strong field geon solutions that either satisfy $f^{'}(0) = 0$ or that can be extended to negative r (i.e. with $-\infty < r < \infty$).

Consider the field equation (see equation (3) in [1] with $\alpha = 0$)

$$R + \beta\Box R = \Lambda \ . \tag{3}$$

In terms of the metric function $f(r)$ this equation can be written as

$$f^{''} + \beta\frac{d}{dr}\left(ff^{'''}\right) = -\Lambda \ . \tag{4}$$

Integrating twice gives

$$f + \beta\left[ff^{''} - \frac{1}{2}\left(f^{'}\right)^2\right] = C_1 + C_2 r - \frac{1}{2}\Lambda r^2 \ , \tag{5}$$

where C_1 and C_2 are integration constants. In terms of $h = \sqrt{f}$ ($f > 0$) the above can be written as

$$h^2 + 2\beta h^3 h^{''} = C_1 + C_2 r - \frac{1}{2}\Lambda r^2 \ . \tag{6}$$

First consider the case $C_2 = \Lambda = 0$. Integrating once gives

$$\frac{1}{2}v^2 + V(h) = E \tag{7}$$

where $v = dh/dr$, E is an integration constant and

$$V(h) = \frac{1}{2\beta}\left[\ln|h| + \frac{C_1}{2h^2}\right] \ . \tag{8}$$

From this equation it is easy to see that $R = -f^{''} \simeq \frac{1}{\beta}\ln f$ for large f (the solution will approach a large vale of f only if $\beta < 0$). Thus, we are only interested in solutions in which f remains finite. The only nontrivial solutions with $h^{'}(0) = 0$ and which satisfy

$h > 0$ occur when β and C_1 are positive. In this case V has one local minimum and goes to ∞ as h goes to zero and ∞. Therefore f will undergo periodic oscillations and these solutions are similar to the solutions (13) in [1], except that they are not restricted to weak fields. Note that we can extend these solutions to negative values of r. These solutions therefore correspond to infinite sequences of geons.

Now consider the case with $C_1 = C_2 = 0$. In this case $f = -\frac{1}{2}\Lambda r^2$ ($h = \sqrt{\frac{1}{2}|\Lambda|}r$) is a solution if $\Lambda < 0$, so that $f > 0$. The behavior of the solutions can be analyzed by letting $h = rZ(r)$ and $x = \ln|r|$. The field equation in these variables is given by

$$\frac{d^2Z}{dx^2} = -\frac{dZ}{dx} - \frac{dU(Z)}{dZ} \ , \tag{9}$$

where the potential is given by

$$U(Z) = \frac{1}{2\beta}\left[\ln|Z| - \frac{\Lambda}{4Z^2}\right] \ . \tag{10}$$

For $\Lambda < 0$ and $\beta > 0$ then $U \to \infty$ as $Z \to 0, \infty$ and there is one minimum at $Z = \sqrt{|\Lambda|/2}$. The damping term $-dZ/dx$ will cause the motion to decay to the solution $Z = \sqrt{|\Lambda|/2}$ at large r. Thus, at large r the function $f(r)$ will approach $\frac{1}{2}|\Lambda|r^2$. Note that this solution can be extended to negative values of r since $Z \to \sqrt{|\Lambda|/2}$ for large x where $x = \ln|r|$. This solution therefore corresponds to a geon with $-\infty < r < \infty$ or with $r \geq 0$ and a reflective boundary condition at $r = 0$ ($f' = 2rZ^2 + 2r^2ZZ'$, so that $f'(0) = 0$ if $Z(0)$ and $Z'(0)$ are finite). At large r the Ricci scalar approaches $-\Lambda$. There are no geon solutions for $\beta < 0$.

Now consider the case $C_1 = \Lambda = 0$. A solution of this equation is $f = C_2 r + \frac{1}{2}\beta C_2^2$. I have been unable to show analytically that, for $\beta > 0$ and $C_2 > 0$, all solutions approach this solution for large r. However, it is easy to show that this solution is stable in the sense that all nearby solutions do converge to it for large r. For simplicity I will take $C_2 = 1$. Now Let

$$f(r) = \left[r + \frac{1}{2}\beta\right](1 + g(r)) \tag{11}$$

The function $g(r)$ satisfies the linearized equation

$$\beta\left[x^2\frac{d^2g}{dx^2} + x\frac{dg}{dx} - g\right] + xg = 0 \ , \tag{12}$$

where $x = r + \beta$. Now define $y = 2\sigma\sqrt{x}$, where $\sigma = \beta^{-1/2}$. In terms of y the equation is

$$y^2\frac{d^2g}{dy^2} + y\frac{dg}{dy} + (y^2 - 4)g = 0 \ . \tag{13}$$

This is a Bessel equation of order two, so the general solution is

$$g(x) = AJ_2\left(2\sigma\sqrt{x}\right) + BY_2\left(2\sigma\sqrt{x}\right) \ , \tag{14}$$

where A and B are constants. Here I have included Y_2, which diverges at the origin, since I am considering only large x. Note that

$$J_2(x) \simeq \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{5}{4}\pi\right) \qquad as \qquad x \to \infty \tag{15}$$

(a similar expression holds for Y_2). Thus, g goes to zero for large r and $f \to r + \frac{1}{2}\beta$ for large r. In this case $R = -f''$ goes to zero at large r. I have studied the differential equation (5) with (11) numerically ($C_1 = 0, C_2 = 1, \Lambda = 0$), using Maple, for the initial values $g(0) \in [-0.9, 3]$ and with $\beta = 1$. The value of $g'(0)$ is determined by setting $f'(0) = 0$. This gives $g'(0) = -2[1 + g(0)]$. The solutions oscillate and decay slowly for large r in a similar fashion to the linearized solutions.

The second class of theories examined in [1] was based on the Lagrangian

$$L = -\sqrt{g}\left[\frac{1}{\phi}R + V(\phi)\right] . \tag{16}$$

It was shown that the field equations could be integrated giving

$$Af' = V(\phi) \qquad with \qquad \phi = \frac{1}{Ar} , \tag{17}$$

where A is a constant. Thus, for a given $f(r)$ it is easy to solve for $V(\phi)$. A simple function that satisfies $f'(0) = 0$ and has Schwarzschild behavior at large r is

$$f(r) = 1 - \frac{2mr^2}{r^3 + 2m\ell^2} , \tag{18}$$

where m and ℓ are constants. The potential is given by

$$V(\phi) = \frac{2mA^2\phi^2\left(1 - 4m\ell^2A^3\phi^3\right)}{\left(1 + 2m\ell^2A^3\phi^3\right)} . \tag{19}$$

Acknowledgements

This research was supported by the Natural Sciences and Engineering Research Council of Canada.

References

[1] Dan N. Vollick Class. Quant. Grav. 25, 175004 (2008)