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Quantum geons

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Abstract

We describe solutions of the Klein-Gordon equation which are spherically symmetric and localized, and may be regarded as massive particles without charge or spin. The proposed model, which is based on the action for a complex scalar field minimally coupled to the electromagnetic and gravitational fields, contains no adjustable parameters and predicts five particle species with masses of the order of the Planck mass. These particles appear to be candidates for dark matter.

1 Introduction

A number of models describing particles as solitons—stable self-bound concentrations of field energy—have been proposed. In an early paper Rosen [14] obtained soliton solutions from the interaction of a complex scalar field and the electromagnetic field. Later Cooperstock and Rosen [6] obtained soliton solutions by coupling a complex scalar field to both electromagnetism and gravity. In the meantime Wheeler [16], using an entirely different theoretical framework, obtained soliton solutions (which he dubbed geons) from a purely classical model of electromagnetism coupled to gravity. More recently Moroz, Penrose and Tod [12] obtained soliton solutions of the Schrödinger-Newton equations. See [13, 4, 3, 2] and references therein for a window into the literature.

The model presented here is similar to that of Cooperstock and Rosen.[1] What differs is the treatment of boundary conditions and the introduction of certain constraints. Cooperstock and Rosen require all fields to be finite and continuous at the center of the particle and they integrate the field equations outward. Parameters are adjusted until the asymptotic wave function vanishes. The resulting model contains several adjustable parameters.

In contrast, we impose asymptotic boundary conditions and integrate the field equations inward, toward the center of the particle. And we require the asymptotically measured mass and charge to equal the mass and charge parameters of the action. The resulting model contains no adjustable parameters.

In Section 2 we write down the action for a complex scalar field minimally coupled to the electromagnetic and gravitational fields, and use the principle of stationary action to derive the corresponding field equations (the Klein-Gordon, Einstein and Maxwell equations). Then we propose a stationary spherically symmetric trial solution for the metric tensor, wave function and vector potential and substitute it into the field equations. We

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[1]Our Eqs. (22)–(25) are equivalent to Eqs. (5.3)–(5.6) of [6].

obtain a system of four coupled nonlinear differential equations, as well as two integral constraints related to the conservation of charge and energy. In Section 3 we search for localized particle-like solutions. After calculating the asymptotic behavior of the equations and imposing asymptotic boundary conditions, we are left with a single adjustable parameter—the system electric charge. By again appealing to the principle of stationary action we find that the charge must vanish. Hence, we arrive at an eigensystem with no adjustable parameters. Using numerical methods we calculate the eigenmodes and find five massive particle species without charge or spin. We name these particles *quantum geons* and show that the spacetime curvature diverges at the center of each geon. In Section 4 we explore the implications of this singularity and argue that it is benign. In Section 5 we discuss our results.

2 Field equations

Consider the action

$$S = \frac{1}{16\pi} \int_V \left[\mathsf{R} - \mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu} + \tfrac{1}{2}\left(\overline{\Psi^{:\mu}}\Psi_{:\mu} + \Psi^{:\mu}\overline{\Psi_{:\mu}} \right) - M^2\overline{\Psi}\Psi \right] \sqrt{} d^4x \tag{1}$$

for a complex scalar field Ψ within a four-dimensional volume V with spacetime curvature R, where a colon denotes the generalized covariant derivative [9, p. 167]

$$\Psi_{:\mu} \equiv \Psi_{;\mu} + iQ\mathsf{A}_\mu\Psi\,, \tag{2}$$

a semicolon denotes the covariant derivative of general relativity, an overline denotes complex conjugation, Q is electric charge, A_μ is the electromagnetic vector potential,

$$\mathsf{F}_{\mu\nu} \equiv \mathsf{A}_{\mu;\nu} - \mathsf{A}_{\nu;\mu} \tag{3}$$

is the electromagnetic field tensor and M is mass (assumed positive). All quantities are expressed in natural units where the speed of light, Planck's reduced constant ($\hbar$), Newton's gravitational constant and Coulomb's electrostatic constant are unity. Our notation and conventions are detailed in Appendix A.

Make small arbitrary variations δS in the action by making small arbitrary variations $\delta\mathsf{g}_{\alpha\beta}$, $\delta\mathsf{A}_\mu$, $\delta\Psi$ and $\delta\overline{\Psi}$ in the fields, and neglect boundary terms arising from integration by parts. Then impose the condition $\delta S = 0$ for stationary action. The terms proportional to $\delta\Psi$ and $\delta\overline{\Psi}$ yield the Klein-Gordon equation

$$\Psi^{:\mu}{}_{:\mu} + M^2\Psi = 0 \tag{4}$$

and its complex conjugate. The terms proportional to $\delta\mathsf{g}_{\alpha\beta}$ yield the Einstein equations

$$\mathsf{G}^{\alpha\beta} = -8\pi\mathsf{T}^{\alpha\beta}\,, \tag{5}$$

where

$$\mathsf{G}^{\alpha\beta} \equiv \mathsf{R}^{\alpha\beta} - \tfrac{1}{2}\mathsf{g}^{\alpha\beta}\mathsf{R} \tag{6}$$

is the Einstein tensor and

$$\begin{aligned} \mathsf{T}^{\alpha\beta} \quad \equiv \quad & \frac{1}{16\pi}\left(\overline{\Psi^{:\alpha}}\Psi^{:\beta} + \Psi^{:\alpha}\overline{\Psi^{:\beta}} - \mathsf{g}^{\alpha\beta}\overline{\Psi^{:\mu}}\Psi_{:\mu} + M^2\mathsf{g}^{\alpha\beta}\overline{\Psi}\Psi \right) \\ & - \frac{1}{4\pi}\left(\mathsf{F}^{\alpha}{}_{\mu}\mathsf{F}^{\beta\mu} - \tfrac{1}{4}\,\mathsf{g}^{\alpha\beta}\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu} \right) \end{aligned} \tag{7}$$

is the energy-momentum tensor. The terms proportional to $\delta\mathsf{A}_\mu$ yield the inhomogeneous Maxwell equations

$$\mathsf{F}^{\mu\nu}{}_{;\nu} = 4\pi\mathsf{J}^\mu\,, \tag{8}$$

where

$$\mathsf{J}^\mu \equiv \frac{iQ}{16\pi}\left(\overline{\Psi}\Psi^{:\mu} - \overline{\Psi^{:\mu}}\Psi\right) \tag{9}$$

is the electromagnetic current vector. The homogeneous Maxwell equations

$$\mathsf{F}_{\mu\nu;\alpha} + \mathsf{F}_{\alpha\mu;\nu} + \mathsf{F}_{\nu\alpha;\mu} = 0 \tag{10}$$

follow immediately from the definition of $\mathsf{F}_{\mu\nu}$.

Let the volume V in Eq. (1) include all of space over some arbitrary time interval T. We will later impose boundary conditions which require R, A_μ and Ψ to vanish asymptotically. Thus we are justified in neglecting boundary terms when integrating by parts in the expression for δS.

When Eqs. (4), (5) and (8) are satisfied the action is stationary and the scalar curvature is

$$\mathsf{R} = 8\pi\mathsf{T} = 2M^2\overline{\Psi}\Psi - \overline{\Psi^{:\mu}}\Psi_{:\mu}\,, \tag{11}$$

where $\mathsf{T} \equiv \mathsf{T}_\mu{}^\mu$. Substituting for R in Eq. (1) gives

$$S = \frac{1}{16\pi}\int_V \left(M^2\overline{\Psi}\Psi - \mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}\right)\sqrt{\;}\,d^4x \tag{12}$$

as the stationary value of the action.

2.1 Trial solution

Our goal is to solve Eqs. (4), (5) and (8) for the fields $\mathsf{g}_{\mu\nu}$, Ψ and A_μ. In order to make the calculations tractable we consider a static spherically symmetric spacetime with spherical polar coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$, and metric tensor

$$\mathsf{g}_{\mu\nu} = \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & -v & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}, \tag{13}$$

where u and v are real functions of r. The corresponding spherically symmetric wave function and vector potential are assumed to have the forms

$$\Psi = R\,e^{-i\omega t} \tag{14}$$

and

$$Q\mathsf{A}_\mu = \begin{pmatrix} \Phi \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{15}$$

where R and Φ are real functions of r and ω is a real constant.

Expressions (13), (14) and (15) for $\mathsf{g}_{\mu\nu}$, Ψ and A_μ comprise a trial solution of field equations (4), (5) and (8) and allow us to calculate all quantities of interest in terms of R, u, v and Φ. We will show that the field equations reduce to a set of four coupled nonlinear differential equations in R, u, v and Φ. Then we will solve these equations subject to physically plausible boundary conditions and constraints. The fact that the field equations are, indeed, satisfied by the assumed trial solution is its ultimate justification.

2.2 Einstein, energy-momentum and electromagnetic tensors

Given trial solution (13), (14) and (15) we can calculate the quantities in field equations (4), (5) and (8). The nonzero elements of the Einstein and energy-momentum tensors are

$$\begin{aligned}
\mathsf{G}_0{}^0 &= -\frac{v'}{rv^2} - \frac{1}{r^2} + \frac{1}{r^2 v} \\
\mathsf{G}_1{}^1 &= \frac{u'}{ruv} - \frac{1}{r^2} + \frac{1}{r^2 v} \\
\mathsf{G}_2{}^2 &= \frac{u''}{2uv} - \frac{(u')^2}{4u^2 v} - \frac{u'v'}{4uv^2} + \frac{u'}{2ruv} - \frac{v'}{2rv^2} \\
\mathsf{G}_3{}^3 &= \mathsf{G}_2{}^2
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\mathsf{T}_0{}^0 &= \frac{M^2 R^2}{16\pi} + \frac{(\omega-\Phi)^2 R^2}{16\pi u} + \frac{(R')^2}{16\pi v} + \frac{(\Phi')^2}{8\pi Q^2 uv} \\
\mathsf{T}_1{}^1 &= \frac{M^2 R^2}{16\pi} - \frac{(\omega-\Phi)^2 R^2}{16\pi u} - \frac{(R')^2}{16\pi v} + \frac{(\Phi')^2}{8\pi Q^2 uv} \\
\mathsf{T}_2{}^2 &= \frac{M^2 R^2}{16\pi} - \frac{(\omega-\Phi)^2 R^2}{16\pi u} + \frac{(R')^2}{16\pi v} - \frac{(\Phi')^2}{8\pi Q^2 uv} \\
\mathsf{T}_3{}^3 &= \mathsf{T}_2{}^2 ,
\end{aligned} \tag{17}$$

where primes denote differentiation with respect to r. The contraction of the energy-momentum tensor is

$$\mathsf{T} = \frac{M^2 R^2}{4\pi} - \frac{(\omega-\Phi)^2 R^2}{8\pi u} + \frac{(R')^2}{8\pi v} . \tag{18}$$

The nonzero elements of the electromagnetic field tensor are

$$Q\mathsf{F}_{01} = -Q\mathsf{F}_{10} = \Phi' \tag{19}$$

and the only nonzero element of the electromagnetic current vector is

$$\mathsf{J}^0 = \frac{Q(\omega-\Phi) R^2}{8\pi u} . \tag{20}$$

We also note that

$$Q^2 \mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu} = -\frac{2(\Phi')^2}{uv} . \tag{21}$$

2.3 Differential equations

The Klein-Gordon equation (4) becomes

$$R'' + \left(\frac{2}{r} + \frac{u'}{2u} - \frac{v'}{2v}\right) R' + \left[\frac{(\omega-\Phi)^2}{u} - M^2\right] vR = 0 . \tag{22}$$

There are four Einstein equations (5), one for each nonzero component of $\mathsf{G}_\mu{}^\nu$. However, the equations corresponding to $\mathsf{G}_2{}^2$ and $\mathsf{G}_3{}^3$ are identical. Furthermore, they can be derived by combining the ${\mathsf{G}_0}^0$, ${\mathsf{G}_1}^1$ and Klein-Gordon equations. So we retain the ${\mathsf{G}_0}^0$, ${\mathsf{G}_1}^1$ and Klein-Gordon equations, and discard the ${\mathsf{G}_2}^2$ and ${\mathsf{G}_3}^3$ equations. It will prove convenient to work with the equations obtained by adding and subtracting the ${\mathsf{G}_0}^0$ and ${\mathsf{G}_1}^1$ equations:

$$\frac{u'}{2u} - \frac{v'}{2v} + \frac{1-v}{r} = -rv \left[\frac{M^2 R^2}{2} + \frac{(\Phi')^2}{Q^2 uv} \right] \tag{23}$$

and

$$\frac{u'}{2u} + \frac{v'}{2v} = \frac{rv}{2} \left[\frac{(\omega - \Phi)^2 R^2}{u} + \frac{(R')^2}{v} \right] . \tag{24}$$

The inhomogeneous Maxwell equations (8) yield the Poisson equation

$$\Phi'' + \left(\frac{2}{r} - \frac{u'}{2u} - \frac{v'}{2v} \right) \Phi' + \frac{Q^2 R^2 v}{2} (\omega - \Phi) = 0 \, . \tag{25}$$

The four coupled nonlinear differential equations (22) through (25) are to be solved for the functions R, u, v and Φ subject to physical constraints and boundary conditions.

2.4 Conserved quantities

In this section we derive expressions for the electric charge, energy, angular momentum and stationary action of our model.

2.4.1 Charge

The electromagnetic current vector satisfies the conservation law

$${\mathsf{J}^\mu}_{;\mu} = (\mathsf{J}^\mu \surd)_{,\mu} = 0 \, . \tag{26}$$

The corresponding conserved quantity

$$Q = k_q \int_\infty \mathsf{J}^0 \surd \, dr \, d\theta \, d\phi \tag{27}$$

is the electric charge, where k_q is a real constant (to be determined later). Substitute J^0 from Eq. (20) and let

$$\surd = r^2 \sqrt{|uv|} \, \sin\theta \tag{28}$$

then integrate over θ and ϕ to get

$$1 = \frac{k_q}{2} \int_0^\infty \frac{r^2 (\omega - \Phi) R^2}{u} \sqrt{|uv|} \, dr \, . \tag{29}$$

Another expression for Q can be obtained as follows. Substituting from Eq. (8) into Eq. (27) and using the identity

$${\mathsf{F}^{\mu\nu}}_{;\nu} \surd = (\mathsf{F}^{\mu\nu} \surd)_{,\nu} \tag{30}$$

gives

$$Q = \frac{k_q}{4\pi} \int_\infty \left(\mathsf{F}^{0\nu} \surd \right)_{,\nu} dr \, d\theta \, d\phi \, . \tag{31}$$

For our static spherically symmetric metric the integrand is

$$\left(\mathsf{F}^{0\nu}\surd\right)_{,\nu} = \frac{d}{dr}\left(-\frac{\Phi'}{Quv}\surd\right) , \tag{32}$$

so

$$Q^2 = \frac{k_q}{4\pi}\int_\infty \frac{d}{dr}\left(-\frac{\Phi'}{uv}\surd\right) dr\, d\theta\, d\phi\,. \tag{33}$$

Substituting expression (28) for $\surd$ then integrating over θ and ϕ gives

$$\begin{aligned} Q^2 &= \int_0^\infty \frac{d}{dr}\left(-\frac{k_q r^2 \Phi' \sqrt{|uv|}}{uv}\right) dr \\ &= \tilde{Q}^2(\infty)\,, \end{aligned} \tag{34}$$

where

$$\tilde{Q}^2(r) \equiv -\frac{k_q r^2 \Phi' \sqrt{|uv|}}{uv} \tag{35}$$

and we assume (to be justified later) $\tilde{Q}^2(0) = 0$. The quantity $\tilde{Q}^2(r_b) - \tilde{Q}^2(r_a)$ is the square of the charge contained in the region between r_a and r_b.

2.4.2 Energy

In our static spherically symmetric spacetime the vector field

$$\xi^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{36}$$

satisfies the Killing equation

$$\xi^{\mu;\nu} + \xi^{\nu;\mu} = 0\,. \tag{37}$$

If we let

$$\mathsf{J}^\mu_\xi \equiv \xi^{\mu;\nu}{}_{;\nu} \tag{38}$$

then the anti-symmetry of $\xi^{\mu;\nu}$ implies

$$\mathsf{J}^\mu_\xi \surd = (\xi^{\mu;\nu}\surd)_{,\nu} \tag{39}$$

and

$$\left(\mathsf{J}^\mu_\xi \surd\right)_{,\mu} = 0\,. \tag{40}$$

Thus J^μ_ξ is a conserved current associated with the time invariance of the metric. The corresponding conserved quantity

$$M = k_\xi \int_\infty \mathsf{J}^0_\xi \surd\, dr\, d\theta\, d\phi \tag{41}$$

is the system mass (energy), where k_ξ is a real constant (to be determined later). Calculation of J_ξ^μ from Eqs. (36) and (38) for our static spherically symmetric spacetime gives

$$\mathsf{J}_\xi^\mu = \begin{pmatrix} \mathsf{R}_0{}^0 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{42}$$

But

$$\mathsf{R}_0{}^0 = -8\pi \left(\mathsf{T}_0{}^0 - \tfrac{1}{2}\mathsf{T} \right) \tag{43}$$

so Eq. (41) becomes

$$M = -8\pi k_\xi \int_\infty \left(\mathsf{T}_0{}^0 - \tfrac{1}{2}\mathsf{T} \right) \surd \, dr \, d\theta \, d\phi \, . \tag{44}$$

Substituting expressions (17), (18) and (28) for $\mathsf{T}_0{}^0$, T and $\surd$ into (44), then integrating over θ and ϕ gives

$$M = -4\pi k_\xi \int_0^\infty r^2 \left[\frac{(\omega - \Phi)^2 R^2}{u} - \frac{M^2 R^2}{2} + \frac{(\Phi')^2}{Q^2 uv} \right] \sqrt{|uv|} \, dr \, . \tag{45}$$

Another expression for M can be obtained as follows. Substituting from Eq. (39) into Eq. (41) gives

$$M = k_\xi \int_\infty \left(\xi^{0;\nu} \surd \right)_{,\nu} \, dr \, d\theta \, d\phi \, . \tag{46}$$

For our static spherically symmetric metric the integrand is

$$\left(\xi^{0;\nu} \surd \right)_{,\nu} = \frac{d}{dr} \left(-\frac{u'}{2uv} \surd \right) , \tag{47}$$

so

$$M = k_\xi \int_\infty \frac{d}{dr} \left(-\frac{u'}{2uv} \surd \right) \, dr \, d\theta \, d\phi \, . \tag{48}$$

Substituting expression (28) for $\surd$ then integrating over θ and ϕ gives

$$\begin{aligned} M &= \int_0^\infty \frac{d}{dr} \left(-\frac{2\pi k_\xi r^2 u' \sqrt{|uv|}}{uv} \right) dr \\ &= \tilde{M}(\infty) \, , \end{aligned} \tag{49}$$

where

$$\tilde{M}(r) \equiv -\frac{2\pi k_\xi r^2 u' \sqrt{|uv|}}{uv} \tag{50}$$

and we assume (to be justified later) $\tilde{M}(0) = 0$. The quantity $\tilde{M}(r_b) - \tilde{M}(r_a)$ is the mass contained in the region between r_a and r_b.

2.4.3 Angular momentum

Based on the Killing field

$$\chi^{\mu} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{51}$$

and using an approach similar to that of Section 2.4.2 one can show that the system angular momentum is zero, as expected for a spherically symmetric model.

2.4.4 Action

From Eqs. (12), (14), (21) and (28) the stationary value of the action is

$$S = \frac{T}{2} \int_0^{\infty} r^2 \left[\frac{M^2 R^2}{2} + \frac{(\Phi')^2}{Q^2 uv} \right] \sqrt{|uv|}\, dr \,. \tag{52}$$

3 Solution of the field equations

3.1 Solution at infinity

Our goal in this section is to determine the asymptotic ($r \to \infty$) expressions for the fields R, u, v and Φ. We seek a particle-like solution with charge and energy distributions localized near the origin and we impose the following boundary conditions:

1. The wave function Ψ and all components of the vector potential A_{μ} asymptotically vanish.

2. Spacetime is asymptotically flat and local measurements by an asymptotic observer agree with the predictions of flat spacetime physics. Specifically, the asymptotic behaviors of u and v reproduce Newtonian gravity, and the asymptotic expressions for Φ and R satisfy the flat spacetime Poisson and Klein-Gordon equations.

These boundary conditions will guide us to candidate expressions for the asymptotic fields. We will then verify by direct substitution that these candidates asymptotically satisfy field equations (22) through (25).

The asymptotic expressions

$$\begin{aligned} u &= 1 - 2Mr^{-1} + Q^2 r^{-2} \\ v &= u^{-1} \end{aligned} \tag{53}$$

correspond to the (Reissner-Nordström) spacetime structure outside a charged nonrotating black hole. They yield flat spacetime (Newtonian) gravity as $r \to \infty$, as required by boundary condition 2. They also imply [see Eq. (50)]

$$\tilde{M}(\infty) = -4\pi k_{\xi} M \,. \tag{54}$$

Comparing this with Eq. (49) gives

$$k_{\xi} = -\frac{1}{4\pi} \,. \tag{55}$$

The asymptotic expression

$$\Phi = Q^2 r^{-1} \tag{56}$$

follows from boundary conditions 1 and 2 and the flat spacetime laws of electrostatics. Expressions (53) and (56) imply [see Eq. (35)]

$$\tilde{Q}^2(\infty) = k_q Q^2 \,. \tag{57}$$

Comparing this with Eq. (34) gives

$$k_q = 1\,. \tag{58}$$

The asymptotic form of R follows from boundary condition 2 and the asymptotic flat spacetime Klein-Gordon equation [obtained by letting $u, v \to 1$ in Eq. (22)]

$$R'' + \frac{2}{r}\,R' + \left[(\omega - \Phi)^2 - M^2\right] R = 0\,, \tag{59}$$

where Φ is given by (56). The solution which asymptotically vanishes (boundary condition 1) is

$$R = \frac{R_\infty\, e^{-kr}}{r^{1+\sigma}}\,, \tag{60}$$

where R_∞ is a real constant,

$$\sigma = \frac{Q^2\omega}{k}\,, \tag{61}$$

and

$$k = \sqrt{M^2 - \omega^2} \tag{62}$$

with $\omega^2 < M^2$.

We now have asymptotic expressions for the fields which satisfy the boundary conditions. But we still need to verify that these expressions are consistent with our field equations. This is accomplished by substituting the asymptotic expressions for R, u, v and Φ into Eqs. (22) through (25) and verifying that the dominant terms satisfy the equations as $r \to \infty$. This procedure reveals that Eqs. (23) through (25) are, indeed, asymptotically satisfied. However, Eq. (22) is satisfied only if we let (see Appendix B)

$$\sigma = \frac{Q^2\omega}{k} + \frac{M}{k}\left(M^2 - 2\omega^2\right)\,, \tag{63}$$

which differs from (61). In order to satisfy boundary condition 2 we must make (61) and (63) consistent, which requires

$$\omega = \pm M/\sqrt{2} \tag{64}$$

and [from (62)]

$$k = M/\sqrt{2}\,. \tag{65}$$

In summary, based on the imposed boundary conditions we expect the asymptotic solution to have the form

$$\begin{aligned}
R &= R_\infty\, r^{-1\mp Q^2} e^{-Mr/\sqrt{2}} \\
u &= 1 - 2Mr^{-1} + Q^2 r^{-2} \\
v &= u^{-1} \\
\Phi &= Q^2 r^{-1}\,.
\end{aligned} \tag{66}$$

3.2 Physical constraints

In addition to satisfying field equations (22) through (25) the solution must also satisfy the charge and energy constraints of Section 2.4. Using results from Section 3.1 in Eqs. (29) and (45) we obtain the constraints

$$1 = \frac{1}{2}\int_0^\infty r^2 \left(\pm\frac{M}{\sqrt{2}} - \Phi\right)\frac{R^2}{u}\sqrt{|uv|}\,dr \tag{67}$$

and

$$1 = \frac{1}{M}\int_0^\infty r^2 \left[\left(\pm\frac{M}{\sqrt{2}} - \Phi\right)^2\frac{R^2}{u} - \frac{M^2R^2}{2} + \frac{(\Phi')^2}{Q^2uv}\right]\sqrt{|uv|}\,dr\,. \tag{68}$$

3.3 Solution strategy

If we are given values of the three parameters Q, M and R_∞ and a choice of sign in Eq. (64) then the asymptotic fields are fully defined by Eqs. (66). A complete solution can be obtained by integrating field equations (22) through (25) from the asymptotic limit ($r \to \infty$) down to the origin ($r \to 0$). Since the two constraints of Section 3.2 must be satisfied, only one of the three parameters is independent. It will be convenient to take Q as the independent parameter and regard M and R_∞ as functions of Q.

Once a solution is found for given Q then the action S can be calculated from Eq. (52). We may regard the action as a function of Q. The field equations, constraints and boundary conditions actually depend on Q^2, not Q, so we may write the action as $S(Q^2)$. Solutions of physical interest have stationary action so they must satisfy

$$\frac{\partial S}{\partial Q} = 2Q\,S'\big(Q^2\big) = 0\,. \tag{69}$$

3.4 Solution for zero charge

We will consider solutions with $Q = 0$. These solutions satisfy Eq. (69) so they are of physical interest.

With $Q = 0$ the asymptotic fields (66) become

$$\begin{aligned} R &= R_\infty\, r^{-1} e^{-Mr/\sqrt{2}} \\ u &= 1 - 2Mr^{-1} \\ v &= u^{-1} \\ \Phi &= 0\,. \end{aligned} \tag{70}$$

Field equation (25) becomes

$$\Phi'' + \left(\frac{2}{r} - \frac{u'}{2u} - \frac{v'}{2v}\right)\Phi' = 0 \tag{71}$$

which, given the asymptotic expression for Φ, yields the trivial solution (valid everywhere)

$$\Phi = 0\,. \tag{72}$$

Then field equations (22) through (24) become

$$R'' + \left(\frac{2}{r} + \frac{u'}{2u} - \frac{v'}{2v}\right) R' + \left(\frac{1}{2u} - 1\right) M^2 v R = 0\,, \tag{73}$$

$$\frac{u'}{2u} - \frac{v'}{2v} + \frac{1-v}{r} = -\frac{M^2 r v R^2}{2} \tag{74}$$

and

$$\frac{u'}{2u} + \frac{v'}{2v} = \frac{rv}{2}\left[\frac{M^2 R^2}{2u} + \frac{(R')^2}{v}\right]\,, \tag{75}$$

and constraints (67) and (68) can be written

$$\Delta \equiv 2\sqrt{2} \mp M \int_0^\infty \frac{r^2 R^2}{u} \sqrt{|uv|}\, dr = 0 \tag{76}$$

and

$$\pm\, 2\sqrt{2} - 2 - M \int_0^\infty r^2 R^2 \sqrt{|uv|}\, dr = 0\,. \tag{77}$$

Since both M and the integral in (77) are positive this constraint can only be satisfied if we choose the upper sign. So we need only consider the upper signs in Eqs. (76) and (77), and [from Eq. (64)] $\omega = M/\sqrt{2}$.

Given values of M and R_∞, and starting with the asymptotic fields (70), Eqs. (73) through (75) can be numerically integrated all the way to the origin. The values of M and R_∞ can be adjusted until constraints (76) and (77) are satisfied.

Figure 1 shows the locus in the M–R_∞ plane which satisfies constraint (77). Figure 2 plots Δ versus M along that locus. Since $\Delta = 0$ when constraint (76) is satisfied the five zero crossings in Fig. 2 correspond to the solutions we seek. These solutions describe uncharged spinless massive particles. Because the masses appear spontaneously as eigenvalues of a self-gravitating quantum field ("mass without mass") we borrow terminology from Wheeler [16] and dub these particles *quantum geons*. We will index the five geons by the integer $n = 0, 1, 2, 3, 4$ in order of increasing mass.

Various features of the geons are plotted in Figs. 3 through 5 and numerical characteristics are tabulated in Table 1. The probability density is

$$P \equiv \frac{\overline{\Psi}\Psi\surd}{\int_\infty \overline{\Psi}\Psi\surd\, dr\, d\theta\, d\phi} = N R^2 \surd\,, \tag{78}$$

where the normalization factor

$$N = \frac{M}{8\pi\left(\sqrt{2}-1\right)} \tag{79}$$

is determined from Eq. (77). The radial probability density

$$P_r = 4\pi N r^2 R^2 \sqrt{|uv|} \tag{80}$$

is plotted in Fig. 6. The stationary value of the action calculated from Eqs. (52) and (77) is

$$S = \frac{\left(\sqrt{2}-1\right) T M}{2} \tag{81}$$

and the action per cycle ($T = 2\pi/\omega$) is $(2-\sqrt{2})\pi$ for each of the five geons.

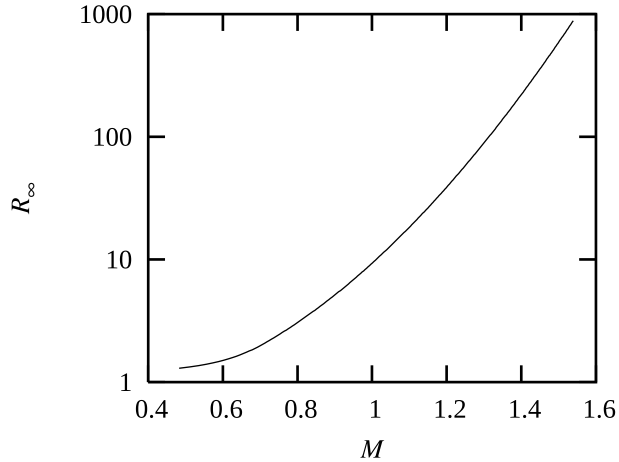

Figure 1: Locus in the M–R_∞ plane which satisfies constraint (77).

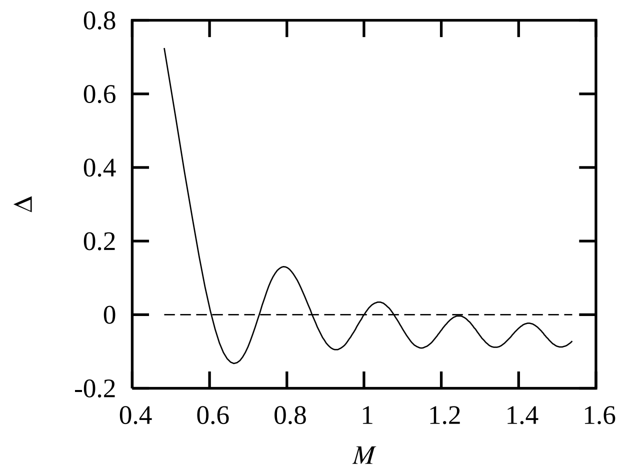

Figure 2: Left-hand side of constraint (76) versus mass for the locus of points of Fig. 1. The five zero crossings correspond to solutions of field equations (73) through (75) which satisfy boundary conditions (70) and constraints (76) and (77). There are no zero crossings in the neighborhood of $M = 1.25$.

Table 1: Numerical characteristics of the geons. Values of M and R_∞ are believed to be accurate to within ± 1 of the least-significant digit.

n	nodes in R	M	R_∞
0	0	0.605	1.51
1	1	0.729	2.22
2	1	0.866	4.26
3	2	1.000	9.3
4	2	1.077	15.6

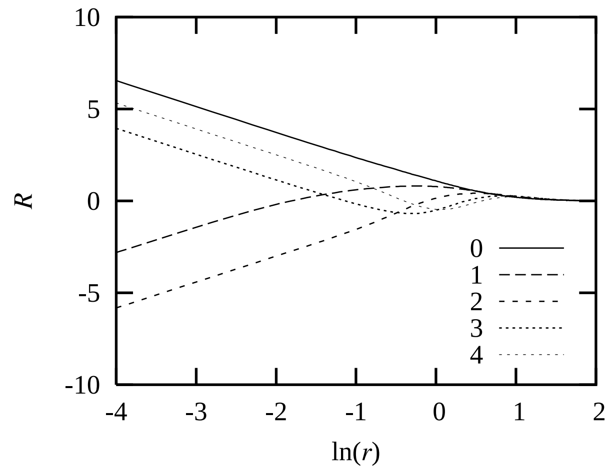

Figure 3: Radial wave function versus radius. Curves labeled by geon index n.

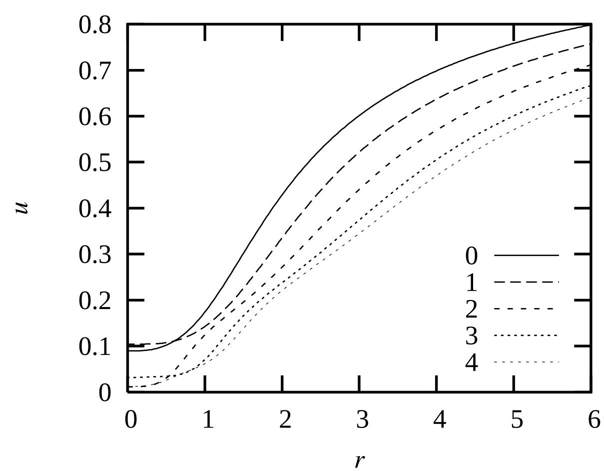

Figure 4: Metric function u versus radius. Curves labeled by geon index n.

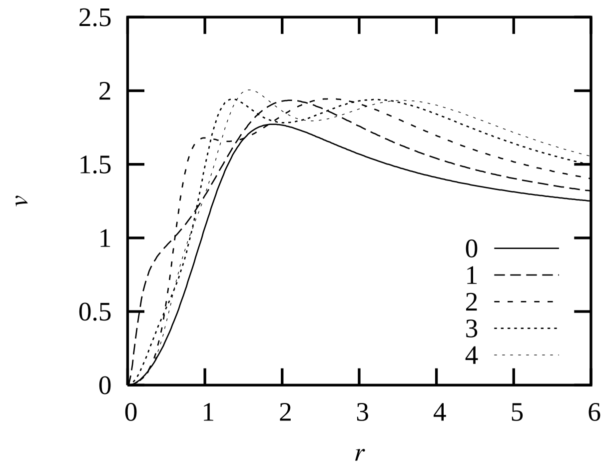

Figure 5: Metric function v versus radius. Curves labeled by geon index n.

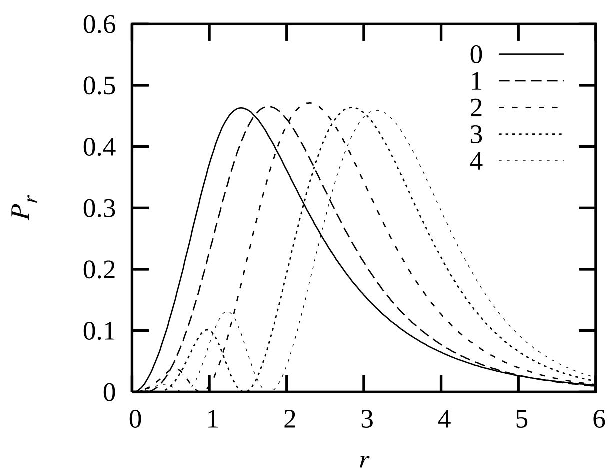

Figure 6: Radial probability density versus radius. Curves labeled by geon index n.

3.5 Solution at the origin

The results of Section 3.4 suggest that in the neighborhood of the origin

$$\begin{aligned} R &= R_0 + R_\ell \ln r \\ u &= u_0 \\ v &= v_2 r^2 \,, \end{aligned} \tag{82}$$

where R_0, R_ℓ, u_0 and v_2 are real constants and $u_0, v_2 > 0$. These expressions satisfy Eqs. (73) through (75) as $r \to 0$ provided

$$R_\ell = \pm\sqrt{2}\,. \tag{83}$$

Now that we know the behavior of the fields as $r \to 0$ we are in a position to justify two earlier assumptions. From expression (35) for $\tilde{Q}^2(r)$ we obtain

$$\lim_{r\to 0} \tilde{Q}^2(r) = \lim_{r\to 0} \left[-\frac{r\,\Phi'(r)}{\sqrt{u_0 v_2}} \right] = 0\,, \tag{84}$$

justifying the assumption $\tilde{Q}^2(0) = 0$. And from expression (50) for $\tilde{M}(r)$ we obtain

$$\lim_{r\to 0} \tilde{M}(r) = \lim_{r\to 0} \left[\frac{r\,u'(r)}{2\sqrt{u_0 v_2}} \right] = 0\,, \tag{85}$$

justifying the assumption $\tilde{M}(0) = 0$.

The wave function R diverges logarithmically at the origin. This singularity is weak enough that terms proportional to R^2 in the components of the energy-momentum tensor (17) are integrable. But terms of the form $(R')^2/v$ diverge as r^{-4} and are not integrable. These nonintegrable terms cancel out of the expression $2\mathsf{T}_0{}^0 - \mathsf{T}$ so the total energy is finite.

Substituting expressions (82) into Eq. (18) for T and using $\mathsf{R} = 8\pi\mathsf{T}$ gives

$$\mathsf{R} = \frac{2}{v_2 r^4} \tag{86}$$

for the dominant behavior of the scalar curvature as $r \to 0$. Thus the scalar curvature diverges at the center of each geon.

4 Implications of infinite curvature

Because the scalar curvature diverges at $r = 0$ it is not possible to establish a locally flat coordinate system there. So the locus $r = 0$ must be excluded from spacetime, leaving a hole in the spacetime manifold. [10] The presence of this spacetime singularity raises the following questions:

1. Do physical parameters (such as mass and charge) diverge?
2. Do arbitrary boundary conditions arise at the singularity?
3. Can particles and photons encounter the singularity and, if so, what happens to them?

Questions 1 and 2 have already been addressed: the geon mass, charge, angular momentum and action are all finite, and the wave function is normalizable; and the conditions $\tilde{Q}^2(0) = 0$ and $\tilde{M}(0) = 0$ are not arbitrary boundary conditions at the singularity—they are consequences of the model [see Eqs. (84) and (85)].

The answer to question 3 is not so clear-cut. In a classical (non-quantum) theory one considers spacetime pathological if the world line of a freely falling test particle (a point particle of negligible mass) does not exist after (or before) a finite interval of proper time. Such pathological spacetimes are said to be timelike *geodesically incomplete*. [10] In a quantum theory, however, no real particle can serve as a test particle in the neighborhood of a singularity, since such a particle would have to be so small (short wavelength, large energy) that it would itself dominate local spacetime structure. Perhaps the concept of geodesic completeness can be extended to the quantum realm [11], but at the moment no consensus exists on how to do this.

So we will not be able to answer question 3 definitively. But we will present a classical analysis which suggests the singularity is benign. In sections 4.1 and 4.2 we calculate timelike and null geodesics—the paths of freely falling particles and photons—in the geon spacetime. We will find that the geon core is sufficiently repulsive to cloak the singularity and leave the spacetime timelike and null geodesically complete.

4.1 Geodesic equations

Consider moving along a geodesic with velocity v^μ. The trajectory obeys

$$\frac{d\mathsf{v}_\alpha}{ds} = \tfrac{1}{2}\mathsf{g}_{\mu\nu,\alpha}\mathsf{v}^\mu\mathsf{v}^\nu \tag{87}$$

subject to the constraint

$$\mathsf{v}^\mu\mathsf{v}_\mu = \zeta\,, \tag{88}$$

where $\zeta = 0, 1$ for null and timelike geodesics, and s (proper time for a timelike geodesic) parameterizes the trajectory. $\mathsf{g}_{\mu\nu}$ is independent of t and ϕ so equation (87) implies v_t and v_ϕ are constant along the trajectory. Given the spherical symmetry we can (without loss of generality) consider trajectories confined to the equatorial plane ($\theta = \pi/2$, $\mathsf{v}_\theta = \mathsf{v}^\theta = 0$). Thus

$$\mathsf{v}_\mu = \begin{pmatrix} \varepsilon \\ -v\dot{r} \\ 0 \\ -\ell \end{pmatrix} \quad \text{and} \quad \mathsf{v}^\mu = \begin{pmatrix} \varepsilon/u \\ \dot{r} \\ 0 \\ \ell/r^2 \end{pmatrix}, \tag{89}$$

where $\dot{r} \equiv dr/ds$, and ε and ℓ are real constants. Then equation (88) gives

$$uv\dot{r}^2 = \varepsilon^2 - V^2\,, \tag{90}$$

where

$$V^2 \equiv u\left(\zeta + \frac{\ell^2}{r^2}\right)\,. \tag{91}$$

The evolution of ϕ is determined by

$$\dot{\phi} \equiv d\phi/ds = v^\phi = \ell/r^2\,, \tag{92}$$

and this equation and (90) completely determine the trajectory. Since $u, v \geq 0$ equation (90) can be satisfied only when $\varepsilon^2 \geq V^2$, and the condition $\varepsilon^2 = V^2$ corresponds to a turning

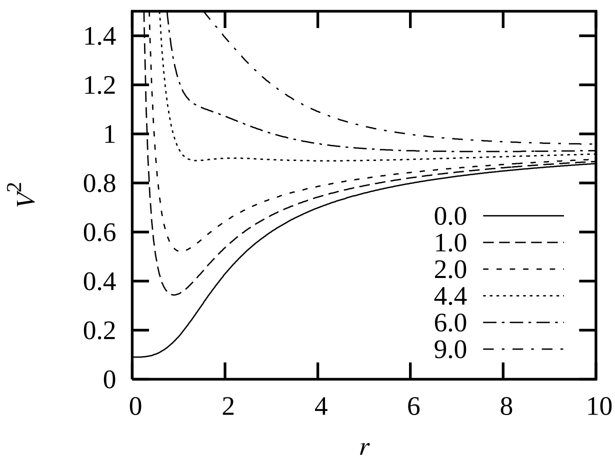

Figure 7: Effective potential for radial motion along timelike geodesics in the spacetime surrounding the $n = 0$ geon. Curves labeled by ℓ^2. $V^2 \to 1$ as $r \to \infty$.

point of the radial motion. We will refer to V^2 as the *effective potential* (for radial motion). The effective potentials for timelike and null geodesics in the spacetime surrounding the $n = 0$ geon are plotted in Figs. 7 and 8.

4.2 Geodesics near the singularity

As $r \to 0$ the metric approaches

$$ds^2 = u_0\, dt^2 - v_2 r^2\, dr^2 - r^2\, d\theta^2 - r^2\, \sin^2\theta\, d\phi^2\, . \tag{93}$$

Using the results of section 4.1 we will explore the equatorial geodesics of this metric.

Let r_0 represent the minimum value of r along a geodesic. By setting $\varepsilon^2 = V^2$ we find

$$r_0 = \sqrt{\frac{u_0 \ell^2}{\varepsilon^2 - u_0 \zeta}}\, . \tag{94}$$

The geodesic trajectory is determined by

$$\frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}} = \pm \frac{1}{\sqrt{v_2}} \sqrt{\left(\frac{r}{r_0}\right)^2 - 1} \tag{95}$$

which can be integrated to give

$$\phi = \pm\, r_0^* \, \ln\left[\frac{r^*}{r_0^*} + \sqrt{\left(\frac{r^*}{r_0^*}\right)^2 - 1}\,\right]\, , \tag{96}$$

where we have introduced the scaled radial coordinate $r^* = \sqrt{v_2}\, r$ and chosen the constant of integration so $\phi = 0$ when $r^* = r_0^*$. Geodesics corresponding to (96) are plotted in terms

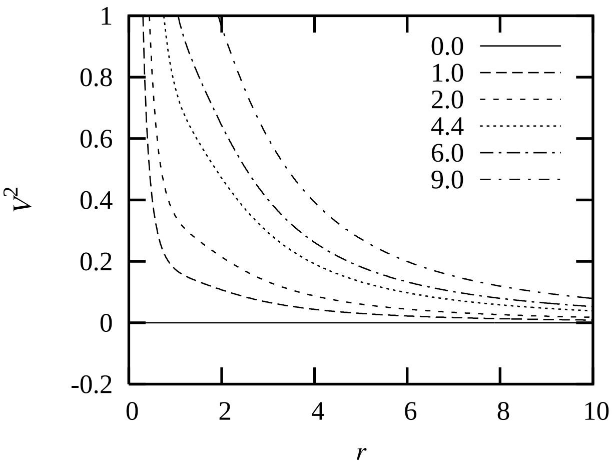

Figure 8: Effective potential for radial motion along null geodesics in the spacetime surrounding the $n = 0$ geon. Curves labeled by ℓ^2. $V^2 \to 0$ as $r \to \infty$.

of the Cartesian coordinates

$$\begin{aligned} x &\equiv r^* \cos\phi \\ y &\equiv r^* \sin\phi \end{aligned} \tag{97}$$

in Fig. 9. Test particles are attracted when $r^* \gg 1$ and repelled when $r^* \ll 1$. In the limit of zero angular momentum ($\ell = r_0^* = 0$) the geodesic is coincident with the positive x axis, corresponding to a test particle rebounding directly backwards.

Now let us calculate the behavior of s as a function of r for particles and photons following the geodesics of Fig. 9. From Eq. (90) we obtain

$$(r\dot{r})^2 = \frac{\varepsilon^2 - V^2}{u_0 v_2} = \kappa^2 \left[1 - \left(\frac{r_0}{r}\right)^2\right] , \tag{98}$$

where

$$\kappa \equiv \sqrt{\frac{\varepsilon^2 - u_0\zeta}{u_0 v_2}} . \tag{99}$$

Integrating (98) we obtain

$$s^* = \pm\frac{(r_0^*)^2}{2}\left\{\frac{r^*}{r_0^*}\sqrt{\left(\frac{r^*}{r_0^*}\right)^2 - 1} + \ln\left[\frac{r^*}{r_0^*} + \sqrt{\left(\frac{r^*}{r_0^*}\right)^2 - 1}\right]\right\} , \tag{100}$$

where we have introduced the scaled parameter $s^* = v_2\kappa s$ and chosen the constant of integration so $s^* = 0$ when $r^* = r_0^*$. Equation (100) is plotted in Fig. 10 for various values of r_0^*.

Figures 9 and 10 suggest that the core of the geon behaves like a repulsive potential barrier, effectively cloaking the singularity. Freely falling test particles with zero angular

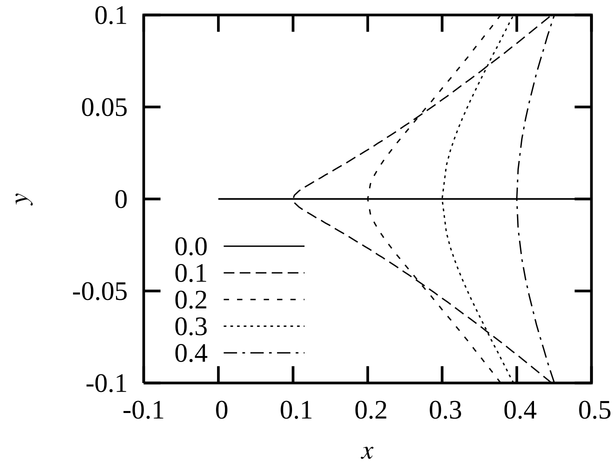

Figure 9: Geodesics associated with the metric (93) near the center of a geon. See Eqs. (97). Curves labeled by r_0^*.

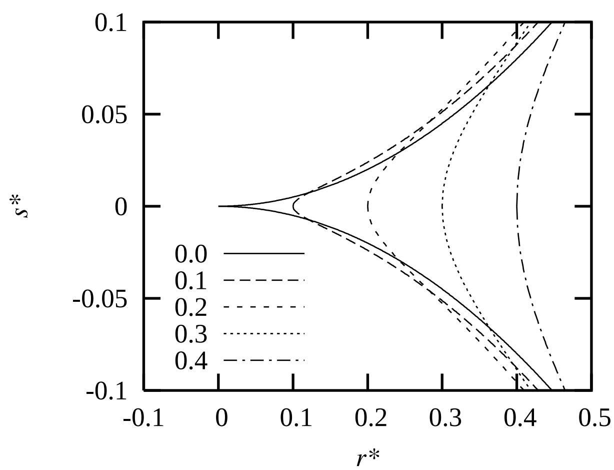

Figure 10: Parameter s^* versus radius r^* for geodesics associated with the metric (93) near the center of a geon. Curves labeled by r_0^*.

momentum ($\ell = r_0^* = 0$) do encounter the singularity, but they rebound—their world lines do not vanish. The spacetime is timelike and null geodesically complete.

One may object that any test particle encountering the singularity will be destroyed by infinite tidal forces. But keep in mind that no real particle can serve as a test particle at the Planck scale, so infinite tidal forces at the singularity do not necessarily pose a physical conundrum.

5 Discussion

The use of boundary condition 2 to force correspondence between Eqs. (61) and (63) can be motivated in the following way. Consider two different localized physical systems A and B with zero angular momentum and identical charges and masses. Suppose A is accurately described by the flat spacetime Klein-Gordon equation (59) while B is described by the curved spacetime equation (22). Viewed from a sufficiently large distance each system appears as a point particle with a wave function given by (60). Using local measurements of the probability density $\overline{\Psi}\Psi$ an asymptotic observer could determine σ and thereby discriminate between the particles, despite the fact that both have identical charge, angular momentum and mass. It is physically appealing to expect the asymptotic behavior of $\overline{\Psi}\Psi$ to be uniquely determined by these three parameters. Indeed, this is what we typically mean by the word "particle". By selecting ω to force σ to be identical for particles A and B we guarantee that the asymptotic behavior of $\overline{\Psi}\Psi$ depends only on charge, angular momentum and mass, and is independent of internal details.

At the Planck scale many physicists expect classical notions of spacetime to fail—a concept conveyed by the phrase (also coined by Wheeler) "spacetime foam". A successful quantum theory of gravity would, it is thought, flesh out the details of spacetime foam and erase the singularities associated with point particles.

The geon model developed here suggests a different perspective. Classical spacetime is assumed valid at the Planck scale and point particles are replaced by eignemodes of a quantum field. Singularities reminiscent of point particles remain, but they do not disturb the geodesic completeness of spacetime. Multi-particle systems would, presumably, correspond to multiple excitations of the geon field [15, 3] and all particle interactions (including those involving wave-packet reduction) would ultimately derive from the action (1). In short, the geon perspective replaces the search for a "quantum theory of gravity" with the search for a "gravitational theory of quanta". [3]

Some questions for future investigation come to mind:

1. Do charged solutions of Eq. (69) exist?

2. Do solutions with nonzero angular momentum exist? The spherically symmetric trial solution [Eqs. (13), (14) and (15)] could be replaced by one with axial symmetry. However, the trial solution would now involve θ and ϕ as well as r and t, and the metric tensor and vector potential would have additional nonzero components, so the solution would be much more challenging. Furthermore, it is not obvious what functional form should replace metric tensor (13).

3. What is the physical significance of the angular frequency $\omega = M/\sqrt{2}$?

4. What is the physical significance of the stationary value of the action and the action per cycle?

5. The action (1) is, arguably, the simplest which includes gravitation, electromagnetism and quantum mechanics. But there is no compelling reason to regard it as correct. [8, p. 144] It would be interesting to see how other terms in the action (such as a cosmological constant or conformal coupling [9, p. 116]) affect geon solutions.

The geons described here are far too massive to correspond to any known particle. They would interact gravitationally with ordinary matter so they appear to be candidates for dark matter. A number of workers have considered Planck-mass particles as dark matter (see [5] and references therein) but it is not clear whether such models can be successfully incorporated into standard cosmology. The density of dark matter within a galactic halo is thought to be about $0.3\,\mathrm{GeV\,cm^{-3}}$ [1] so, if all dark matter is composed of Plank-mass (1.2×10^{19} GeV) geons, the local geon number density is about $0.3 \times 10^{-19}\,\mathrm{cm^{-3}}$. This tiny density and weak coupling to ordinary matter would make the detection of such particles difficult.

A Notation and conventions

Our notation follows Dirac [7]. Greek indices take on the values $0, 1, 2, 3$ and repeated indices are summed. The spacetime coordinates are x^μ with $\mathsf{x}^0 = t$ and the metric signature is $+---$.

The curvature tensor is

$$\mathsf{R}^\alpha{}_{\mu\nu\beta} \equiv -\Gamma^\alpha{}_{\mu\nu,\beta} + \Gamma^\alpha{}_{\mu\beta,\nu} - \Gamma^\sigma{}_{\mu\nu}\Gamma^\alpha{}_{\sigma\beta} + \Gamma^\sigma{}_{\mu\beta}\Gamma^\alpha{}_{\sigma\nu}\,, \tag{101}$$

the Ricci tensor is $\mathsf{R}_{\mu\nu} \equiv \mathsf{R}^\alpha{}_{\mu\nu\alpha}$ and the scalar curvature is $\mathsf{R} \equiv \mathsf{R}_\mu{}^\mu$, where

$$\Gamma_{\alpha\mu\nu} \equiv \tfrac{1}{2}\left(\mathsf{g}_{\alpha\mu,\nu} + \mathsf{g}_{\alpha\nu,\mu} - \mathsf{g}_{\mu\nu,\alpha}\right) \tag{102}$$

is the Christoffel symbol and a comma preceding some lower index μ denotes partial differentiation with respect to x^μ. The metric tensor $\mathsf{g}_{\mu\nu}$ is symmetric, its contraction is

$$\mathsf{g}_{\mu\nu}\mathsf{g}^{\mu\nu} = \mathsf{g}_\mu{}^\mu = 4\,, \tag{103}$$

and, for notational brevity,

$$\surd \equiv \sqrt{|\det(\mathsf{g}_{\mu\nu})\,|}\,. \tag{104}$$

A semicolon or colon preceding some lower index μ denotes a covariant derivative with respect to x^μ. A semicolon denotes the covariant derivative of general relativity. For a scalar, covariant vector and contravariant vector

$$\begin{aligned} \Psi_{;\mu} &\equiv \Psi_{,\mu} \\ \mathsf{V}_{\mu;\nu} &\equiv \mathsf{V}_{\mu,\nu} - \Gamma^\alpha{}_{\mu\nu}\mathsf{V}_\alpha \\ \mathsf{V}^\mu{}_{;\nu} &\equiv \mathsf{V}^\mu{}_{,\nu} + \Gamma^\mu{}_{\alpha\nu}\mathsf{V}^\alpha \end{aligned} \tag{105}$$

and the procedure generalizes in the usual way to tensors of any rank and mixture of covariant and contravariant indices. A colon denotes the generalized covariant derivative

$$\mathsf{U}_{:\mu} \equiv \mathsf{U}_{;\mu} + iQ\mathsf{A}_\mu\mathsf{U} \tag{106}$$

for an arbitrary tensor U.

B Asymptotic wave function

In the asymptotic regime [see Eqs. (53) and (56)]

$$\Phi = Q^2 r^{-1} + O(r^{-2}) \tag{107}$$

and

$$u = 1 - 2Mr^{-1} + Q^2 r^{-2} + O(r^{-3})\,. \tag{108}$$

By expanding u^{-1} in terms of r^{-1} as $r \to \infty$ we obtain

$$v = 1 + 2Mr^{-1} + \left(4M^2 - Q^2\right) r^{-2} + O(r^{-3})\,. \tag{109}$$

Consider a trial solution for R of the form

$$R = R_\infty \left[1 + ar^{-1} + O(r^{-2})\right] r^{-1-\sigma} e^{-kr}\,, \tag{110}$$

where a is a real constant. Substituting the above expressions into Eq. (22) and expanding in terms of r^{-1} as $r \to \infty$ gives

$$0 = \kappa + \left[\kappa a - 2M\left(M^2 - 2\omega^2\right) - 2Q^2\omega + 2k\sigma\right] r^{-1} + O(r^{-2})\,, \tag{111}$$

where

$$\kappa \equiv k^2 + \omega^2 - M^2\,. \tag{112}$$

Equation (111) must be satisfied term-by-term so $\kappa = 0$ and we obtain Eqs. (62) and (63).

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