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Logarithmic potential for the gravitational field of Schwarzschild black holes

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ABSTRACT

Approximate gravitational potentials are often used to describe analytically the motion of particles near black holes (BHs), as well as to study the structure of an accretion disc. Such 'pseudo-Newtonian' potentials are used with the flat-metric equations. Here we consider the motion of a free particle near a non-rotating BH in the context of an exact 'logarithmic' gravitational potential. We show how the logarithmic potential gives an exact solution for a mechanical problem and present the relativistic Bernoulli equation for the fluid in the Schwarzschild metric.

Key words: stars: black holes - gravitation.

1 INTRODUCTION

In the Newton celestial mechanics, a gravitational potential is one of the basic concepts. In the General Relativity (GR) there is generally no such concept as a gravitational potential. In some special cases, however, it is possible to use such a concept, as we show in this work. This gravitational potential is different from what is usually termed as a pseudo-Newtonian potential.

To describe analytically and in a simple way the dynamics of particles near a BH, as well as to study the structure of an accretion disc, approximate approaches are frequently used. For example, it is common to utilize pseudo-Newtonian gravitational potentials in the equations written in the flat-metric. For a non-rotation BH, the potential by Paczynsky & Wiita (1980) is used (hereafter, 'PW potential'). For a rotating black hole (BH), Artemova, Bjoernsson & Novikov (1996) proposed a formula for a pseudo-Newtonian gravitational force acting on particles near Kerr BH.

Here we consider a non-rotating BH and a 'logarithmic' gravitational potential. This gravitational potential, together with an allowance for the curvature of the space-time, provide the laws of motion for a free particle, which are identical to those derived in the GR.

In Section 2, the pseudo-Newtonian gravitational potentials are very briefly reviewed. We introduce the logarithmic potential in Section 3. In Section 4, we consider the equation of motion of a particle in a curved space–time and derive the conserved value of energy. We obtain the law of motion for the logarithmic potential and consider its consequences in Section 5. The relativistic Bernoulli equation for a stationary fluid around a Schwarschild BH is derived in Section 6.

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2 PSEUDO-NEWTONIAN GRAVITATIONAL POTENTIALS

Near a BH, the curvature of the space–time is a decisive factor affecting the structure of an accretion disc. For a non-rotating BH, the radius of the innermost stable circular orbit $r_{\rm ISCO} = 3 R_{\rm g}$, where the Schwarzschild radius $R_{\rm g}$ is the event horizon of a non-rotating BH and is given by

$$R_{\rm g} = 2 \, G \, M/c^2 \, .$$

To approximate effects of the GR in the vicinity of a non-rotating BH, the Paczynski–Wiita potential can be used (Paczynsky & Wiita 1980):

$$\Phi_{\rm PW} = -\frac{G\,M}{r-R_{\rm g}}\,.\tag{1}$$

For free particles in circular orbits, the velocities can be found from the radial component of the Navier–Stokes equation

$$\frac{v_{\varphi}^2}{r} = \frac{\mathrm{d}\Phi}{\mathrm{d}r} \,. \tag{2}$$

As a result, one obtains the orbital velocity

$$\frac{v_{\varphi}^{\rm PW}}{c} = \frac{1}{\sqrt{2}} \frac{\sqrt{r R_{\rm g}}}{(r - R_{\rm g})} \, .$$

and the specific angular momentum of a test particle in the Paczynski-Wiita potential

$$h^{\rm PW} = v_{\varphi}^{\rm PW} r = \sqrt{\frac{G M r}{(1 - \frac{R_{\rm g}}{r})^2}}.$$
 (3)

The modified potential (equation 1) is often used in hydrodynamic and magnetohydrodynamic numerical codes, since it approximates quite well the curvature effects of the space-time metric around a Schwarzschild BH (Ohsuga & Mineshige 2011; Jiang,



Figure 1. Illustration of the 'shrinking' of a coordinate element dr, corresponding to an element of distance dl, measured by a local static or a fiducial observer ('FIDO' of Thorne et al. 1986).

Stone & Davis 2014; Yuan & Narayan 2014). Other approximate potentials, in particular such applicable to the case of rotating BHs, can be found in Artemova et al. (1996), Kato, Fukue & Mineshige (1998), Witzany, Semerák & Suková (2015) and, especially, Witzany & Lammerzahl (2017).

3 LOGARITHMIC POTENTIAL

To describe the relativistic motion in the vicinity of a Schwarzschild BH we may use the following 'logarithmic' potential (Landau & Lifshitz 1975; Thorne, Price & MacDonald 1986):

$$\Phi = \frac{c^2}{2} \ln\left(1 - \frac{R_g}{r}\right) = c^2 \ln\sqrt{1 - \frac{R_g}{r}}.$$
 (4)

Below, we will show how the logarithmic potential gives an exact solution for a mechanical problem. This will require consideration of the space–time curvature near a Schwarzschild BH.

Note that Artemova et al. (1996) treated the logarithmic potential as a pseudo-Newtonian potential and this provided an approximate result, with an order of accuracy comparable to that of the PW potential.

4 EQUATION OF MOTION WITH LOGARITHMIC POTENTIAL

Let us write down the Schwarzschild stationary metric as the square of an interval between two events separated in time and space:

$$ds^{2} = -(1 - R_{g}/r) dt^{2} + (1 - R_{g}/r)^{-1} dr^{2} + r^{2}(d\theta + \sin^{2}\theta d\varphi).$$

Here, t, r, θ , and φ are the Schwarzschild coordinates. Due to the curvature of the space–time near a BH, the distance element dl along the radius, as measured by a local observer, is longer than the corresponding coordinate element dr (see Fig. 1):

$$\mathrm{d}l = \frac{\mathrm{d}r}{\sqrt{1 - R_{\mathrm{g}}/r}} \,.$$

Inherited by equation (4), $\sqrt{1 - R_g/r}$ is a lapse function in the Schwarzschild metric. It determines the redshift of a signal emitted from the vicinity of a BH and the difference between two time intervals, one of which, dt, is measured at infinity and the other, $d\tau_l$, by an observer in the local stationary reference frame:

$$\mathrm{d}\tau_l/\mathrm{d}t = \sqrt{1 - R_{\rm g}/r} \,. \tag{5}$$

The time measured in the frame of a moving particle is related to the time measured by a local stationary observer as

$$d\tau_{\rm p}/d\tau_l = \sqrt{1 - v^2/c^2} \,. \tag{6}$$

Let us consider a relativistic particle with the rest mass m_o . Its momentum p and energy E_{local} , relative to the local stationary observer, are

$$p = rac{m_o \, v}{\sqrt{1 - v^2/c^2}}$$
 and $E_{\mathrm{local}} = rac{m_o \, c^2}{\sqrt{1 - v^2/c^2}},$

respectively, where the square velocity $v^2 = v_r^2 + v_{\varphi}^2$ for particles moving in the equatorial plane.

We may also introduce the notion of 'energy at infinity' E. This value remains unchanged along the particle trajectory. Let us determine it.

Consider a particle travelling past a stationary observer who is located at a distance from a BH. The equation of particle motion in the reference system of this observer can be written as follows(Landau & Lifshitz 1975):

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau_l} = -\frac{m_o}{\sqrt{1 - v^2/c^2}} \,\nabla\Phi\,.\tag{7}$$

As it is done in mechanics, the energy of a particle can be found from the equation of motion by multiplying scalarly equation (7) by v:

$$\boldsymbol{v} \frac{\mathrm{d}}{\mathrm{d}\tau_l} \left(\frac{m_o \boldsymbol{v}}{\sqrt{1 - v^2/c^2}} \right) = -\frac{m_o \boldsymbol{v}}{\sqrt{1 - v^2/c^2}} \nabla \Phi$$

or, noting that the potential Φ is spherically symmetric,

$$\boldsymbol{v} \frac{\mathrm{d}}{\mathrm{d}\tau_l} \left(\frac{m_o \boldsymbol{v}}{\sqrt{1 - v^2/c^2}} \right) = -\frac{m_o \boldsymbol{v} \boldsymbol{e}_r}{\sqrt{1 - v^2/c^2}} \frac{\mathrm{d}\Phi}{\mathrm{d}l} , \qquad (8)$$

where e_r is a unit radial vector in the Cartesian reference system of the local observer. Further, we differentiate the left-hand part of equation (8):

$$\frac{1}{2} \frac{m_o}{\sqrt{1 - v^2/c^2}} \frac{\mathrm{d}v^2}{\mathrm{d}\tau_l} + \frac{1}{2} \frac{m_o v^2/c^2}{(1 - v^2/c^2)^{3/2}} \frac{\mathrm{d}v^2}{\mathrm{d}\tau_l}$$
$$= -\frac{m_o v e_r}{\sqrt{1 - v^2/c^2}} \frac{\mathrm{d}\Phi}{\mathrm{d}l} .$$

When multiplying this by $(1 - v^2/c^2)^{3/2}$, cancelling out the two equal terms with opposite signs in the left-hand part of the equation and using the equality $v_r = dl/d\tau_l$ for the radial velocity, we obtain

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\tau_l} \left(1 - v^2/c^2\right) = \left(1 - v^2/c^2\right) \frac{\mathrm{d}l}{\mathrm{d}\tau_l} \frac{\mathrm{d}}{\mathrm{d}l} \ln(1 - R_{\mathrm{g}}/r)^{1/2},$$

which is equivalent to the following equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau_l}\ln(1-v^2/c^2) = \frac{\mathrm{d}}{\mathrm{d}\tau_l}\ln(1-R_{\mathrm{g}}/r)\,.$$

As a result, we obtain the following relationship:

$$(1 - R_{\rm g}/r) / (1 - v^2/c^2) = const.$$

Hence, the value

$$E = \frac{m_o c^2}{\sqrt{1 - v^2/c^2}} \sqrt{1 - \frac{R_g}{r}} = E_{\text{local}} \sqrt{1 - \frac{R_g}{r}} = const$$
(9)

does not change for a freely moving particle, while the locally measured energy E_{local} varies in the gravitational field of the BH. This value *E* is termed 'energy-at-infinity' (Thorne et al. 1986). In

GR, the value E corresponds to the time component of the four-vector impulse (Landau & Lifshitz 1975).

For a photon, the rest mass of which is $m_o = 0$, equation (9) yields a relation between its frequency v_o in the reference system of the local observer, and its frequency detected at infinity $v_{\infty} = v_o \sqrt{1 - R_g/r}$. This relation describes the redshift effect.

In the non-relativistic approximation, energy \mathscr{E}_N of a particle has the well-known form

$$E - m_o c^2 \equiv \mathscr{E}_{\rm N} = m_o v^2 / 2 - m_o G M / r .$$
 (10)

Let us underline a difference between post-Newtonian approximations and the approach that we use here. A pseudo-Newtonian potential enters (equation 10) in place of the Newtonian potential and is a term of a sum, while the exact expression for the conserved energy (equation 9) is a product of two terms.

5 VELOCITIES AND BINDING ENERGY

Let us now determine the components of the particle velocity in the equatorial plane. A freely moving particle with mass m_o in the spherically symmetrical gravitational potential keeps its angular momentum unchanged (Landau & Lifshitz 1975)

$$h_{\rm p} = \frac{m_o \, v_{\varphi} \, r}{\sqrt{1 - v^2/c^2}} \,. \tag{11}$$

When taking into consideration that $v^2 = v_r^2 + v_{\varphi}^2$, equations (9) and (11) yield

$$\frac{v_r^2}{c^2} = 1 - \frac{m_o^2 c^4}{E^2} \left(\frac{h_p^2}{r^2 m_o^2 c^2} + 1 \right) \left(1 - \frac{R_g}{r} \right).$$
(12)

Multiplying by a factor $E^2/(m_o^2 c^4)$ and using equations (6) and (9) together with the relation

$$\frac{v_r^2}{c^2} = \frac{1}{c^2} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau_\mathrm{p}}\right)^2 \frac{m_o^2 \, c^4}{E^2} \,,$$

we may rewrite the last expression. As a result, we obtain the law of motion for a particle with energy E, which is identical to the exact solution in GR, see Shapiro & Teukolsky (1983):

$$\frac{1}{c^2} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau_{\rm p}} \right)^2 = \frac{E^2}{m_o^2 \, c^4} - \left(\frac{h_{\rm p}^2}{r^2 \, m_o^2 \, c^2} + 1 \right) \, \left(1 - \frac{R_{\rm g}}{r} \right) \, .$$

Note that in the approximation of a Newtonian potential, this law of motion looks like:

$$v_r^2 = \frac{2}{m_o} \left(\mathscr{E}_{\mathrm{N}} + m_o \, \frac{G \, M}{r} \right) - \frac{h_{\mathrm{N}}^2}{r^2 \, m_o^2},$$

where $h_{\rm N} = m_o v_{\varphi} r = const$.

Let us consider particles moving in circular orbits around a Schwarzschild BH. For such motion, both v_r and $dr/d\tau_p$ become zero. For the sake of convenience, we may introduce an effective potential

$$V(r) = \left(\frac{h_{\rm p}^2}{r^2 m_o^2 c^2} + 1\right) \left(1 - \frac{R_{\rm g}}{r}\right) \,.$$

For circular orbits, the first derivative of this potential becomes zero (the potential has an extremum). The system of equations

$$\frac{\mathrm{d}r}{\mathrm{d}\tau_{\mathrm{p}}} = 0\,, \qquad \qquad \frac{\partial V(r)}{\partial r} = 0$$

yields the following angular momentum in a circular orbit:

$$h_{\rm p}^2 = \frac{m_o^2 r R_{\rm g} c^2}{2 - 3R_{\rm g}/r} \,. \tag{13}$$

After squaring equation (11) and using equation (13), we obtain the tangential velocity as measured by the local observer

$$\frac{v_{\varphi}}{c} = \frac{1}{\sqrt{2}} \sqrt{\frac{R_{\rm g}}{r - R_{\rm g}}} \,. \tag{14}$$

For the local observer, the angular velocity of a particle is

$$\omega_l = \frac{v_{\varphi}}{r} = \frac{c}{\sqrt{2}r} \sqrt{\frac{R_g}{r - R_g}} \,. \tag{15}$$

Using time dilation (equation 5), we obtain the angular velocity measured by an observer at infinity:

$$\omega = \frac{c \sqrt{R_g}}{\sqrt{2}r^{3/2}} = \frac{\sqrt{G M}}{r^{3/2}},$$
(16)

that is, the classical expression following from Kepler's law.

According to the Rayleigh criterion (Rayleigh 1917), stable circular orbits cannot exist where $dh_p/dr < 0$. This criterion implies that the innermost stable circular orbit has a radius $r_{ISCO} = 3 R_g$.

When substituting the velocity $v_{\varphi} = c/2$ that corresponds to $r_{\rm ISCO}$, into equation (9), we determine the energy of a particle rotating in the last possible stable orbit. The energy of this particle, $E = m_o c^2 2\sqrt{2}/3$, is less than its rest energy at infinity, $m_0 c^2$. This means that when a particle moves from infinity towards the Schwarzschild BH, i.e. in the process of accretion, the released energy is $(m_0 c^2 - E) \approx 0.0572 m_0 c^2$. Thus, the energy conversion efficiency in the accretion process on to a non-rotating BH is equal to ~ 6 per cent. A calculation using the Kerr metric shows that the binding energy of the particles is the largest for a maximally rotating BH and equals to $1 - \sqrt{1/3} \approx 0.423$ times the rest energy (Kato, Fukue & Mineshige 2008).

Extracting the square root of equation (13), we find the specific angular momentum of a particle in circular orbit in the Schwarzschild metric:

$$h = \frac{h_{\rm p}}{m_o} = \frac{\sqrt{G \, M \, r}}{\sqrt{1 - \frac{3 \, G \, M}{c^2 \, r}}} \,. \tag{17}$$

Fig. 2 shows the dependence of the specific angular momentum of a test particle on the radius of the orbit in the gravitational field of the BH. In addition, the respective dependencies are shown in the Newtonian potential (dashed line) and in the Paczynski–Wiita potential (dotted line). In the gravitational field of the Schwarzschild BH, the specific angular momentum *h* becomes minimum at the radius of the innermost stable circular orbit 6 $G M/c^2$. In contrast to the case of the Newtonian potential, the first derivative of the specific angular momentum, dh/dr, vanishes at this radius (see Fig. 2).

We notice that the innermost stable orbit for the logarithmic potential *treated as a pseudo-Newtonian potential* within the classical approach (Artemova et al. 1996) has radius 2 R_g , and the normalized binding energy at this orbit is 0.096. This is an evidently much worse result, comparing to the accuracy provided by the Paczynski–Wiita potential. For the Paczynski–Wiita potential, the radius of the last stable orbit coincides with the GR result, 3 R_g , although the binding energy exceeds by ~9 per cent the exact value (see Table 1).



Figure 2. Specific angular momentum *h* of a test particle in the gravitational field of a BH. The inner radius of the disc is $r_{in} = 3R_g = 6 G M/c^2$. The solid lines show the dependence in the exact logarithmic potential (equation 4), dotted lines show the same in the Paczynski–Wiita potential, dashed lines – in the Newtonian approximation.

Table 1. The normalized binding energy of a particle at the innermost stable circular orbit in different gravitational potentials.

	$(m_0 c^2 - E)/(m_0 c^2)$
Newtonian potential	1/12 = 0.08(3)
Paczynski–Wiita potential	1/16 = 0.0625
Logarithmic potential	
as a pseudo-Newtonian potential	0.096
Logarithmic potential	
in Schwarzschild metric	$1 - 2\sqrt{2}/3 \approx 0.0572$

6 RELATIVISTIC BERNOULLI EQUATION

We have considered above the mechanical characteristics of moving particles. Hydrodynamic equations can be also written for the case of fluid motion in the gravitational filed of a Schwarzschild BH, using the concept of gravitational potential. Here we consider the Bernoulli equation.¹

For an isentropic stationary motion of a fluid we can write an Euler equation in a relativistic form

$$\gamma (\boldsymbol{v} \nabla)(\gamma w \boldsymbol{v}) + c^2 \nabla w = -\gamma w \nabla \Phi.$$
⁽¹⁸⁾

Here ω is a 'specific' dimensionless enthalpy (per one particle). For $v \ll c$, we have $w = 1 + \frac{w_{\text{NR}}}{c^2}$, where w_{NR} is a non-relativistic enthalpy. For the ideal gas,

$$w_{\rm NR} = \frac{n}{n+1} \frac{P}{\rho} \,,$$

where *P* is the pressure, ρ is the density, and *n* is the adiabatic index $(P \propto \rho^n)$. Equation (18) is obtained from a relativistic equation for the energy conservation in a fluid (see section 134, Chap. XV of Landau & Lifshitz 1987) by adding the term $(-\gamma w \nabla \Phi)$ to its right-hand side, which allows for the action of the gravitational force. In this section, we use the following designation: $\gamma = (1 - v^2/c^2)^{-1/2}$.

Following the usual rules for transformations with the operator ∇ (Korn & Korn 1961), we rewrite the first term in equation (18) as

$$\gamma \boldsymbol{v} (\gamma \boldsymbol{v} \cdot \nabla w) + w(\gamma \boldsymbol{v} \nabla) \gamma \boldsymbol{v} .$$
⁽¹⁹⁾

¹This section was added after we had received essential comments from the anonymous referee.

Using the rules for a double vector product, the first term in equation (19) can be transformed into

$$\gamma \boldsymbol{v} (\gamma \boldsymbol{v} \cdot \nabla w) = \gamma \boldsymbol{v} \times [\gamma \boldsymbol{v} \times \nabla w] + \gamma^2 c^2 \nabla w$$

The second term of equation (19) can be transformed using another formula of the vector analysis (Korn & Korn 1961) as

$$(\gamma \boldsymbol{v} \nabla) \gamma \boldsymbol{v} = \frac{1}{2} \nabla \gamma^2 v^2 - \gamma \boldsymbol{v} \times [\nabla \times \gamma \boldsymbol{v}].$$
(20)

Now let us convert the first term in the right-hand side of equation (20):

$$\frac{1}{2} \nabla \gamma^2 v^2 = \frac{1}{2} \nabla (c^2 + \gamma^2 v^2) = \frac{1}{2} \nabla \left(c^2 + \frac{v^2}{1 - v^2/c^2} \right)$$
$$= \frac{1}{2} \nabla \frac{c^2}{1 - v^2/c^2} = \frac{c^2}{2} \nabla \gamma^2.$$

We divide equation (18) by $\gamma^2 w$ and, applying the above manipulations, obtain

$$\frac{c^2}{w}\nabla w + \boldsymbol{v} \times \left[\boldsymbol{v} \times \frac{\nabla w}{w}\right] + \frac{c^2 \nabla \gamma}{\gamma} - \frac{1}{\gamma^2} [\gamma \, \boldsymbol{v} \times [\nabla \times \gamma \boldsymbol{v}]] = -\nabla \Phi.$$
(21)

Now let us scalarly multiply equation (21) by \boldsymbol{v} . Vectors $\left[\boldsymbol{v} \times \left[\boldsymbol{v} \times \frac{\nabla w}{w}\right]\right]$ and $\left[\gamma \boldsymbol{v} \times [\boldsymbol{\nabla} \gamma \boldsymbol{v}]\right]$ are orthogonal to the velocity vector \boldsymbol{v} . Thus, their projections to the direction of the motion is zero and the scalar product of equation (21) by \boldsymbol{v} yields

$$\boldsymbol{v} \cdot (\nabla \ln w + \nabla \ln \gamma + \frac{1}{c^2} \nabla \Phi) = 0.$$
⁽²²⁾

Taking into account the form for the gravitational potential Φ (equation 4), we rewrite the last expression as

$$\boldsymbol{v}\cdot\nabla\gamma\ w\left(1-\frac{R_{\rm g}}{r}\right)^{1/2}=0.$$

We thus obtain the following result. Along the flow lines, the following value is conserved:

$$m_o c^2 w \gamma \left(1 - \frac{R_g}{r}\right)^{1/2} = m_o c^2 w \frac{\left(1 - \frac{R_g}{r}\right)^{1/2}}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} = const.$$

This is a relativistic Bernoulli equation, written for the case of the Schwarzschild metric.

An elegant derivation of the relativistic Bernoulli equation, performed taking into account the properties of the Killing vector field, can be found in Gourgoulhon (2006, 2007).

7 SUMMARY

The BH gravitation causes the curvature of space around it. A logarithmic potential can be introduced to describe the motion of particles in such gravitational landscape. In contrast with pseudo-Newtonian potentials, which can give only approximate results, the logarithmic potential provides the exact laws of motion. For this, we consider the logarithmic potential within a different approach, which represents the 3+1 decomposition of the Schwarzschild space–time near a BH. The advantage of such an approach for GR problems is that it allows using the physical concepts analogous to those in the classical physics.

In particular, the energy of a particle can be derived from the equation of motion using the logarithmic potential. We show that the derived velocity of a particle, physically measured by a local observer, is correct in the sense that it is identical to that in GR. The relativistic Bernoulli equation for a fluid in the Schwarzschild metric is obtained.

The choice of a potential and a method to deal with it depends on a desired accuracy of a problem. For considerations, which are not very precise, one can use the classical Newtonian mechanics and the Paczynski–Wiita's potential. It is not advised to use the logarithmic potential in the framework of the classical mechanics, since it gives less accurate results comparing to those obtained with the Paczynski–Wiita's potential (see discussion at the end of Section 5).

One can also use the potential approach in the framework of classical mechanics to approximate the motion of a particle in the Kerr metric, using a more sophisticated formula for a potential (see, for example, Artemova et al. 1996; Kato et al. 2008, where index β is introduced). However, the exact consideration of a particle motion in the Kerr metric implies the existence of a gravitomagnetic force, which is analogous to the Lorentz force in the electromagnetic theory and which is not conservative, that is, it cannot be determined by a potential (see, for example, Thorne et al. 1986, equations 3–18 and 3–19 abc). The exact force in the Kerr metric can be written out in the context of problem 1 of paragraph 88 in Landau & Lifshitz (1975) (see equation 3 there). This task could be a subject of another study.

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