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Electrostatic equilibrium of two spherical charged masses in general relativity

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Abstract. Approximate solutions representing the gravitational–electrostatic balance of two arbitrary point sources in general relativity have led to contradictory arguments in the literature with respect to the condition of balance. Up to the present time, the only known exact solutions which can be interpreted as the nonlinear superposition of two spherically symmetric (Reissner–Nordström) bodies without an intervening strut have been for critically charged masses, $M_i^2 = Q_i^2$. In the present paper, an exact electrostatic solution of the Einstein–Maxwell equations representing the exterior field of two arbitrary charged Reissner–Nordström bodies in equilibrium is studied. The invariant physical charge for each source is found by direct integration of Maxwell's equations. The physical mass for each source is defined invariantly in a manner similar to the way in which the charge was found. It is shown through numerical methods that balance without tension or strut can occur for non-critically charged bodies. It is demonstrated that other authors have not identified the correct physical parameters for the mass and charge of the sources. Further properties of the solution, including the multipole structure and comparison with other parametrizations, are examined.

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1. Introduction

In a recent paper by Bonnor [1], the equilibrium conditions for a charged test particle in the field of a spherically symmetric charged mass (Reissner–Nordström solution) were investigated. He found that the classical condition for equilibrium

$$M_1 M_2 = Q_1 Q_2 \tag{1.1}$$

for which the separation between the particles is arbitrary, was neither necessary nor sufficient for electrostatic balance of two spherical masses. This is in conflict with the earlier results of Barker and O'Connell [2] and Kimura and Ohta [3] who used different approximation methods. Barker and O'Connell claimed that in the post-Newtonian approximation, the equation

$$(M_1 Q_2 - M_2 Q_1)(Q_1 - Q_2) = 0 \tag{1.2}$$

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had to be satisfied in addition to (1.1). Kimura and Ohta claimed that in the post-post-Newtonian approximation, the necessary and sufficient condition for balance is that each mass must be 'critically' charged,

$$M_i = |Q_i| \qquad i = 1, 2 \tag{1.3}$$

and balance can occur for arbitrary separation of the sources. Up to the present time, the problem of gravitational–electrostatic balance of two spherical bodies in general relativity without an intervening Weyl line singularity (strut or tension) has been solved *exactly* only for critically charged masses [4–6]. A balanced solution was originally thought to have been found [7] within the Herlt class for both sources having $M_i > |Q_i|$, but it was subsequently shown that the intervening line singularity could not be removed [8]. Kramer [9] presented an exact solution for the electrostatic counterpart of the double Kerr–NUT solution with zero spin parameter. He found that condition (1.1) holds for electrostatic balance. However, he stated that his solution cannot be interpreted as the nonlinear superposition of two Reissner–Nordström solutions and thus the masses are not spherically symmetric.

In the present paper, an exact electrostatic solution of the Einstein–Maxwell equations representing the exterior field of two arbitrary charged nonlinearly superposed Reissner–Nordström sources in equilibrium is given. It is obtained with the aid of Sibgatullin's [10] method for constructing the complex Ernst potentials [11]. It is mathematically equivalent to the solutions of Manko *et al* [12] and Chamorro *et al* [13], henceforth referred to as papers I and II, respectively (with their spin parameters set to zero) and they are all special cases of the general mathematical solution given by Ernst [14]. It is of primary importance that the parameters in the solution be related to a *physical* set of parameters in order for any subsequent analysis of the solution to have any significant physical meaning. For a physical set of parameters, one would prefer to use the individual masses and charges of each source and the distance between the sources. The invariant charge enclosed by a spacelike hypersurface can be found by the direct integration of Maxwell's equations. For spacetimes with a timelike Killing vector, a conserved quantity which can be interpreted as the contribution to the total mass from each body can be invariantly defined in analogy with the charge (see, for example, [9, 15, 16]). This paper follows Kramer [9] for the definition of the individual mass of each body. In section 2, the integrals of charge and mass are given and they are applied to the Weyl-class solution for two Reissner–Nordström bodies in section 3. Section 4 presents the solution for a parametrization of the non-Weyl-class double Reissner–Nordström solution based on the Weyl-class parametrization. These are then compared to the parametrizations proposed in papers I and II. It is shown that the parametrizations employed in papers I and II do not represent the physical masses or charges of the individual sources even in the Weyl-class limit (except for the special case of identical bodies in paper I). Due to the complexities of the parametrization, a rendering of the solution in terms of the individual masses and charges as given in section 2 has not yet been accomplished. However, numerical analysis of the physical masses and charges is possible for a given set of parameters. In section 5, balance without a strut or tension for numerical values of the physical mass and charge is examined. It is found that there are balance conditions for which neither body is critically charged and the Newtonian balance condition does not hold. This is in accordance with Bonnor's [1] test particle analysis. The dependence of the balance condition on the separation of the bodies is not yet known. A discussion of the results and conclusions are given in sections 6 and 7.

2. Mass and charge

For a static axially symmetric spacetime, the mass M_i and charge Q_i of a source inside a closed 2-surface σ_i are given by the integrals† [9]

$$M_i \equiv -\frac{1}{8\pi}\oint_{\sigma_i} K^{ab}\sqrt{-g}\,\mathrm{d}f^*_{ab} \tag{2.1}$$

$$Q_i = -\frac{1}{8\pi}\oint_{\sigma_i} F^{ab}\sqrt{-g}\,\mathrm{d}f^*_{ab} \tag{2.2}$$

where

$$K^{ab} \equiv \xi^{a;b} + \Phi F^{ab}. \tag{2.3}$$

The timelike Killing vector is ξ^a, F_{ab} is the electromagnetic field tensor, Φ is the electrostatic potential, g is the determinant of the metric and $\mathrm{d}f^*_{ab}$ is the dual to the surface element 2-form $\mathrm{d}f^{ab}$,

$$\mathrm{d}f^*_{ab} = \tfrac{1}{2}e_{abcd}\,\mathrm{d}f^{cd} \tag{2.4}$$

(here e_{abcd} is the flat space Levi-Civita permutation symbol). The above integral conservation laws follow from the local conservation laws

$$F^{ab}{}_{;b;a} = 0 \qquad K^{ab}{}_{;b;a} = 0, \tag{2.5}$$

the first, following from the conservation of charge and the second from the existence of the timelike Killing vector ξ^a and the restriction to a static axially symmetric spacetime metric. Since the Einstein–Maxwell equations also imply

$$F^{ab}{}_{;b} = 0 \qquad K^{ab}{}_{;b} = 0, \tag{2.6}$$

in a source-free region, any deformation of the surface σ_i in the electrovacuum region outside the sources does not change the values of the integrals M_i and Q_i.

3. The Weyl-class two-body solution

To investigate the structure of spacetimes with two sources, the Weyl-class double Reissner–Nordström solution provides a suitable yet mathematically uncumbersome framework from which to proceed. The solution is easily found through the method presented in [4]. The metric for a static axially symmetric spacetime can be written in the canonical form

$$\mathrm{d}s^2 = \mathrm{e}^w\,\mathrm{d}t^2 - \mathrm{e}^{v-w}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) - \rho^2\mathrm{e}^{-w}\,\mathrm{d}\phi^2, \tag{3.1}$$

where w and v are functions of the cylindrical coordinates ρ and z. The Weyl-class solutions are characterized by the metric function w which is a function of the electrostatic potential, i.e. $w = w(\Phi)$ so that the gravitational and electrostatic equipotential surfaces overlap. For asymptotically flat boundary conditions, the unique functional relationship between e^w and Φ is [18]

$$\mathrm{e}^w = 1 - 2\frac{m_\mathrm{T}}{q_\mathrm{T}}\Phi + \Phi^2, \tag{3.2}$$

where Φ is the electrostatic potential and m_T and q_T are the total mass and charge, respectively. The solution representing two 'undercharged' ($M_i > |Q_i|$) Reissner–Nordström bodies (or 'black holes') is given by

$$\Phi = a\frac{f-1}{a^2 f - 1}, \tag{3.3}$$

† Notations and conventions used are those of [17].

where

$$f = \left(\frac{R_1 + R_2 - 2l_1}{R_1 + R_2 + 2l_1}\right)\left(\frac{R_3 + R_4 - 2l_2}{R_3 + R_4 + 2l_2}\right), \tag{3.4}$$

$$R_1^2 \equiv (z - d - 2l_1)^2 + \rho^2, \tag{3.5}$$

$$R_2^2 \equiv (z - d)^2 + \rho^2, \tag{3.6}$$

$$R_3^2 \equiv (z + d)^2 + \rho^2, \tag{3.7}$$

$$R_4^2 \equiv (z + d + 2l_2)^2 + \rho^2. \tag{3.8}$$

The constant parameters $2d$ and $2l_1, 2l_2$ are the coordinate distance between the horizons and the 'lengths' of the horizons (Weyl 'rods'), respectively (see figure 1). The parameter a is defined through the equation

$$\frac{1 + a^2}{a} = \frac{2m_{\mathrm{T}}}{q_{\mathrm{T}}}. \tag{3.9}$$

The metric function e^w is found through equation (3.2). The metric function e^v is

$$e^v = \frac{(R_1 + R_2)^2 - 4l_1^2}{4R_1R_2}\,\frac{(R_3 + R_4)^2 - 4l_2^2}{4R_3R_4}\left[\frac{((l_1 + l_2 + d)\,R_1 + (l_2 + d)\,R_2 - l_1R_4)\,d}{((l_1 + d)\,R_1 + R_2d - l_1R_3)\,(l_2 + d)}\right]^2.$$

Choosing the surface σ_1 to encompass body 1 and the surface σ_2 to encompass body 2 of figure 1, the mass and charge integrals of equations (2.1) and (2.2) yield

$$\begin{aligned} M_1 &= \frac{1 + a^2}{1 - a^2}l_1 \qquad & Q_1 &= \frac{2a}{1 - a^2}l_1 \\ M_2 &= \frac{1 + a^2}{1 - a^2}l_2 \qquad & Q_2 &= \frac{2a}{1 - a^2}l_2. \end{aligned} \tag{3.10}$$

The above form of the individual mass and charge for each Reissner–Nordström body is similar to the form proposed in [4] for the mass and charge decomposition of two charged Curzon particles. It was stated in [7] that the conjectured charge decomposition for both the double Reissner–Nordström and Curzon cases were verified by direct calculation through

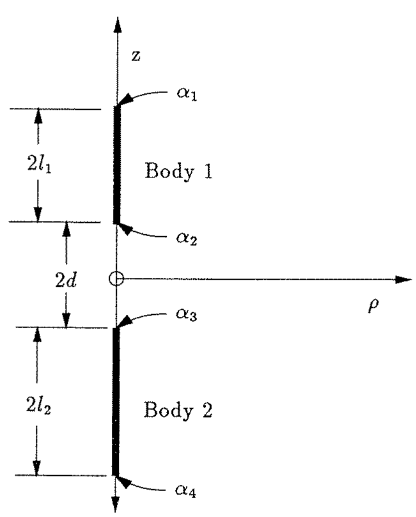

Figure 1. Schematic of two Reissner–Nordström black holes in cylindrical coordinates. The thick lines are the Weyl 'rods' which show the locations of the event horizon surfaces.

equation (2.2). It is straightforward to verify that equation (2.1) yields the conjectured mass decomposition for the double Curzon solution. Because of the functional relationship between the gravitational potential and the electrostatic potential, not all of the parameters M_1, M_2, Q_1, Q_2 are independent. Thus the Weyl class is also characterized by the constraint

$$M_1 Q_2 = M_2 Q_1. \tag{3.11}$$

Removal of the line singularity between the bodies yields equation (1.1) as an additional condition on the parameters. As a result, the parameters also satisfy equation (1.3). Thus equilibrium without a strut or tension occurs for 'critically' charged sources and this balance is found to be independent of the separation distance [4].

4. Non-Weyl parametrizations

Generalizing the Weyl-class double Reissner–Nordström solution to the case in which the gravitational and electrostatic equipotential surfaces no longer overlap has usually been attempted through the means of generating techniques (see, for example, [6, 7, 9]). In these techniques, new solutions are generated from old ones rather than by solving the equations directly. Recently, considerable interest has focused upon a method [10] which constructs the Ernst potentials [11] from initial data on the symmetry axis. The complex Ernst potentials $\mathcal{E}(\rho, z)$ and $\Psi(\rho, z)$ of all stationary axisymmetric electrovacuum spacetimes with axis data of the form

$$\mathcal{E}(z, \rho = 0) = \frac{U - W}{U + W}, \qquad \Psi(z, \rho = 0) = \frac{V}{U + W}, \tag{4.1}$$

where

$$U = z^2 + U_1 z + U_2 \tag{4.2}$$

$$V = V_1 z + V_2 \tag{4.3}$$

$$W = W_1 z + W_2 \tag{4.4}$$

and $U_1, U_2, V_1, V_2, W_1, W_2$ are complex constants, have been found [14]. However, a mathematical solution to the Einstein–Maxwell field equations does not imply a well understood physical interpretation of the solution. Sibgatullin's method of constructing the Ernst potentials aids in obtaining the physically meaningful parametrization which is sought for the two-body case in question.

In Sibgatullin's method, it is required that the Ernst potentials along the z-axis be specified. Our choice was [19, 20]

$$\begin{aligned} \mathcal{E}(\rho = 0, z) \equiv e(z) &= 1 - \frac{2(m_1(z + z_2) + m_2(z + z_1))}{(z + z_1 + m_1)(z + z_2 + m_2) - q_1 q_2}, \\ \Psi(\rho = 0, z) \equiv F(z) &= \frac{q_1(z + z_2) + q_2(z + z_1)}{(z + z_1 + m_1)(z + z_2 + m_2) - q_1 q_2}. \end{aligned} \tag{4.5}$$

It has the form of the Weyl-class double Reissner–Nordström axis data. If the additional Weyl-class constraint

$$m_1 q_2 - m_2 q_1 = 0 \tag{4.6}$$

is placed on the functions $e(z)$ and $F(z)$, then Sibgatullin's method yields the Weyl-class double Reissner–Nordström solution (in an alternate form to [4]) and the parameters m_1, m_2, q_1, q_2 are the physical masses and charges as defined by (2.1) and (2.2) (i.e.

$M_1 = m_1, Q_1 = q_1, M_2 = m_2, Q_2 = q_2$). For the solution of two Weyl-class Reissner–Nordström black holes (given in section 3), figure 1 shows the coordinate positions of the centres of the 'rods' as $d + l_1$ for body 1 and $-d - l_2$ for body 2. The parameters z_1, z_2 identify the negative of the coordinate positions of the centres of the 'rods', i.e.

$$z_1 = -d - l_1, \qquad z_2 = d + l_2.$$

If condition (4.6) is not imposed, $w \neq w(\Phi)$, i.e. the gravitational and electrostatic equipotential surfaces no longer overlap. In section 5 it will be shown that the parameters m_1, m_2, q_1, q_2 then no longer carry the suggested physical meaning and the parameters z_1, z_2 no longer coincide with the centres of the 'rods' when the Weyl-class constraint (4.6) is not imposed.

The full Ernst potentials $\mathcal{E}(\rho, z)$ and $\Psi(\rho, z)$ for the axis data of equation (4.5), expressed in terms of the cylindrical coordinates (ρ, z), are found to be (the details of the method can be found in [10, 12, 21] and in the review article [22])

$$\mathcal{E} = \frac{A - B}{A + B}, \qquad \Psi = \frac{C}{A + B}, \tag{4.7}$$

where

$$A \equiv \sum_{i<j}^{4} a_{ij} r_i r_j, \qquad B \equiv \sum_{i=1}^{4} b_i r_i, \qquad C \equiv \sum_{i=1}^{4} c_i r_i,$$
$$r_n \equiv \sqrt{\rho^2 + (z - \alpha_n)^2}, \qquad (n = 1 \to 4).$$

The constants α_n in equation (4.7) are the roots of the equation

$$e(z) + [F(z)]^2 = 0 \tag{4.8}$$

and can only be real or complex conjugate pairs. The remaining constants a_{ij}, b_i and c_i are defined as follows:

$$a_{ij} \equiv (-1)^{i+j+1} s_i s_j t_i t_j (s_i t_j - s_j t_i) \begin{vmatrix} s_k v_k & s_l v_l \\ t_k u_k & t_l u_l \end{vmatrix},$$
$$(i < j;\ k < l;\ k, l \neq i, j;\ i, k = 1 \to 3;\ j, l = 2 \to 4);$$

$$b_i \equiv (-1)^i s_i t_i (s_i - t_i) \begin{vmatrix} s_k^2 t_k^2 & s_l^2 t_l^2 & s_m^2 t_m^2 \\ s_k v_k & s_l v_l & s_m v_m \\ t_k u_k & t_l u_l & t_m u_m \end{vmatrix},$$
$$(k < l < m;\ k, l, m \neq i;\ i = 1 \to 4;\ k = 1, 2;\ l = 2, 3;\ m = 3, 4);$$

$$c_i \equiv (-1)^{i+1} s_i t_i (s_i - t_i)(K_3 G_i + K_4 H_i),$$

$$G_i \equiv \begin{vmatrix} s_k t_k^2 & s_l t_l^2 & s_m t_m^2 \\ s_k v_k & s_l v_l & s_m v_m \\ t_k u_k & t_l u_l & t_m u_m \end{vmatrix}, \qquad H_i \equiv \begin{vmatrix} s_k^2 t_k & s_l^2 t_l & s_m^2 t_m \\ s_k v_k & s_l v_l & s_m v_m \\ t_k u_k & t_l u_l & t_m u_m \end{vmatrix}, \tag{4.9}$$
$$(k < l < m;\ k, l, m \neq i;\ i = 1 \to 4;\ k = 1, 2;\ l = 2, 3;\ m = 3, 4);$$

$$s_i \equiv \beta_1 - \alpha_i, \qquad t_i \equiv \beta_2 - \alpha_i,$$

$$u_i \equiv K_1 s_i t_i + K_3^2 t_i + K_3 K_4 s_i, \qquad v_i \equiv K_2 s_i t_i + K_4^2 s_i + K_3 K_4 t_i,$$

$$K_1 \equiv \frac{m_1 z_2 + m_2 z_1 + (m_1 + m_2)\beta_1}{\beta_1 - \beta_2}, \qquad K_2 \equiv \frac{m_1 z_2 + m_2 z_1 + (m_1 + m_2)\beta_2}{\beta_2 - \beta_1},$$

$$K_3 \equiv \frac{q_1 z_2 + q_2 z_1 + (q_1 + q_2)\beta_1}{\beta_1 - \beta_2}, \qquad K_4 \equiv \frac{q_1 z_2 + q_2 z_1 + (q_1 + q_2)\beta_2}{\beta_2 - \beta_1},$$

$$\beta_1 \equiv -\tfrac{1}{2}\left(z_1 + m_1 + z_2 + m_2 - \sqrt{(z_1 - z_2 + m_1 - m_2)^2 + 4q_1q_2}\right),$$

$$\beta_2 \equiv -\tfrac{1}{2}\left(z_1 + m_1 + z_2 + m_2 + \sqrt{(z_1 - z_2 + m_1 - m_2)^2 + 4q_1q_2}\right),$$

where all of the subsequent quantities introduced are constants ultimately defined in terms of m_i, q_i, z_i, $i = 1, 2$, which specify the character and locations of the sources in the Weyl-class limit only.

The expressions for $\mathcal{E}$ and Ψ are in Kinnersley's [23] form and this permits one to write the corresponding metric functions as

$$\mathrm{e}^{w} = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{(A + B)(\bar{A} + \bar{B})}, \qquad \mathrm{e}^{v} = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{K_0 r_1 r_2 r_3 r_4}, \tag{4.10}$$

where

$$K_0 = \left(\sum_{i<j}^{4} a_{ij}\right)\left(\sum_{i<j}^{4} \bar{a}_{ij}\right) \tag{4.11}$$

and a bar denotes complex conjugation. For a static metric, the electrostatic potential Φ is equal to the Ernst potential Ψ and this completes the solution.

With the knowledge of the full Ernst potentials and the metric functions, the next step would be to evaluate the true mass and charge integrals in terms of the parameters $m_1, m_2, q_1, q_2, z_1, z_2$. It is to be stressed that outside of the Weyl class, these parameters no longer carry the suggested physical meaning. For the metric (3.1), the integrals (2.1) and (2.2) can be written as relations in flat 3-space ($i = 1, 2$):

$$M_i = \frac{1}{8\pi} \oint_{\sigma_i} \mathrm{e}^{-w} \mathcal{E}_{,\alpha} n^{\alpha} \, \mathrm{d}A \tag{4.12}$$

$$Q_i = -\frac{1}{4\pi} \oint_{\sigma_i} \mathrm{e}^{-w} \Phi_{,\alpha} n^{\alpha} \, \mathrm{d}A, \tag{4.13}$$

where n^{α} (α runs from 1 to 3) is the unit vector orthogonal to the surface and $\mathrm{d}A$ denotes the invariant (flat) surface element (see also [9] and references therein).

We can extend the Weyl-class definitions of the coordinate positions of the bodies to the non-Weyl-class solution. There are three distinct types of sources of interest. They are characterized by the transition between a source with an event horizon to one without an event horizon. As mentioned previously, the constants α_n, $n = 1 \to 4$ in equation (4.7) are either real or complex conjugate pairs. By definition, we choose $\alpha_1 \geqslant \alpha_2 > \alpha_3 \geqslant \alpha_4$. A Reissner–Nordström 'black hole' is characterized by real pairs of α_n. Figure 1 shows that in the Weyl canonical coordinate system α_n indicates the end points of a 'Weyl rod', which itself is the event horizon surface. A 'superextreme' object [12] or 'naked singularity' is characterized by a complex conjugate pair of α_n. Body 2 of figure 2 illustrates the manifestation of a 'superextreme' body in the spacetime. An 'extreme' object, for example, would be characterized by real α_n for which $\alpha_1 = \alpha_2$. Therefore we have the following definitions for the coordinate positions of the sources:

(i) For a Reissner–Nordström 'black hole', we define $-Z_i$ to be the coordinate position of the centre of the 'Weyl rod'. For example, the coordinate position of body 1 of figure 2 is

$$-Z_1 = \tfrac{1}{2}\left(\alpha_1 + \alpha_2\right).$$

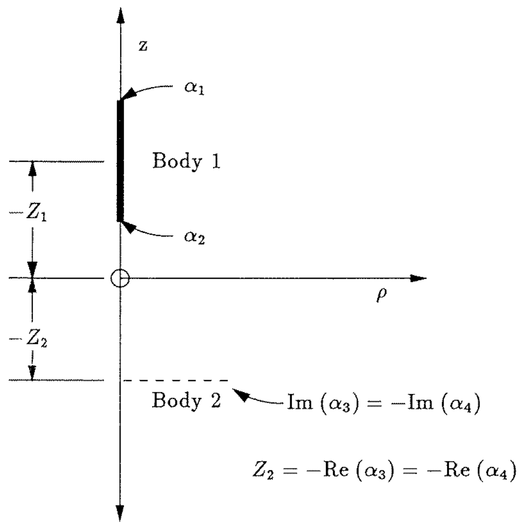

Figure 2. Schematic of a Reissner–Nordström black hole and a Reissner–Nordström superextreme body. The dotted line is a 'complex Weyl rod'. The intersection of the 'rod' with the z-axis is defined as the coordinate position of body 2.

(ii) For a 'superextreme' object, we define $-Z_i$ to be the coordinate position of the real part of α_n. For example, body 2 of figure 2 is a 'superextreme' object. Therefore its coordinate position is

$$-Z_2 = \mathrm{Re}(\alpha_3) = \mathrm{Re}(\alpha_4).$$

(One could consider the imaginary part of α_n as the end points of a 'complex Weyl rod' with the coordinate position of this 'complex rod' being defined as its intersection with the real axis (z-axis).)

(iii) For an 'extreme object', we define $-Z_i$ to be the coordinate position of the point locating the zero 'length' Weyl 'rod'. For example, if body 1 was an 'extreme' object, then $\alpha_1 = \alpha_2$ and $-Z_1 = \alpha_1$.

We also define

$$\mathrm{Re}(\alpha_2) > \mathrm{Re}(\alpha_3) \tag{4.14}$$

as the condition for having two separated bodies irrespective of the type of object.

With the above integrals and the coordinate positions as defined above evaluated in terms of $m_1, m_2, q_1, q_2, z_1, z_2$, it would then be possible, in principle, to invert these equations and hence write the solution (4.7)–(4.9) in terms of the true physical parameters M_i, Q_i and the coordinate positions Z_i, $i = 1, 2$. Ideally, the coordinate positions of the sources should be replaced with the proper separation of the sources. The complexity of the above Ernst potentials makes the analytic evaluation of the integrals (4.12), (4.13) and the proper separation difficult. Consequently, this goal has not yet been achieved. However, it is possible to numerically integrate equations (2.1) and (2.2) for a given set $\{m_1, m_2, q_1, q_2, z_1, z_2\}$. This will prove to be useful in studying balance conditions without a strut in section 5.

Although the numerical evaluation of the physical mass and charge can be achieved from the parametrizations of paper I or II, it was hoped that the parametrization proposed in this paper, based on the Weyl-class solution, would facilitate the analytic evaluation of

Table 1. The values of the parameters in the parametrizations of papers I and II is shown given the Weyl-class values.

Weyl class	Paper I	Paper II
$m_1 = 8$	$\tilde{m}_1 = 14.17$	$\hat{m}_1 = 3.58$
$q_1 = 8$	$\tilde{q}_1 = 14.17$	$\hat{q}_1 = 3.58$
$m_2 = 3$	$\tilde{m}_2 = -3.17$	$\hat{m}_2 = 7.42$
$q_2 = 3$	$\tilde{q}_2 = -3.17$	$\hat{q}_2 = 7.42$
$z_1 = -7$	$\tilde{z}_1 = -2.02$	$\hat{z}_1 = -4.7$
$z_2 = 7$	$\tilde{z}_2 = 2.02$	$\hat{z}_2 = 4.7$

the integrals. It is not difficult to show that the parametrizations in papers I or II do not correctly identify the individual masses, charges of each source. We stated earlier that our parametrization $\{m_1, m_2, q_1, q_2, z_1, z_2\}$ only represents the physical masses and charges and coordinate positions of each source when the Weyl-class condition (equation (3.2) or (4.6)) is imposed (i.e. $\{M_1 = m_1,\ M_2 = m_2,\ Q_1 = q_1,\ Q_2 = q_2,\ Z_1 = z_1,\ Z_2 = z_2\}$). We can best demonstrate the problems with the parametrizations of papers I and II by comparing the representation of a properly parametrized Weyl-class solution with each of the other parametrizations. Let the set $\{m_1, m_2, q_1, q_2, z_1, z_2\}$ represent the physical Weyl-class parameters under the condition $m_1 q_2 = m_2 q_1$. Then the relationships between the three parametrizations are found by solving the set of equations (setting the spin parameters found in papers I and II to zero):

$$\begin{array}{ccccc} \text{Weyl class} & & \text{Paper I} & & \text{Paper II} \\ m_1 + m_2 & = & \tilde{m}_1 + \tilde{m}_2 & = & \hat{m}_1 + \hat{m}_2 \\ q_1 + q_2 & = & \tilde{q}_1 + \tilde{q}_2 & = & \hat{q}_1 + \hat{q}_2 \\ z_1 + z_2 & = & \tilde{z}_1 + \tilde{z}_2 & = & \hat{z}_1 + \hat{z}_2 \\ m_1 z_2 + m_2 z_1 & = & \tilde{m}_1 \tilde{z}_2 + \tilde{m}_2 \tilde{z}_1 & = & \hat{m}_1 \hat{z}_2 + \hat{m}_2 \hat{z}_1 + 2\hat{m}_1 \hat{m}_2 \\ q_1 z_2 + q_2 z_1 & = & \tilde{q}_1 \tilde{z}_2 + \tilde{q}_2 \tilde{z}_1 & = & \hat{q}_1 \hat{z}_2 + \hat{q}_2 \hat{z}_1 + \hat{q}_1 \hat{m}_2 + \hat{q}_2 \hat{m}_1 \\ z_1 z_2 + m_1 m_2 - q_1 q_2 & = & \tilde{z}_1 \tilde{z}_2 + \tilde{m}_1 \tilde{m}_2 & = & \hat{z}_1 \hat{z}_2 - \hat{m}_1 \hat{m}_2\ . \end{array} \tag{4.15}$$

The tilded and careted parameters are the parametrizations of papers I and II, respectively. Table 1 summarizes the results of solving the system (4.15) given the values shown in column 1. The solution represents two Weyl-class Reissner–Nordström ‘critically charged’ bodies without an intervening line singularity. It is clear that none of the parameter values in the latter two columns match the physical Weyl-class values. In fact, one has to assign negative values to $\tilde{m}_2, \tilde{q}_2$ in order to obtain a *positive* physical mass and charge for source 2. Thus, apart from one special case, neither paper I nor paper II parametrizations can be interpreted as the invariant physical parameters. The only exception is for identical bodies (with or without a line singularity) in the parametrization of paper I. In this very special case of the Weyl class, the parameters $\tilde{m}_1 = \tilde{m}_2,\ \tilde{q}_1 = \tilde{q}_2$ are the physical masses and charges. However, $\tilde{z}_1$ and $\tilde{z}_2$ do not identify the coordinate positions of the bodies as defined earlier. The paper II parametrization is not physical even for identical bodies.

It is the demand for the inclusion of the Weyl-class solution in [4] which led to our form of $e(z)$ and $F(z)$. It should be emphasized that our parametrization contains as a special case, the simplest clearly individually spherical two-body balance solution of two critically charged bodies. This can be best illustrated by examining the Simon [24, 25] relativistic multipole moments of each parametrization. The first five Simon relativistic mass and charge multipole moments for our parametrization are

$$\begin{aligned}
\mathcal{M}_0 &= m_1 + m_2, \\
\mathcal{M}_1 &= -m_1 z_1 - m_2 z_2, \\
\mathcal{M}_2 &= m_1 z_1^2 + m_2 z_2^2 - (m_1 m_2 - q_1 q_2)(m_1 + m_2), \\
\mathcal{M}_3 &= -m_1 z_1^3 - m_2 z_2^3 + (m_1 m_2 - q_1 q_2)(2m_1 z_1 + 2m_2 z_2 + z_1 m_2 + z_2 m_1), \\
\mathcal{M}_4 &= m_1 z_1^4 + m_2 z_2^4 - (m_1 m_2 - q_1 q_2)\big[\,(m_1 + m_2)(q_1 q_2 - m_1 m_2) \\
&\quad +2\left(m_1 z_1^2 + m_2 z_2^2\right) + (m_1 + m_2)(z_1 + z_2)^2 \\
&\quad +\tfrac{1}{7}(m_1 + m_2)\left((q_1 + q_2)^2 - (m_1 + m_2)^2\right)\big] \\
&\quad -\tfrac{1}{210}(z_1 - z_2)\big[16(z_1 - z_2)(m_1 + m_2)(m_1 q_2 - m_2 q_1)^2 \\
&\quad +z_1\left(30 m_1\left(m_1 m_2 + m_2^2 - q_2^2\right) - 3q_1(3m_2 q_1 + 7q_2 m_1)\right) \\
&\quad -z_2\left(30 m_2\left(m_1 m_2 + m_1^2 - q_1^2\right) - 3q_2(3m_1 q_2 + 7q_1 m_2)\right)\big]
\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}
\mathcal{Q}_0 &= q_1 + q_2, \\
\mathcal{Q}_1 &= -q_1 z_1 - q_2 z_2, \\
\mathcal{Q}_2 &= q_1 z_1^2 + q_2 z_2^2 - (m_1 m_2 - q_1 q_2)(q_1 + q_2), \\
\mathcal{Q}_3 &= -q_1 z_1^3 - q_2 z_2^3 + (m_1 m_2 - q_1 q_2)(2q_1 z_1 + 2q_2 z_2 + z_1 q_2 + z_2 q_1). \\
\mathcal{Q}_4 &= q_1 z_1^4 + q_2 z_2^4 - (m_1 m_2 - q_1 q_2)\big[\,(q_1 + q_2)(q_1 q_2 - m_1 m_2) \\
&\quad +2\left(q_1 z_1^2 + q_2 z_2^2\right) + (q_1 + q_2)(z_1 + z_2)^2 \\
&\quad +\tfrac{1}{7}(q_1 + q_2)\left((q_1 + q_2)^2 - (m_1 + m_2)^2\right)\big] \\
&\quad -\tfrac{1}{210}(z_1 - z_2)\big[16(z_1 - z_2)(q_1 + q_2)(m_1 q_2 - m_2 q_1)^2 \\
&\quad -z_1\left(30 q_2\left(q_1 q_2 - m_1 m_2 + q_1^2\right) - 3m_1(13m_1 q_2 - 3m_2 q_1)\right) \\
&\quad +z_2\left(30 q_1\left(q_1 q_2 - m_1 m_2 + q_1^2\right) - 3m_2(13m_2 q_1 - 3m_1 q_2)\right)\big]
\end{aligned} \tag{4.17}$$

respectively. In Newtonian physics, a system of two monopoles at positions z_1, z_2 has multipole moments

$$\mathcal{M}_n = m_1 z_1^n + m_2 z_2^n, \qquad \mathcal{Q}_n = q_1 z_1^n + q_2 z_2^n. \tag{4.18}$$

It is interesting to observe that this is also the relativistic multipole structure for two Weyl-class critically charged bodies, at least up to $\mathcal{M}_4$, $\mathcal{Q}_4$. There is an inherent asphericity imposed upon each, since the two bodies are interacting in a line. For nonlinearly interacting sources in a line, one would not expect to realize perfect sphericity of the individual sources. (It is yet to be explained why the sphericity is maintained in the Weyl class, at least up to $\mathcal{M}_4$, $\mathcal{Q}_4$.) Once the solution is written analytically in terms of the physically meaningful constants M_i, Q_i and the coordinate positions Z_i, $i = 1, 2$, one will be able to examine the general multipole structure of nonlinearly interacting spherical bodies.

For comparison, the first four Simon relativistic mass and charge multipole moments for the parametrization of paper I (with their spin parameters $a_i = 0, i = 1, 2$) are

$$\begin{aligned}
\mathcal{M}_0 &= \tilde{m}_1 + \tilde{m}_2,\\
\mathcal{M}_1 &= -\tilde{m}_1\tilde{z}_1 - \tilde{m}_2\tilde{z}_2,\\
\mathcal{M}_2 &= \tilde{m}_1\tilde{z}_1^2 + \tilde{m}_2\tilde{z}_2^2 - \tilde{m}_1\tilde{m}_2(\tilde{m}_1 + \tilde{m}_2),\\
\mathcal{M}_3 &= -\tilde{m}_1\tilde{z}_1^3 - \tilde{m}_2\tilde{z}_2^3 + \tilde{m}_1\tilde{m}_2\,(2\tilde{m}_1\tilde{z}_1 + 2\tilde{m}_2\tilde{z}_2 + \tilde{z}_1\tilde{m}_2 + \tilde{z}_2\tilde{m}_1)
\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
\mathcal{Q}_0 &= \tilde{q}_1 + \tilde{q}_2,\\
\mathcal{Q}_1 &= -\tilde{q}_1\tilde{z}_1 - \tilde{q}_2\tilde{z}_2,\\
\mathcal{Q}_2 &= \tilde{q}_1\tilde{z}_1^2 + \tilde{q}_2\tilde{z}_2^2 - \tilde{m}_1\tilde{m}_2(\tilde{q}_1 + \tilde{q}_2),\\
\mathcal{Q}_3 &= -\tilde{q}_1\tilde{z}_1^3 - \tilde{q}_2\tilde{z}_2^3 + \tilde{m}_1\tilde{m}_2(2\tilde{q}_1\tilde{z}_1 + 2\tilde{q}_2\tilde{z}_2 + \tilde{z}_1\tilde{q}_2 + \tilde{z}_2\tilde{q}_1).
\end{aligned} \tag{4.20}$$

The first four Simon relativistic mass and charge multipole moments for the parametrization of paper II (with their spin parameters $a_i = 0,\ i = 1, 2$) are

$$\begin{aligned}
\mathcal{M}_0 &= \hat{m}_1 + \hat{m}_2,\\
\mathcal{M}_1 &= -\hat{m}_1\hat{z}_1 - \hat{m}_2\hat{z}_2 + 2\hat{m}_1\hat{m}_2,\\
\mathcal{M}_2 &= \hat{m}_1\hat{z}_1^2 + \hat{m}_2\hat{z}_2^2 + \hat{m}_1\hat{m}_2(\hat{m}_1 + \hat{m}_2 - 2\hat{z}_1 - 2\hat{z}_2),\\
\mathcal{M}_3 &= -\hat{m}_1\hat{z}_1^3 - \hat{m}_2\hat{z}_2^3 + \hat{m}_1\hat{m}_2\left(2\hat{m}_1\hat{m}_2 + 2\hat{z}_1\hat{z}_2 + 2\hat{z}_1^2 + 2\hat{z}_2^2\right.\\
&\qquad \left.-\hat{m}_1\hat{z}_2 - \hat{m}_2\hat{z}_1 - 2\hat{m}_1\hat{z}_1 - 2\hat{m}_2\hat{z}_2\right)
\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}
\mathcal{Q}_0 &= \hat{q}_1 + \hat{q}_2,\\
\mathcal{Q}_1 &= -\hat{q}_1\hat{z}_1 - \hat{q}_2\hat{z}_2 + \hat{m}_1\hat{q}_2 + \hat{m}_2\hat{q}_1,\\
\mathcal{Q}_2 &= \hat{q}_1\hat{z}_1^2 + \hat{q}_2\hat{z}_2^2 + \hat{m}_1\hat{m}_2\left(\hat{q}_1 + \hat{q}_2\right) - \left(\hat{q}_1\hat{m}_2 + \hat{q}_2\hat{m}_1\right)(\hat{z}_1 + \hat{z}_2),\\
\mathcal{Q}_3 &= -\hat{q}_1\hat{z}_1^3 - \hat{q}_2\hat{z}_2^3 - \hat{m}_1\hat{m}_2\left(2\hat{q}_1\hat{z}_1 + 2\hat{q}_2\hat{z}_2 + \hat{z}_1\hat{q}_2 + \hat{z}_2\hat{q}_1\right)\\
&\qquad + \left(\hat{q}_1\hat{m}_2 + \hat{q}_2\hat{m}_1\right)(\hat{m}_1\hat{m}_2 + \hat{z}_1\hat{z}_2 + \hat{z}_1^2 + \hat{z}_2^2).
\end{aligned} \tag{4.22}$$

If the above parametrizations did represent the physical mass and charge, it is evident that the multipole structure would not be that of Newtonian spherical bodies even for critically charged bodies. As stated earlier, it should be noted that in the parametrization of paper I, it can be shown that only in the case of identical bodies, the parameters $\tilde{m}_1 = \tilde{m}_2, \tilde{q}_1 = \tilde{q}_2$ are the physical mass and charge. However, in this case the multipoles still do not have the form of (4.18) since the parameters $\tilde{z}_1$ and $\tilde{z}_2$ do not identify the positions of the bodies as defined earlier. A simple transformation would correct the multipoles in this case.

5. The equilibrium condition

In order for the spacetime to be regular on the z-axis between the sources (removal of the Weyl line singularity or imposition of the condition for elementary flatness [26]), it is required that the metric function

$$\upsilon(z, \rho = 0) = 0 \tag{5.1}$$

between the sources. If the origin of the coordinate system is located between the sources (i.e. $\mathrm{Re}(\alpha_2) > 0$, $\mathrm{Re}(\alpha_3) < 0$), then application of equation (5.1), after some simplification, yields the balance equation

$$K \equiv \frac{a_{12}\,(\bar{a}_{13} + \bar{a}_{14}) + \bar{a}_{12}\,(a_{13} + a_{14})}{K_0} = 0. \tag{5.2}$$

Three cases were examined: (i) two Reissner–Nordström black holes, (ii) two Reissner–Nordström superextreme bodies and (iii) one black hole and one superextreme body.

The procedure for testing for equilibrium without an intervening strut or tension will be as follows:

(i) Assign numerical values to five of the six parameters from the unphysical set $\{m_1, m_2, q_1, q_2, z_1, z_2\}$.

(ii) Solve equation (5.2) for the unknown variable.

(iii) If a real root of equation (5.2) exists, then evaluate equations (2.1) and (2.2) to determine the physical mass and charge parameters.

The results for each of the three cases are as follows:

5.1. Two Reissner–Nordström black holes

Numerous sets of the parameters $\{m_1, m_2, q_1, q_2, z_1, z_2\}$, such that the constants α_n, $n = 1 \to 4$ are real, were investigated. No roots of equation (5.2) were found. For example, choosing $m_1 = 9.0$, $q_1 = 3.0$, $z_1 = -15.0$, $m_2 = 8.0$, $q_2 = 2.0$, no balance for $0 \leqslant z_2 \leqslant 10^{10}$ was found. These findings are consistent with other results [6, 8, 15] that two Reissner–Nordström black holes cannot be found in equilibrium without an intervening strut or tension.

5.2. Two Reissner–Nordström superextreme bodies

Numerous sets of the parameters $\{m_1, m_2, q_1, q_2, z_1, z_2\}$, such that the constants α_n, $n = 1 \to 4$ are complex conjugate pairs, were investigated. No roots of equation (5.2) were found. For example, in choosing $m_1 = 3.0$, $q_1 = 9.0$, $z_1 = -15.0$, $m_2 = 2.0$, $q_2 = 8.0$, no balance for $0 \leqslant z_2 \leqslant 10^{10}$ was found. These findings suggest that two Reissner–Nordström superextreme bodies cannot be found in equilibrium without a strut or tension.

5.3. One black hole and one superextreme body

The following three different cases were found for which equation (5.2) has a real root. Each case has the configuration illustrated in figure 2.

Case A. For $m_1 = 6.0$, $q_1 = 2.0$, $z_1 = -5.0$, $m_2 = -0.7$, $q_2 = 4.0$, balance at approximately $z_2 = 2.08$ was found†. The values of α_n are $\alpha_1 = 10.3$, $\alpha_2 = 1.74$, $\alpha_3 = -3.11 + \mathrm{i}4.30$, $\alpha_4 = -3.11 - \mathrm{i}4.30$. Using equations (4.12) and (4.13), the physical masses and charges are $M_1 = 3.95$, $Q_1 = -0.887$, $M_2 = 1.35$, $Q_2 = 6.89$. Using the definitions of coordinate positions described in section 4, it was found that $Z_1 = -6.03$ and $Z_2 = 3.11$. Thus balance has occurred for $M_1 M_2 > Q_1 Q_2$, $Q_1 Q_2 < 0$ at a coordinate separation of

† In cases A–C, equation (5.2) has been solved to a precision of $|K| < 10^{-50}$ using highly refined values of z_2.

$\mathcal{S} \equiv Z_2 - Z_1 = 9.13$. Note that the parameter m_2 is negative but both physical masses are positive. The parametrizations of papers I and II yield, respectively,

Paper I	Paper II
$\tilde{m}_1 = 4.96$	$\hat{m}_1 = 4.36$
$\tilde{q}_1 = 2.31$	$\hat{q}_1 = -1.05$
$\tilde{m}_2 = 0.34$	$\hat{m}_2 = 0.94$
$\tilde{q}_2 = 3.69$	$\hat{q}_2 = 7.05$
$\tilde{z}_1 = -6.60$	$\hat{z}_1 = -6.00$
$\tilde{z}_2 = 3.68$	$\hat{z}_2 = 3.08$

which do not agree with the integrated values of equations (2.1) and (2.2). This demonstrates that, in general, none of the analytic parametrizations proposed, including our own, are suitable choices for the individual masses and charges of the sources.

Case B. For $m_1 = 9.0$, $q_1 = 3.0$, $z_1 = -40.0$, $m_2 = 2.5$, $q_2 = 8.0$, balance was found at approximately $z_2 = 34.6$. The values of α_n are $\alpha_1 = 48.4$, $\alpha_2 = 31.61$, $\alpha_3 = -34.62 + \mathrm{i}7.65$, $\alpha_4 = -34.62 - \mathrm{i}7.65$. The physical masses and charges are $M_1 = 8.87$, $Q_1 = 2.00$, $M_2 = 2.63$, $Q_2 = 9.00$. The coordinate positions are $-Z_1 = 40.01$, $-Z_2 = -34.6$ Thus balance has occurred for $M_1M_2 > Q_1Q_2,\ Q_1Q_2 > 0$ at a coordinate separation of $\mathcal{S} = 74.6$.

Case C. For $m_1 = 900.0$, $q_1 = 300.0$, $z_1 = -865.0$, $m_2 = 0.025$, $q_2 = 0.080$, balance was found at approximately $z_2 = 21.581$. The values of α_n are $\alpha_1 = 1713.5$, $\alpha_2 = 16.474$, $\alpha_3 = -21.582 + \mathrm{i}0.262\,26$, $\alpha_4 = -21.582 - \mathrm{i}0.262\,26$. The physical masses and charges are $M_1 = 899.71$, $Q_1 = 298.25$, $M_2 = 0.318\,97$, $Q_2 = 1.8254$. The coordinate positions are $-Z_1 = 865.00$, $-Z_2 = -21.582$ Thus balance has occurred for $Q_1Q_2 > M_1M_2,\ Q_1Q_2 > 0$ at a coordinate separation of $\mathcal{S} = 886.58$.

5.4. Comparison with test particle analysis

Bonnor's [1] examination of a test particle in the field of a Reissner–Nordström source yielded a wide variety of balance conditions. The following cases for separation-independent equilibrium were examined (note M, Q characterize the Reissner–Nordström spacetime and m, q are the test body parameters):

Case 1. For $q = \epsilon m,\ Q = \eta M,\ \epsilon, \eta = \pm 1$, balance occurs if $\epsilon = \eta$.

Case 2. If $m = |q|,\ M \neq |Q|$, or $m \neq |q|,\ M = |Q|$, no equilibrium is possible.

Case 3. If $mM = qQ$ but $m \neq |q|$, then no equilibrium is possible.

Since the exact solution under study contains the Weyl-class solution as a special case, we also find Bonnor's case 1 as a separation-independent equilibrium condition. Case 2 or 3 cannot be readily tested by our numerical procedure. In order to do so, one would have to have the good fortune of correctly choosing the set $\{m_1, m_2, q_1, q_2, z_1, z_2\}$ such that the physical masses and charges satisfy the given conditions (i.e. $M_1 = |Q_1|$ etc). Then to test the dependence on separation, one would need to choose a new set of unphysical parameters such that the proper separation changes while the physical masses and charges remain the same.

The following separation-dependent cases were also found in [1]:

Case 4. If $|Q| > M,\ mM = -qQ$ and $m^2 \neq q^2$ with $qQ < 0$, then an equilibrium exists at

$$r = \frac{Q^2}{2M}.$$

Case 5. If $|Q| > M,\ |q| < m,\ qQ < 0$ or

Case 6. If $|Q| > M,\ |q| < m,\ qQ > 0,\ qQ < mM$ then an equilibrium position exists at

$$r = \frac{Q^2\left(M\left(m^2 - q^2\right) + q\sqrt{\left(m^2 - q^2\right)\left(Q^2 - M^2\right)}\right)}{m^2M^2 - q^2Q^2}.$$

Case 7. If $|Q| < M,\ |q| > m,\ qQ > 0,\ qQ > mM$ then an equilibrium position exists at

$$r = \frac{Q^2\left(M\left(m^2 - q^2\right) - q\sqrt{\left(m^2 - q^2\right)\left(Q^2 - M^2\right)}\right)}{m^2M^2 - q^2Q^2}.$$

Thus we have found a direct correspondence between cases A–C of the exact solution and cases 5–7 of Bonnor's test particle analysis. The separation dependence of cases 4–7 cannot be studied in the exact solution using the present methods for the same reasons cases 2–3 cannot be studied. Since the separation dependence cannot be tested using the present methods, there is little value in numerically calculating the proper separation of the sources in cases A–C.

The physical parameters in case C could approximate a test body in a strong gravitational field. Using these values in case 7 and transforming from spherical coordinates to cylindrical coordinates for a single Reissner–Nordström body using the transformation (with $\theta = 0$)

$$z = (r - M)\cos\theta, \qquad \rho = \sqrt{r^2 - 2Mr + Q^2}\,\sin\theta. \tag{5.3}$$

Bonnor's method yields a coordinate separation of $\mathcal{S} = 1465.5$. Since the separation of the bodies from these two methods are not consistent, it would appear that case C does not sufficiently approximate a test body.

6. Discussion

The essential departure in the present paper from previous work is the attempt to parametrize the solution in terms of true physical constants of the spacetime. For a static axially symmetric solution of the Einstein–Maxwell equations, the integrals of equations (2.1) and (2.2) provide the invariant parameters required for meaningful analysis of the properties of the solution.

There are three cases of the exact solution which have not been examined. They are an extreme body with, respectively, a Reissner–Nordström black hole, a superextreme body and another extreme body for which the solution is not of the Weyl class. Knowledge of the solution analytically in terms of the physical parameters is required to analyse these cases adequately.

The terms 'undercharged', 'overcharged' and 'critically charged' are defined in [4] as follows ($i = 1, 2$):

$$M_i^2 > Q_i^2 \qquad \text{'undercharged'} \tag{6.1}$$

$$M_i^2 < Q_i^2 \qquad \text{'overcharged'} \tag{6.2}$$

$$M_i^2 = Q_i^2 \qquad \text{'critically charged'.} \tag{6.3}$$

For the Weyl class, the 'lengths' of the Weyl rods are $2l_i = 2\sqrt{M_i^2 - Q_i^2}$, $i = 1, 2$ [4]. If body 1 is 'critically charged'†, then $\alpha_1 = \alpha_2$ $(= d)$ since $l_1 = 0$ (see figure 1). This implies that the terminology 'critically charged' body and 'extreme' body may be used interchangeably for Weyl-class solutions. If body 1 is 'undercharged', α_1 $(= d + 2l_1)$ and α_2 $(= d)$ are real quantities. Thus 'undercharged body' and 'black hole' are synonymous terms in the Weyl class. Finally, if body 1 is 'overcharged', α_1 $(= d + l_1)$ and α_2 $(= d + \bar{l}_1)$ are complex conjugates. Thus the terms 'overcharged' and 'superextreme' are equivalent descriptions in the Weyl class. Unlike the Weyl-class solutions where the 'lengths' of the Weyl rods (real or complex) depend only upon the mass and charge of that source, it is strongly suggested from the analysis of section 5 that for the general (non-Weyl-class) solution, the 'lengths' of the rods also depend on the mass and charge of the other source and the distance separating the bodies as well. It would thus be possible to have a 'critically' charged body (according to equation (6.3)) for which the 'rod' is either of non-zero 'length' or 'complex'. This is important in terms of nomenclature for describing the physics of the spacetime. Since the transition of a pair (e.g. (α_1, α_2)) from real values to a complex conjugate pair in Sibgatullin's [10] method defines a differentiation of an object with a horizon to one without, it would seem that the appropriate description would be, respectively, a black hole (horizon), 'extreme' body (zero 'length' Weyl rod) and 'superextreme' body (no horizon or naked singularity) as described in paper I. The descriptions 'under', 'over' and 'critically' charged body should be reserved for the relations $M_i^2 > Q_i^2$, $M_i^2 < Q_i^2$ and $M_i^2 = Q_i^2$, respectively, between the individual masses and charges. This classification scheme would describe equilibrium conditions more precisely once all are identified. The appropriateness of such a scheme would become apparent when the analytic physical parametrization of the solution is known.

Bonnor's [1] test particle analysis has been modified [28] in such a way that the equilibrium conditions of a charged test particle in the field of a Kerr–Newman source can be studied. The generalization of the mathematically exact solution to two spinning sources (Kerr–Newman sources) is already known [12–14]. One is able to invariantly define angular momentum for a stationary spacetime in a manner similar to (2.1) and (2.2) because of the presence of a spacelike Killing vector (rotational symmetry) (see [15] and references therein for definitions of mass and angular momentum of stationary vacuum fields). It is unknown how the subsequent analysis of two identical spinning bodies in paper I based on the invariant definitions will affect their results, if at all. However, it is clear that the parametrization given is inadequate for the physical analysis of the general case (non-identical bodies).

† It should be noted that having identical roots $\alpha_1 = \alpha_2$ is not sufficient for identifying critically charged bodies even in the Weyl class. The Curzon particle is such an object with $\alpha_1 = \alpha_2$ but it is not necessarily critically charged (see [27, 4]).

7. Conclusions

The solution derived in papers I, II and this paper is a generalization of the Weyl-class double Reissner–Nordström solution. However, the analytic parametrizations presented in papers I, II and this paper cannot in all cases be interpreted as the true physical constants of the spacetime. The invariant physical charge for each source is found by direct integration of Maxwell's equations. The physical mass is invariantly defined [9] in a manner similar to which the charge was found. Numerical methods were used to evaluate the invariant individual masses and charges for the axially symmetric superposition of two Reissner–Nordström bodies. It was found that neither the Newtonian balance condition nor critically charged bodies are necessary for electrostatic equilibrium. The dependence of the balance condition on the separation of the bodies is not yet known due to the complexities involved in expressing the solution analytically in terms of the true physical set of parameters. However, all the balance conditions found are consistent with Bonnor's test particle analysis. This suggests that there exist equilibrium conditions which depend on the separation of the sources. The parametrization of this paper is manifestly physical in the Weyl-class limit.

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