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On Geometrodynamics and Null Fields*

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The possibility of describing null electromagnetic fields by purely metric concepts has recently been subject to some doubt. Following a method devised by Hlavatý, we here investigate the relations that a Riemannian manifold must satisfy in order to correspond to a null electromagnetic field. It is shown that in most cases the fulfilment of five geometrical relations is a necessary and sufficient condition for the existence of a null electromagnetic field. The latter is unique, except for an arbitrary constant phase factor (as in the case of non-null fields). However, in some exceptional cases, there is a larger degree of arbitrariness in the null electromagnetic field that corresponds to a given metric. Such fields (which always possess wave fronts) are not reducible to metric concepts. We then turn to examine how it can occur that null electromagnetic fields require the fulfilment of five relations, rather than three, as non-null ones. In order to settle this question, we make an attempt to consider null fields as a limiting case of non-null ones, by superimposing an arbitrary infinitesimal non-null field on a finite null one. It is then shown that the Rainich vector of such a field does not have a well defined limit, when the perturbing non-null field tends to zero. It is thereby inferred that null electromagnetic fields really have a special status within the frame of geometrodynamics.

Geometrodynamics is the description of gravitational and electromagnetic fields by purely metric concepts.

It was shown long ago by Rainich (*1*) that it is usually possible to eliminate the Maxwell tensor $F^{\mu\nu}$ from the Maxwell–Einstein equations, so as to describe the gravitational and electromagnetic fields by means of the symmetric tensor $g_{\mu\nu}$ only. However, Rainich's unified field theory lay dormant during more than 30 years, because the restriction to gravity and electromagnetism without other sources seemed too severe. Its recent revival is essentially due to Wheeler's conception of geons and wormholes (*2*): if proper account is taken of the space-time topology, then the theory is capable of producing idealized classical models of charged, massive particles constructed out of the fields (*3*). Recently, it has been shown that this can be generalized to the neutrino field (*4*) and to real and complex scalar fields (*5*). In this form, geometrodynamics provides a moderately

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rich idealization of classical physics, without appeal to phenomenological descriptions of unanalyzed elements.

Of course, the relevance of the theory to the actual physical world can only be investigated after its quantization. Now, in the usual procedure of quantizing field equations, the infinite plane monochromatic waves play an essential role. It is well known that this concept already leads to some difficulties in pure gravitational theory (*6*). In geometrodynamics, the situation seems even worse, since its usual form cannot be applied at all to null fields (*1*, *3*).

Null fields, however, have recently been the subject of much attention (*7*) and the purpose of this paper is to investigate whether it is possible to generalize geometrodynamics so as to include in it null electromagnetic fields. We here follow a method devised by Hlavatý (*8*), and show that in most cases the fulfilment of five geometrical relations is a necessary and sufficient condition for the existence of a null electromagnetic field. The latter is unique, except for an arbitrary constant phase factor (as in the case of non-null fields).

There remains however one exceptional case where the electromagnetic field is *not* determined locally by the geometry, namely, when the electromagnetic field is not only null (and consequently defines a propagation vector L_μ) but also possesses wave fronts in the sense that $L_\mu = \lambda \partial U/\partial x^\mu$.

The outline of this paper is as follows: after a review of the original Rainich problem (in Section 1), we describe the necessary algebraic apparatus for the study of the null case (Section 2), and complete the analysis of the null case with a restatement of the Maxwell–Einstein equations in geometric form, when this is possible (Section 3). The exceptional case referred to above is discussed in Section 4, and the results summarized and compared with the non-null case in Section 5. Finally, Sections 6 and 7 deal with "almost null" fields, and explain why it is not possible to deal with null fields as a limiting case of non-null ones.

1. INTRODUCTION TO THE RAINICH PROBLEM

From the mathematical point of view, the Rainich problem consists in the elimination of the Maxwell tensor $F^{\mu\nu}$ from the Maxwell–Einstein equations[1]

$$F^{\mu\nu}{}_{\nu} = 0, \tag{1}$$

$$\epsilon^{\alpha\lambda\mu\nu} F_{\lambda\mu\nu} = 0, \tag{2}$$

$$R_\mu{}^\nu - \tfrac{1}{2}\delta_\mu{}^\nu R = -F_{\mu\lambda}F^{\lambda\nu} + \tfrac{1}{4}\delta_\mu{}^\nu F_{\alpha\beta}F^{\beta\alpha}. \tag{3}$$

[1] Greek indices run from 0 to 3. The signature of the metric $g_{\mu\nu}$ is taken as $(+---)$, unless otherwise stated. An index placed after an already defined symbol means *covariant* differentiation. The pseudotensor $\epsilon^{\alpha\beta\gamma\delta}$ is defined by $\epsilon^{\alpha\beta\gamma\delta} = +1$ if $\alpha\beta\gamma\delta$ is an even permutation of 0123, $\epsilon^{\alpha\beta\gamma\delta} = -1$ if $\alpha\beta\gamma\delta$ is an odd permutation of 0123, and $\epsilon^{\alpha\beta\gamma\delta} = 0$ if any two indices are equal. Throughout this paper, we shall use natural units: $c = 1$ and $8\pi G = 1$.

The investigation of these equations is greatly simplified by the introduction of the tensor

$$H_{\mu\nu} = 2^{-1/2}(F_{\mu\nu} + \tfrac{1}{2}g^{1/2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}), \tag{4}$$

where g is the determinant of the $g_{\mu\nu}$. Since $g^{1/2}$ is purely imaginary, one has

$$F_{\mu\nu} = 2^{-1/2}(H_{\mu\nu} + \bar{H}_{\mu\nu}). \tag{5}$$

From (4) it follows that

$$H_{\mu\nu} = \tfrac{1}{2}g^{1/2}\epsilon_{\mu\nu\alpha\beta}H^{\alpha\beta}, \tag{6}$$

so that $H_{\mu\nu}$ is a self-dual tensor. It therefore satisfies the following identities

$$H_{\mu\nu}\bar{H}^{\mu\nu} = 0, \tag{7}$$

$$H_{\mu\lambda}H^{\lambda\nu} = \tfrac{1}{4}\delta_{\mu}^{\ \nu}H_{\alpha\beta}H^{\beta\alpha}, \tag{8}$$

$$H_{\mu\lambda}\bar{H}^{\lambda\nu} = \bar{H}_{\mu\lambda}H^{\lambda\nu}. \tag{9}$$

(These generally covariant identities can easily be proved by taking a locally Minkowskian coordinate system, where $g^{1/2} = i$, and

$$H_{23} = H^{23} = iH^{01} = -iH_{01}, \tag{10}$$

with two similar equations obtained by cyclic permutation of 123.)

With help of these identities and of Eq. (4), we can now rewrite Eqs. (1–3) as

$$H^{\mu\nu}{}_{,\nu} = 0, \tag{11}$$

$$R_{\mu}^{\ \nu} - \tfrac{1}{2}\delta_{\mu}^{\ \nu}R = -H_{\mu\lambda}\bar{H}^{\lambda\nu}. \tag{12}$$

Equation (12) can be further simplified, since its contraction leads to

$$R = 0, \tag{13}$$

by virtue of (7). One thus has

$$R_{\mu}^{\ \nu} = -H_{\mu\lambda}\bar{H}^{\lambda\nu}. \tag{14}$$

It further follows, with the help of (8) and (9) that

$$R_{\mu\lambda}R^{\lambda\nu} = H_{\mu\alpha}\bar{H}^{\alpha\lambda}\bar{H}_{\lambda\beta}H^{\beta\nu}, \tag{15}$$

$$= \tfrac{1}{16}\delta_{\mu}^{\ \nu}H_{\alpha\beta}H^{\beta\alpha}\bar{H}_{\gamma\lambda}\bar{H}^{\lambda\gamma}, \tag{16}$$

whence

$$R_{\mu}^{\ \lambda}R_{\lambda}^{\ \nu} = \tfrac{1}{4}\delta_{\mu}^{\ \nu}R_{\alpha}^{\ \beta}R_{\beta}^{\ \alpha}. \tag{17}$$

It was first shown by Rainich (*1*), and later by Misner and Wheeler (*3*), that the fulfilment of relations (13) and (17) entails the existence of an infinity of self-dual tensors $K^{\mu\nu}$ such that

$$R_{\mu}^{\ \nu} = -K_{\mu\lambda}\bar{K}^{\lambda\nu}. \tag{18}$$

Any two such self-dual tensors differ only by an arbitrary phase factor and/or an inversion of the sign of i. Since Eqs. (9), (11), and (12) are invariant under such an inversion, we can always write $H^{\mu\nu}$, if it exists, as

$$H^{\mu\nu} = e^{i\phi} K^{\mu\nu}, \tag{19}$$

where ϕ is real. It then follows from (11) that

$$K^{\mu\nu}\phi_\nu = iK^{\mu\nu}{}_{;\nu}. \tag{20}$$

If the $K^{\mu\nu}$ matrix is not singular, this equation can easily be solved for ϕ_ν, e.g., by multiplying it by $K_{\lambda\mu}$ and making use of (8). It can then be shown (*1*, *3*) that the resulting equations are integrable if, and only if, the Rainich vector

$$S_\lambda = g^{-1/2} g_{\lambda\alpha} g_{\mu\delta} \epsilon^{\alpha\beta\gamma\delta} R^\nu{}_{\beta;\gamma} R^\mu{}_\nu / R_\xi{}^\omega R_\omega{}^\xi, \tag{21}$$

is a gradient, i.e., if

$$S_{\lambda;\mu} - S_{\mu;\lambda} = 0. \tag{22}$$

In this case, ϕ is determined by the metric, up to an arbitrary real additive constant.

If, however, the $K^{\mu\nu}$ matrix is singular, the above method breaks down. It is seen from (8) and (16) that this occurs if

$$R_\mu{}^\lambda R_\lambda{}^\nu = 0, \tag{23}$$

so that Eq. (21) is then meaningless. Such electromagnetic fields are called *null fields*.

This apparent failure of geometrodynamics has been for some time rather troublesome, and it was even suggested (*9*) that nontrivial null fields should be ruled off by the combined Maxwell–Einstein equations. Such fields, however, do exist (*10*, *11*) and one has to decide whether geometrodynamics (in some modified form) can be applied to them, or not.

Recently, Hlavatý (*8*)[2] developed a rather sophisticated method enabling the treatment of non-null and null fields on almost the same footing. He found the vector ϕ_ν and stressed that it must be a gradient, without going into detail about this requirement.

In this paper, we derive necessary and sufficient conditions for ϕ_ν to be a gradient, which enable us to find ϕ_ν explicitly, as well as to discuss exceptional cases connected with this problem.

We here tackle the Rainich problem for null electromagnetic fields by two different approaches. First, we solve Eq. (20) in a straightforward manner. The result indicates that null fields really possess a kind of privileged status. In order

[2] I am greatly indebted to Prof. V. Hlavatý for making his results available to me prior to publication.

to clarify this point, we then try to consider them as a limiting case of non-null electromagnetic fields. Both methods lean heavily upon the existence of a tetrad of (nonorthogonal) isotropic vectors which is inherent to the problem, and the importance of which has been emphasized by Hlavatý (*8*).

Our philosophy here will be essentially constructive: we shall take special pains to establish how each of the quantities involved in the computations can be explicitly constructed from geometrical concepts.

2. THE INTRINSIC TETRAD

From (13) and (23) it follows that all the invariants of the $R_\mu^{\ \nu}$ matrix vanish. Let us temporarily take a locally Minkowskian coordinate system and then define an imaginary time coordinate, the metric tensor thereby becoming proportional to the unit matrix, and therefore invariant under complex orthogonal transformations. It can then be shown that any symmetric complex matrix, all the invariants of which vanish (such as $R_{\mu\nu}$) can be brought by orthogonal complex transformations to one of the following canonical forms (*12*):

$$\begin{pmatrix} u & iu & 0 & 0 \\ iu & -u & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} u & iu & 0 & 0 \\ iu & -u & 0 & 0 \\ 0 & 0 & v & iv \\ 0 & 0 & iv & -v \end{pmatrix}, \tag{24 a, b}$$

$$\begin{pmatrix} 0 & iu & 0 & 0 \\ iu & 0 & u & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & u & iu & 0 \\ u & iu & u & -iu \\ iu & u & -iu & u \\ 0 & -iu & u & 0 \end{pmatrix}. \tag{25 a, b}$$

However, the matrices (25 a, b) do not satisfy Eq. (23), so that we remain with (24 a, b) only.

Let us now return to a real time coordinate, so that $R_{\mu\nu}$ is real and R_{00} negative definite. It is easily seen that (24 a) can be written as

$$R_{\mu\nu} = -L_\mu L_\nu , \tag{26}$$

where L_μ is an isotropic vector:

$$L^\mu L_\mu = 0. \tag{27}$$

In any coordinate system, its components are uniquely defined by

$$L_\mu = (-R_{\mu\mu})^{1/2}, \tag{28}$$

and the choice of the sign of L_0, which we shall take as positive, by definition.

Similarly, (24 b) can be written as

$$R_{\mu\nu} = -L_\mu L_\nu - K_\mu K_\nu , \tag{29}$$

where K_μ is another isotropic vector, orthogonal to L_μ. However, two real isotropic vectors in Minkowski space cannot be orthogonal, unless they are also parallel. This is easily seen by writing L_μ and K_μ as $(L, \mathbf{L})$ and $(K, \mathbf{K})$, where L and K are the three-dimensional lengths of $\mathbf{L}$ and $\mathbf{K}$. The orthogonality of L_μ and K_μ implies $LK - \mathbf{L}\cdot\mathbf{K} = 0$, so that $\mathbf{K} = K\mathbf{L}/L$. This is not, however, what is meant by the matrix (24 b), so that it should also be ruled off, because of the reality of $R_{\mu\nu}$.

We thus remain with (24 a) only, i.e., with (26) and (27).

We further introduce two complex conjugate isotropic vectors M_μ and $\bar{M}_\mu$, which satisfy the relations

$$L^\mu M_\mu = L^\mu \bar{M}_\mu = 0, \tag{30}$$

$$M^\mu M_\mu = \bar{M}^\mu \bar{M}_\mu = 0, \tag{31}$$

$$M_\mu \bar{M}^\mu = -1. \tag{32}$$

These are only five equations for eight components, and we remain with three arbitrary parameters at our disposal. In fact, if Eqs. (30–32) are satisfied by some M_μ, they will also be satisfied by

$$M'_\mu = e^{i\phi} M_\mu + \psi L_\mu, \tag{33}$$

where ϕ is a real and ψ a complex parameter. We shall call the transformation (33) a T-transformation, for lack of a more appropriate name. A T-transformation involving only ϕ will be called a ϕ-transformation, while one involving only ψ will be called a ψ-transformation.

In order to construct M_μ in some local Minkowski frame, let us choose any three-dimensional vector $\mathbf{K}$, not parallel to $\mathbf{L}$, and let us form $\mathbf{P} = \mathbf{L} \times \mathbf{K}$ and $\mathbf{Q} = \mathbf{L} \times \mathbf{P}$. It is then easily seen that (30–32) are satisfied by $M_\mu = (0, \mathbf{M})$, where

$$\mathbf{M} = 2^{-1/2}[(\mathbf{P}/P) + i(\mathbf{Q}/Q)]. \tag{34}$$

Finally, we can perform on this M_μ an arbitrary T-transformation.

We further define

$$K^{\mu\nu} = L^\mu M^\nu - L^\nu M^\mu. \tag{35}$$

Notice that $K^{\mu\nu}$ is ψ-invariant, but not ϕ-invariant.

We shall now prove that $K^{\mu\nu}$ is self-dual. To show this, it is sufficient to consider the particular M_μ given by (34), since self-duality is a linear property and therefore cannot be affected by a ϕ-transformation.

We take as basis the orthogonal unit vectors $\mathbf{i} = \mathbf{L}/L$, $\mathbf{j} = \mathbf{P}/P$, and $\mathbf{k} = \mathbf{Q}/Q$.

Then L_μ and M_μ take the form

$$L_\mu = (L, L, 0, 0), \tag{36}$$

$$M_\mu = (0, 0, 2^{-1/2}, 2^{-1/2}i), \tag{37}$$

and it is seen, by simple inspection, that $K^{\mu\nu}$ satisfies relations like (10), i.e., is self-dual.

We then notice that, by virtue of (26), (30), and (32),

$$K_{\mu\lambda}\bar{K}^{\lambda\nu} = L_\mu L^\nu = -R_\mu{}^\nu. \tag{38}$$

This formula is T-invariant, and solves the algebraic part of the Rainich problem

Let us now consider the vector $g^{1/2}\epsilon_{\alpha\beta\gamma\delta}L^\beta M^\gamma \bar{M}^\delta$. This vector is real, T-invariant orthogonal to L_μ and isotropic. It must therefore be parallel to L_μ. In fact, one easily sees, with the help of (36) and (37), that

$$g^{1/2}\epsilon_{\alpha\beta\gamma\delta}L^\beta M^\gamma \bar{M}^\delta = -L_\alpha. \tag{39}$$

Finally, let us introduce the real isotropic vector N_μ by

$$N_\mu N^\mu = N_\mu M^\mu = N_\mu \bar{M}^\mu = 0, \tag{40}$$

$$N_\mu L^\mu = 1. \tag{41}$$

Since there are four equations for four components, N_μ is thereby defined. It is ϕ-invariant, but not ψ-invariant. If

$$M'_\mu = M_\mu + \psi L_\mu, \tag{42}$$

then

$$N'_\mu = N_\mu + \bar{\psi}M_\mu + \psi\bar{M}_\mu + \psi\bar{\psi}L_\mu, \tag{43}$$

as may be easily verified by substituting into (40) and (41).

In order to construct N_μ, let us choose any real vector A_μ not orthogonal to L_μ and let us define

$$B_\alpha = g^{1/2}\epsilon_{\alpha\beta\gamma\delta}A^\beta M^\gamma \bar{M}^\delta / A_\mu L^\mu. \tag{44}$$

B_α is orthogonal to M^α and $\bar{M}^\alpha$, and satisfies $B_\alpha L^\alpha = 1$, by virtue of (39). It follows that

$$N_\mu = B_\mu - \tfrac{1}{2}B_\nu B^\nu L_\mu \tag{45}$$

satisfies (40) and (41).

From (39) and (41), one obtains the important relation

$$g^{1/2}\epsilon_{\alpha\beta\gamma\delta}L^\alpha M^\beta \bar{M}^\gamma N^\delta = 1. \tag{46}$$

These four vectors are therefore linearly independent, and can serve as a new basis. For instance, one can write any vector V_μ as

$$V_\mu = AL_\mu + BM_\mu + C\bar{M}_\mu + DN_\mu , \tag{47}$$

where

$$A = V_\mu N^\mu, \qquad B = -V_\mu \bar{M}^\mu, \qquad C = -V_\mu M^\mu, \qquad D = V_\mu L^\mu. \tag{48}$$

It is also possible to split tensors into sums of products of our new basic vectors. In order to work out this formalism systematically, let us denote our tetrad by

$$h^m{}_\mu = (L_\mu , M_\mu , \bar{M}_\mu , N_\mu). \tag{49}$$

Here, latin indices are *enumerators*. They also take the values 0123 and the usual summation rule is to be followed. Let us define

$$g^{mn} = h^m{}_\mu h^n{}_\nu g^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{50}$$

and let

$$g_{mn} g^{ns} = \delta_n{}^s, \tag{51}$$

so that g_{mn} is numerically equal to g^{mn}. Let

$$h_{m\mu} = G_{mn} h^n{}_\mu = (N_\mu , -\bar{M}_\mu , -M_\mu , L_\mu). \tag{52}$$

It is easily seen that

$$h_{m\mu} h^{n\mu} = \delta_m{}^n , \tag{53}$$

whence

$$h_{m\mu} h^{m\nu} = \delta_\mu{}^\nu. \tag{54}$$

For any vector V_μ one has

$$V_\mu = h^m{}_\mu V_m , \qquad V_m = h_m{}^\mu V_\mu . \tag{55}$$

Similarly, for a tensor:

$$T_{\mu\nu} = h^m{}_\mu h^n{}_\nu T_{mn} , \qquad T_{mn} = h_m{}^\mu h_n{}^\nu T_{\mu\nu}. \tag{56}$$

For instance

$$H_{mn} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{57}$$

and

$$R_{mn} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{58}$$

The latin indices are invariant under coordinate transformations, but they transform according to a linear law under T-transformations. However, components with only lower 0 indices and/or upper 3 indices are T-invariant, since L^μ is T-invariant.

3. SOLUTION OF THE RAINICH PROBLEM

We now return to Eq. (20), which we try to solve in the case of null fields. The solution, if it exists, can be written as

$$\phi_\nu = AL_\nu + BM_\nu + \bar{B}\bar{M}_\nu + CN_\nu , \tag{59}$$

where A and C are real. It then follows from (20) and (35) that

$$-\bar{B}L^\mu - CM^\mu = i(L^\mu{}_\nu M^\nu + L^\mu M^\nu{}_\nu - L^\nu{}_\nu M^\mu - L^\nu M^\mu{}_\nu). \tag{60}$$

Multiplying this in turn by L_μ, M_μ, $\bar{M}_\mu$, and N_μ, we obtain:

$$L^\mu L^\nu M_{\mu\nu} = 0, \tag{61}$$

$$M^\mu M^\nu L_{\mu\nu} = 0, \tag{62}$$

$$C = i(L^\nu{}_\nu + L_{\mu\nu}\bar{M}^\mu M^\nu - L^\nu \bar{M}_\mu M^\mu{}_\nu), \tag{63}$$

$$\bar{B} = -i(M^\nu{}_\nu + N_\mu L^\mu{}_\nu M^\nu - N_\mu L^\nu M^\mu{}_\nu). \tag{64}$$

Equation (61) does not bring anything new, since by virtue of (30) and of the Bianchi identities

$$(L^\mu L^\nu)_\nu = L^\mu{}_\nu L^\nu + L^\mu L^\nu{}_\nu = 0, \tag{65}$$

we can write

$$L^\mu L^\nu M_{\mu\nu} = -L^\nu M_\mu L^\mu{}_\nu = L^\nu{}_\nu M_\mu L^\mu = 0. \tag{66}$$

On the other hand, (62) represents *two real conditions imposed on the metric.*

We now pass to (63). Since C must be real and since, by virtue of (32)

$$\bar{M}_\mu M^\mu{}_\nu = -\bar{M}_{\mu\nu}M^\mu = -M_\mu \bar{M}^\mu{}_\nu , \tag{67}$$

is purely imaginary, then it follows that

$$C = -i[L^\nu \bar{M}_\mu M^\mu{}_\nu + \tfrac{1}{2}L_{\mu\nu}(M^\mu \bar{M}^\nu - M^\nu \bar{M}^\mu)], \tag{68}$$

and

$$L^{\nu}{}_{\nu} + \tfrac{1}{2}L_{\mu\nu}(M^{\mu}\bar{M}^{\nu} + M^{\nu}\bar{M}^{\mu}) = 0, \tag{69}$$

which is another real condition. Notice that (62) and (69) are T-invariant, by virtue of (27) and (65). This fact may be enhanced as follows.

We first introduce

$$W_{\mu\nu} = \tfrac{1}{2}(L_{\mu\nu} + L_{\nu\mu}) - g_{\mu\nu}L_{\lambda}{}^{\lambda}. \tag{70}$$

It then follows from (56), (62), (65), and (69) that

$$\begin{aligned} W_{\mu\nu} = PL_{\mu}L_{\nu} + Q(L_{\mu}M_{\nu} + L_{\nu}M_{\mu}) + \bar{Q}(L_{\mu}\bar{M}_{\nu} + L_{\nu}\bar{M}_{\mu}) \\ + \tfrac{1}{2}W_{\lambda}{}^{\lambda}(L_{\mu}N_{\nu} + L_{\nu}N_{\mu}), \end{aligned} \tag{71}$$

whence

$$W_{\mu\lambda}W^{\lambda\nu} = \tfrac{1}{2}W_{\lambda}{}^{\lambda}W_{\mu}{}^{\nu} - SL_{\mu}L^{\nu}, \tag{72}$$

where S is a *scalar*. Equation (72) contains only T-invariant quantities and its trace can be written as

$$L^{\mu\nu}(L_{\mu\nu} + L_{\nu\mu}) = 5(L_{\mu}{}^{\mu})^2, \tag{73}$$

a relation first derived by Robinson and Sachs (7).

We now return to Eq. (59). The values of B, $\bar{B}$, and C are already known, but A is still arbitrary. Let us define $V_{\mu\nu}$ by

$$V_{\mu\nu} = \phi_{\mu\nu} - \phi_{\nu\mu}, \tag{74}$$

$$\begin{aligned} &= A(L_{\mu\nu} - L_{\nu\mu}) + B(M_{\mu\nu} - M_{\nu\mu}) + \bar{B}(\bar{M}_{\mu\nu} - \bar{M}_{\nu\mu}) + C(N_{\mu\nu} - N_{\nu\mu}) \\ &\quad + L_{\mu}A_{\nu} - L_{\nu}A_{\mu} + M_{\mu}B_{\nu} - M_{\nu}B_{\mu} + \bar{M}_{\mu}\bar{B}_{\nu} - \bar{M}_{\nu}\bar{B}_{\mu} + N_{\mu}C_{\nu} - N_{\nu}C_{\mu}. \end{aligned} \tag{75}$$

The integrability conditions of (59) are

$$V_{\mu\nu} = 0. \tag{76}$$

These six equations are not independent, since, by virtue of (74), $V_{\mu\nu}$ satisfies the four identities

$$\epsilon^{\alpha\beta\gamma\delta}V_{\beta\gamma\delta} = 0. \tag{77}$$

These four identities themselves are not independent, since for any antisymmetric $W_{\mu\nu}$, one has $(\epsilon^{\alpha\beta\gamma\delta}W_{\alpha\beta\gamma})_{\delta} = 0$. Therefore (76) actually represents $6 - 4 + 1 = 3$ independent relations, and one may as well equate to zero three independent linear combinations of the $V_{\mu\nu}$.

Two such relations are

$$L^{\mu}M^{\nu}V_{\mu\nu} = 0, \tag{78}$$

and the complex conjugate equation, which do not contain A nor derivatives of A. Explicitly, one has, with the help of (50), (62), and (65):

$$\bar{B}L^\mu M^\nu(\bar{M}_{\mu\nu} - \bar{M}_{\nu\mu}) + CL^\mu M^\nu(N_{\mu\nu} - N_{\nu\mu}) + L^\mu \bar{B}_\mu + M^\nu C_\nu = 0. \tag{79}$$

Now, it follows from (30), (32), (40), (41), (62), and (64) that

$$L^\mu M^\nu(\bar{M}_{\mu\nu} - \bar{M}_{\nu\mu}) = L_\mu^{\ \mu} + iC, \tag{80}$$

and

$$L^\mu M^\nu(N_{\mu\nu} - N_{\nu\mu}) = M_\nu^{\ \nu} - i\bar{B}, \tag{81}$$

so that (79) simply becomes

$$(\bar{B}L^\lambda + CM^\lambda)_\lambda = 0. \tag{82}$$

In fact, this result directly follows from (20) and (35), because $\bar{B} = -M^\mu\phi_\mu$ and $C = L^\mu\phi_\mu$.

It may easily be seen that (82), like (62) and (69), is T-invariant. However, it seems impossible to express (82) as a function of L_μ only, so that a seemingly foreign element has to be introduced. (Such phenomena often occur in mathematical theories. For instance, the general solution of the cubic equation necessitates the use of complex numbers, even if all three roots are real).

A third relation, besides (78), may be

$$M^\mu\bar{M}^\nu V_{\mu\nu} = 0, \tag{83}$$

i.e.,

$$\begin{aligned} AM^\mu\bar{M}^\nu(L_{\mu\nu} - L_{\nu\mu}) - BM^\mu\bar{M}^\nu M_{\nu\mu} + \bar{B}M^\mu\bar{M}^\nu\bar{M}_{\mu\nu} \\ + CM^\mu\bar{M}^\nu(N_{\mu\nu} - N_{\nu\mu}) + M^\mu B_\mu - \bar{M}^\nu\bar{B}_\nu = 0. \end{aligned} \tag{84}$$

If $M^\mu\bar{M}^\nu(L_{\mu\nu} - L_{\nu\mu}) \neq 0$, this relation determines A, and the problem is solved: if (62), (69), and (82) are satisfied, then

$$\phi = \int (AL_\mu + BM_\mu + \bar{B}\bar{M}_\mu + CN_\mu)\, dx^\mu \tag{85}$$

is a single-valued function (we suppose that space-time is simply connected) which satisfies Eq. (20). As in the case of non-null fields, ϕ contains an arbitrary additive constant.

4. THE EXCEPTIONAL CASE[3]

If

$$M^\mu\bar{M}^\nu(L_{\mu\nu} - L_{\nu\mu}) = 0, \tag{86}$$

[3] This case was not discussed by Hlavatý (8), who did not consider the problem of finding A (ξ by Hlavatý) at all.

then (84) represents a seventh condition imposed on the metric. (The six others are (62) and its complex conjugate, (69), (82) and its complex conjugate, and (86) itself.) On the other hand, we still have to determine A.

Now, the fulfilment of (82), (84), and (86) implies that

$$\xi_\mu = BM_\mu + \bar{B}\bar{M}_\mu + CN_\mu \tag{87}$$

must be the gradient of some scalar ξ, since these relations are three independent integrability conditions for (75), with $A = 0$. We can therefore take $A = 0$. However, in this case, A is not unique, as will presently be shown.

If A does not vanish, then it follows from (75) that

$$A(L_{\mu\nu} - L_{\nu\mu}) + L_\mu A_\nu - L_\nu A_\mu = 0, \tag{88}$$

so that

$$\epsilon^{\alpha\beta\gamma\delta} L_\beta L_{\gamma\delta} = 0. \tag{89}$$

Moreover, it can be shown (*13*) that (89) is not only a necessary, but also a sufficient condition for the integrability of (88).

We now show that (89) necessarily follows from (86). First, we notice that, by virtue of (27) and (65),

$$L^\mu(L_{\mu\nu} - L_{\nu\mu}) = L_\nu L_\mu{}^\mu, \tag{90}$$

so that $(L_{\mu\nu} - L_{\nu\mu})$ must be of the form

$$\begin{aligned} L_{\mu\nu} - L_{\nu\mu} = \alpha(L_\mu M_\nu - L_\nu M_\mu) + \bar{\alpha}(L_\mu \bar{M}_\nu - L_\nu \bar{M}_\mu) \\ - L_\lambda{}^\lambda (L_\mu N_\nu - L_\nu N_\mu) + \beta(M_\mu \bar{M}_\nu - M_\nu \bar{M}_\mu). \end{aligned} \tag{91}$$

Since (86) implies that β vanishes, then (89) must hold if (86) does, and vice versa.

One can then easily show (*13*) that L_μ must have the form

$$L_\mu = \lambda U_\mu , \tag{92}$$

where λ and U are two scalars, and that the solution of (88) is

$$A = f(U)/\lambda, \tag{93}$$

where $f(U)$ is an arbitrary function of U.

The hypersurfaces $U =$ const. are null hypersurfaces (wave fronts), and the final solution is

$$\phi = f(U) + \int (BM_\mu + \bar{B}\bar{M}_\mu + CN_\mu)\, dx^\mu. \tag{94}$$

Let us now choose a new system of coordinates y^μ, such that $y^0 = U$. The other y^μ will be denoted as y^k, so that $k = 1, 2, 3$. In this new coordinate system,

$$L_0 = \lambda, \qquad L_k = 0. \tag{95}$$

It then follows from (27) that

$$g^{00} = 0, \tag{96}$$

and from (73) that

$$g^{0k}\lambda_k = 0, \tag{97}$$

whence, in general

$$L_\mu{}^\mu = 0, \tag{98}$$

and

$$L^\mu \lambda_\mu = 0. \tag{99}$$

Moreover, since

$$\partial U/\partial y^\mu = \delta_\mu{}^0 = \text{const.} \tag{100}$$

one has, in general,

$$U_{\mu\nu} = 0. \tag{101}$$

The isotropic vector U_μ is covariantly constant.

As an example, let us consider the metric (*11*)

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 + 2f(x, y, z + t)(dz + dt)^2.$$

We have $U = z + t$, and

$$\lambda = (f_{xx} + f_{yy})^{1/2}. \tag{103}$$

The intrinsic tetrad may be taken as

$$L^\mu = (\lambda, 0, 0, -\lambda), \tag{104}$$

$$M^\mu = (0, 2^{-1/2}, 2^{-1/2}i, 0), \tag{105}$$

$$\bar{M}^\mu = (0, 2^{-1/2}, -2^{-1/2}i, 0), \tag{106}$$

$$N^\mu = (\tfrac{1}{2}\lambda^{-1}, 0, 0, \tfrac{1}{2}\lambda^{-1}). \tag{107}$$

One may easily verify that (62), (69), (82), and (86) are satisfied. On the other hand, (84) gives

$$(\ln \lambda)_{xx} + (\ln \lambda)_{yy} = 0. \tag{108}$$

and the final solution is

$$\phi = F(z + t) + \tfrac{1}{2} \int (\lambda_x/\lambda)\, dy - (\lambda_y/\lambda)\, dx. \qquad (109)$$

5. DISCUSSION OF THE RESULTS

We have hitherto found two essential differences between null electromagnetic fields and non-null ones. If Eqs. (13) and (17) are satisfied, and if the field is not null, then we need three more independent conditions (22) in order to ascertain that the Maxwell equations (11) are satisfied.

On the other hand, in the case of null fields, we need five independent conditions: (62) and its complex conjugate, (69) and (82) and its complex conjugate. However, the Bianchi identities (65) are not independent, since, for any null vector V_μ, one has

$$V_\mu(V^\mu{}_\nu V^\nu + V^\mu V^\nu{}_\nu) = 0. \qquad (110)$$

This might explain the need of one additional condition, but not of two additional ones.

The second, and much more important difference between null electromagnetic fields and non-null ones, is the existence of the exceptional case discussed in the previous section, and characterized by the fulfilment of the additional conditions (84) and (86). We have called this case exceptional, because the phase of $H^{\mu\nu}$ is left arbitrary to a large extent, rather than being fixed (up to an additive constant) by the metric field.

At first sight, such conclusions are rather surprising, since in the special theory of relativity, it is always possible to superimpose an arbitrary infinitesimal non-null field on a finite null one, so that there is a continuous transition from non-null fields to null ones. Now, all previous considerations were *special relativistic*, since the non-Euclicean character of the metric appeared nowhere in our argument.[4] One could therefore think of solving the Rainich problem by considering a sequence of non-null electromagnetic fields having a null field as its limit. The conditions for the existence of a null field would then be some appropriate limit of the usual Rainich relations.

This program will be carried out in the next two sections, and the above suggestion will turn out to be too naive. In fact we shall see that the superposition of an infinitesimal non-null field on a finite null one determines the phase of the latter no matter how small the perturbing field is, and in particular that *finite* phase differences of $H^{\mu\nu}$ can be caused by *infinitesimal* variations of the metric field.

[4] The only role of the Ricci tensor was that of an energy-momentum tensor (with reversed sign). Its relation to curvature was nowhere used. Even the Bianchi identities (65) can be deduced from Eqs. (6), (7), (11), and (12).

6. ALMOST NULL FIELDS: ALGEBRA

As suggested by Blancheton (*14*) and Bertotti (*15*), we shall now consider null electromagnetic fields as a limiting case of non-null fields. From the physical point of view, this may be described as the superposition of a small arbitrary non-null electromagnetic field, of order ϵ say, on a finite null field.

To put this in mathematical terms, let us consider a one parameter family of metrics $g_{\mu\nu}(\epsilon)$, and let us suppose that $g_{\mu\nu}$ can be expanded as

$$g_{\mu\nu}(\epsilon) = {}_0g_{\mu\nu} + \epsilon\, {}_1g_{\mu\nu} + \epsilon^2\, {}_2g_{\mu\nu} + \cdots . \tag{111}$$

Similarly we expand the *mixed* Ricci tensor

$$R_\mu{}^\nu = {}_0R_\mu{}^\nu + \epsilon\, {}_1R_\mu{}^\nu + \epsilon^2\, {}_2R_\mu{}^\nu + \cdots . \tag{112}$$

We further suppose that

$${}_0R_\mu{}^\nu = -L_\mu L^\nu, \tag{113}$$

where

$$L_\mu = {}_0g_{\mu\nu}L^\nu, \tag{114}$$

and

$$L_\mu L^\mu = 0. \tag{115}$$

We now require that

$$R = 0, \tag{116}$$

$$R_\mu{}^\lambda R_\lambda{}^\nu = \tfrac{1}{4}\delta_\mu{}^\nu R_\alpha{}^\beta R_\beta{}^\alpha \neq 0, \tag{117}$$

and

$$S_{\lambda\mu} - S_{\mu\lambda} = 0, \tag{118}$$

where S_λ is defined by (21).

It then follows that $g_{\mu\nu}(\epsilon)$ describes a non-null electromagnetic field if $\epsilon \neq 0$, and that this field becomes a null one for $\epsilon = 0$. Our problems is the investigation of the relations imposed on ${}_0g_{\mu\nu}$ by Eqs. (111–118). In this section, we shall expand (116) and (117) into powers of ϵ. The discussion of (118) will be postponed to the next section. As will be seen from the sequel, it is sufficient to consider the first and second order terms only.

We shall find it convenient to raise and lower the indices of ${}_1R_\mu{}^\nu$ and ${}_2R_\mu{}^\nu$ with the help of the ${}_0g_{\mu\nu}$ tensor, as in Eq. (114), e.g.,

$${}_1R_{\mu\nu} = {}_1R_\mu{}^\lambda\, {}_0g_{\lambda\nu} . \tag{119}$$

(Thus ${}_1R_{\mu\nu}$ is *not* the second term of an expansion $R_{\mu\nu} = L_\mu L_\nu + \epsilon\, {}_1R_\mu{}^\nu + \cdots$).

In the first order, Eqs. (116) and (117) give

$$ {}_1R_\mu{}^\mu = 0, \tag{120} $$

and, by virtue of (113):

$$ L_\mu L^\lambda \, {}_1R_{\lambda\nu} + {}_1R_{\mu\lambda}L^\lambda L_\nu = \tfrac{1}{2} \, {}_0g_{\mu\nu}L^\alpha L^\beta \, {}_1R_{\alpha\beta} , \tag{121} $$

$$ = 0, \tag{122} $$

the last step being obtained by contracting both sides of (121) with L^μ. It follows that

$$ {}_1R_{\mu\nu}L^\nu = 0, \tag{123} $$

so that, if ${}_1R_{\mu\nu}$ is expanded as in (56), no terms with N_ν occur:

$$ \begin{aligned} {}_1R_{\mu\nu} = {} & AL_\mu L_\nu + B(L_\mu M_\nu + L_\nu M_\mu) + \bar{B}(L_\mu \bar{M}_\nu + L_\nu \bar{M}_\mu) \\ & + CM_\mu M_\nu + \bar{C}\bar{M}_\mu \bar{M}_\nu + D(M_\mu \bar{M}_\nu + M_\nu \bar{M}_\mu). \end{aligned} \tag{124} $$

Here, M_μ, $\bar{M}_\mu$, and N_μ are defined as in Section 2, with the help of ${}_0g_{\mu\nu}$. From (120), it follows that $D = 0$. On the other hand, we can always take $C \neq 0$, since if C vanished, we would have

$$ R_\mu{}^\nu = -L'_\mu L'^\nu + O(\epsilon^2), \tag{125} $$

where

$$ L'_\mu = L_\mu - \epsilon(\tfrac{1}{2} AL_\mu - BM_\mu - \bar{B}\bar{M}_\mu). \tag{126} $$

This would mean that the non-null part of the field is of order ϵ^2, and not of order ϵ, as initially supposed.

Let us now perform the following ψ-transformation:

$$ M'_\mu = M_\mu + (B/C)L_\mu . \tag{127} $$

We obtain

$$ {}_1R_{\mu\nu} = [A - (B^2/C) - (\bar{B}^2/\bar{C})]L_\mu L_\nu + CM'_\mu M'_\nu + \bar{C}\bar{M}'_\mu \bar{M}'_\nu . \tag{128} $$

The first term on the right-hand side can be omitted, since it can always be combined with ${}_0R_{\mu\nu}$. Finally, a ϕ-transformation

$$ M''_\mu = (C/\bar{C})^{1/4}M'_\mu , \tag{129} $$

gives

$$ {}_1R_{\mu\nu} = \gamma(M_\mu M_\nu + \bar{M}_\mu \bar{M}_\nu), \tag{130} $$

where $\gamma = (C\bar{C})^{1/2}$ is real, and primes have been dropped. It follows that

$$ {}_1R_\mu{}^\lambda \, {}_1R_{\lambda\nu} = -\gamma^2(M_\mu \bar{M}_\nu + M_\nu \bar{M}_\mu). \tag{131} $$

We are now ready to pass to the second order. Equations (116) and (117) give

$$ {}_2R_\mu{}^\mu = 0, \tag{132} $$

and, by virtue of (32), (113), and (131)

$$ L_\mu L^\lambda\, {}_2R_{\lambda\nu} + {}_2R_{\mu\lambda}L^\lambda L_\nu + \gamma^2(M_\mu\bar{M}_\nu + M_\nu\bar{M}_\mu) = \tfrac{1}{2}\, g_{\mu\nu}(L^\alpha L^\beta\, {}_2R_{\alpha\beta} - \gamma^2). \tag{133} $$

Contraction of this equation with L_μ gives

$$ L^\alpha L^\beta\, {}_2R_{\alpha\beta} = -\gamma^2, \tag{134} $$

whence

$$ L_\mu L^\lambda\, {}_2R_{\lambda\nu} + {}_2R_{\mu\lambda}L^\lambda L_\nu = -\gamma^2(g_{\mu\nu} + M_\mu\bar{M}_\nu + M_\nu\bar{M}_\mu), \tag{135} $$

$$ = -\gamma^2(L_\mu N_\nu + L_\nu N_\mu), \tag{136} $$

where use has been made of (50) and (56). It follows that

$$ L^\lambda\, {}_2R_{\lambda\nu} = -\gamma^2 N_\nu\,, \tag{137} $$

or

$$ L^\lambda({}_2R_{\lambda\nu} + \gamma^2 N_\lambda N_\nu) = 0. \tag{138} $$

Comparison of this result with (123) leads, when account is taken of (132), to the expansion

$$ \begin{aligned} {}_2R_{\mu\nu} = AL_\mu L_\nu + B(L_\mu M_\nu + L_\nu M_\mu) + \bar{B}(L_\mu\bar{M}_\nu + L_\nu\bar{M}_\mu) \\ + CM_\mu M_\nu + \bar{C}\bar{M}_\mu\bar{M}_\nu - \gamma^2 N_\mu N_\nu\,. \end{aligned} \tag{139} $$

We have used the same letters A, B, and C as before, since ${}_1R_{\mu\nu}$ is now given by (130) and no confusion can arise. Once more, the term $AL_\mu L_\nu$ can be combined with ${}_0R_{\mu\nu}$ and therefore can be omitted. Similarly, one can assume that C is purely imaginary, since its real part can be combined with γ in ${}_1R_{\mu\nu}$. However, it is impossible to further simplify (139), since the tetrad is already fixed.

7. ALMOST NULL FIELDS: ANALYSIS

As already stated, the vector

$$ S_\lambda = g^{-1/2} g_{\lambda\alpha} g_{\mu\delta}\epsilon^{\alpha\beta\gamma\delta} R^\nu{}_{\beta\gamma} R^\mu{}_\nu / R_\xi{}^\omega R_\omega{}^\xi, \tag{140} $$

has the form 0/0 for null fields, and is therefore meaningless. Nevertheless, it would be reasonable to expect that it has a well-defined limit for almost null fields, when ϵ tends to 0. (In fact, we shall show that there is *no* such limit). From (131), (133), and (134), it follows that the denominator is

$$ \epsilon^2({}_1R_\mu{}^\nu\, {}_1R_\nu{}^\mu - 2L^\mu L^\nu\, {}_2R_{\mu\nu}) = 4\epsilon^2\gamma^2. \tag{141} $$

Our perturbation procedure will therefore be consistent if, and only if, the following two conditions are satisfied:

(*a*) The first order terms in the numerator of (140) vanish.

(*b*) The second order terms in the numerator of (140) that are not proportional to γ^2 vanish (Otherwise, there would be unwarranted limitations on the perturbing field).

If both conditions are fulfilled, then S_λ has a definite limit S'_λ, which is independent of the perturbing field, and one can write the integrability equations

$$S'_{\lambda\mu} - S'_{\mu\lambda} = 0. \tag{142}$$

Before we start examining these conditions, we still have to expand $R^\nu{}_{\beta\gamma}$. Let us define

$$\Delta^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - {}_0\Gamma^\lambda{}_{\mu\nu}. \tag{143}$$

Notice that $\Delta^\lambda{}_{\mu\nu}$ is a *tensor.* It can be expanded as

$$\Delta^\lambda{}_{\mu\nu} = \epsilon\, {}_1\Delta^\lambda{}_{\mu\nu} + \epsilon^2\, {}_2\Delta^\lambda{}_{\mu\nu} + \cdots. \tag{144}$$

Let us now denote by a stroke, e.g., $L_{\mu/\nu}$ or $R^\nu{}_{\beta/\gamma}$, etc., the covariant derivative with respect to the ${}_0\Gamma^\lambda{}_{\mu\nu}$ affinity. We then have

$$R^\nu{}_{\beta\gamma} = R^\nu{}_{\beta/\gamma} + R^\omega{}_\beta \Delta^\nu{}_{\omega\gamma} - R^\nu{}_\xi \Delta^\xi{}_{\beta\gamma}. \tag{145}$$

(The last term does not contribute to the final result, because of $\epsilon^{\alpha\beta\gamma\delta}$). Notice that since we raise or lower indices with the help of ${}_0g_{\mu\nu}$, these operations commute with the stroke.

We now turn to examine the various terms in the numerator of (140). First, we notice that, by virtue of (27) and (65)

$$\epsilon^{\alpha\beta\gamma\delta}(L^\nu L_\beta)_{/\gamma} L^\mu L_\nu = 0, \tag{146}$$

so that the vanishing of the first-order terms implies

$${}_0g_{\mu\delta}\epsilon^{\alpha\beta\gamma\delta}[({}_1R^\nu{}_{\beta/\gamma} + {}_0R^\omega{}_\beta\, {}_1\Delta^\nu{}_{\omega\gamma})\,{}_0R^\mu{}_\nu + {}_0R^\nu{}_{\beta/\gamma}\, {}_1R^\mu{}_\nu] = 0. \tag{147}$$

With the help of (113), (123), and (130), this leads to

$$\epsilon^{\alpha\beta\gamma\delta} L^\nu{}_{/\gamma} L_\beta (M_\delta M_\nu + \bar{M}_\delta \bar{M}_\nu) = 0. \tag{148}$$

We now recall the discussion of Eq. (35), where we have shown that $\epsilon^{\alpha\beta\gamma\delta}L_\beta M_\delta$ is a self-dual tensor, and similarly $\epsilon^{\alpha\beta\gamma\delta}L_\beta\bar{M}_\delta$ an anti-dual tensor. It follows that

$$L^\nu{}_{/\gamma}[M_\nu(L^\alpha M^\gamma - L^\gamma M^\alpha) - \bar{M}_\nu(L^\alpha \bar{M}^\gamma - L^\gamma \bar{M}^\alpha)] = 0. \tag{149}$$

However, from the zero order Bianchi identities

$$(L^\mu L^\nu)_{/\nu} = 0, \tag{150}$$

and from Eq. (30) it follows that

$$L^{\nu}{}_{/\gamma}L^{\gamma}M_{\nu} = 0, \tag{151}$$

so that (149) is equivalent to

$$L_{\nu/\gamma}(M^{\nu}M^{\gamma} - \bar{M}^{\nu}\bar{M}^{\gamma}) = 0. \tag{152}$$

This equation is ψ-invariant, by virtue of (151), but it is not ϕ-invariant. We shall now show that it implies the stronger conditions

$$L_{\nu/\gamma}M^{\nu}M^{\gamma} = 0, \qquad L_{\nu/\gamma}\bar{M}^{\nu}\bar{M}^{\gamma} = 0, \tag{153}$$

which are also ϕ-invariant.

Multiply (152) by

$$\gamma = {}_1R_{\alpha\beta}M^{\alpha}M^{\beta} = {}_1R_{\alpha\beta}\bar{M}^{\alpha}\bar{M}^{\beta}, \tag{154}$$

(which follows from (130) and the fact that γ is real). One obtains

$${}_1R_{\alpha\beta}L_{\nu/\gamma}(M^{\nu}M^{\gamma}\bar{M}^{\alpha}\bar{M}^{\beta} - \bar{M}^{\nu}\bar{M}^{\gamma}M^{\alpha}M^{\beta}) = 0. \tag{155}$$

which is obviously ϕ-invariant, and also ψ-invariant, by virtue of (123) and (151). Now, ${}_1R_{\alpha\beta}$ is not an arbitrary tensor, because of (120) and (123). However, if V_{α} is an arbitrary vector, and D an arbitrary scalar, then

$$Q_{\alpha\beta} = {}_1R_{\alpha\beta} + V_{\alpha}N_{\beta} + V_{\beta}N_{\alpha} + D(M_{\alpha}\bar{M}_{\beta} + M_{\beta}\bar{M}_{\alpha}) \tag{156}$$

is an arbitrary symmetric tensor. Furthermore, it follows from (31), (40), and (155) that

$$Q_{\alpha\beta}L_{\nu/\gamma}(M^{\nu}M^{\gamma}\bar{M}^{\alpha}\bar{M}^{\beta} - \bar{M}^{\nu}\bar{M}^{\gamma}M^{\alpha}M^{\beta}) = 0. \tag{157}$$

Since $Q_{\alpha\beta}$ is arbitrary, then it follows that $L_{\nu/\gamma}(M^{\nu}M^{\gamma}\bar{M}^{\alpha}\bar{M}^{\beta} - \bar{M}^{\nu}\bar{M}^{\gamma}M^{\alpha}M^{\beta})$ must be antisymmetric in $\alpha\beta$, which is possible only if this expression vanishes. Multiplying it by $M_{\alpha}M_{\beta}$, or $\bar{M}_{\alpha}\bar{M}_{\beta}$, we finally obtain (153).

In fact, we have just given a new derivation of Eq. (62). This partial success may give us increased confidence in the present method.

We can now pass to the second order. We first notice that we can take, in the numerator of (140), $g^{-1/2}g_{\lambda\alpha} = {}_0g^{-1/2}\,{}_0g_{\lambda\alpha}$, since the factor

$$g_{\mu\delta}\epsilon^{\alpha\beta\gamma\delta}(R^{\nu}{}_{\beta/\gamma} + R^{\omega}{}_{\beta}\Delta^{\nu}{}_{\omega\gamma})R^{\mu}{}_{\nu} \tag{158}$$

must vanish in the first order. We thus have to compute (158) in the second order. In this expression, ${}_2R_{\mu\nu}$ occurs only in the following terms:

$$-\epsilon^{\alpha\beta\gamma\delta}[{}_2R^{\nu}{}_{\beta/\gamma}L_{\delta}L_{\nu} + (L^{\nu}L_{\beta})_{/\gamma}\,{}_2R_{\delta\nu}], \tag{159}$$

where use has been made of (113). Now, by virtue of (137), one has

$${}_2R^{\nu}{}_{\beta/\gamma}L_{\nu} = -(\gamma^2N_{\beta})_{/\gamma} - {}_2R^{\nu}{}_{\beta}L_{\nu/\gamma}, \tag{160}$$

so that (159) becomes

$$\epsilon^{\alpha\beta\gamma\delta}[(\gamma^2 N_\beta)_{/\gamma} L_\delta + {}_2R^\nu{}_\beta L_{\nu/\gamma} L_\delta - {}_2R_{\delta\nu} L^\nu L_{\beta/\gamma} - {}_2R_{\delta\nu} L^\nu{}_{/\gamma} L_\beta]$$
$$= \epsilon^{\alpha\beta\gamma\delta}[(\gamma^2 N_\beta)_{/\gamma} L_\delta + \gamma^2 N_\delta L_{\beta/\gamma} + 2{}_2R_{\nu\beta} L_\delta L^\nu{}_{/\gamma}], \quad (161)$$

where use has again been made of (137).

We now recall that ${}_2R_{\nu\beta}$ is given by (139). It is easily seen that the terms proportional to A and B do not contribute to (161). The terms proportional to C also do not contribute, because $K^{\mu\nu}$, as defined by (35), is self-dual, and therefore

$$g^{-1/2} \epsilon^{\alpha\beta\gamma\delta} M_\nu M_\beta L_\delta L^\nu{}_{/\gamma} = M_\nu L^\nu{}_{/\gamma}(L^\alpha M^\gamma - M^\alpha L^\gamma) = 0, \quad (162)$$

by virtue of (151) and (153). Thus, the only contribution of ${}_2R_{\mu\nu}$ to (158) is

$$\epsilon^{\alpha\beta\gamma\delta}[(\gamma^2 N_\beta)_{/\gamma} L_\delta + \gamma^2 N_\delta L_{\beta/\gamma} - 2\gamma^2 N_\nu N_\beta L_\delta L^\nu{}_{/\gamma}]. \quad (163)$$

Moreover, it is easily seen, with the help of (113), that ${}_2\Delta^\nu{}_{\omega\gamma}$ does not contribute to (158) at our order, and therefore the only remaining terms, besides (163), are

$$\epsilon^{\alpha\beta\gamma\delta}\{{}_1g_{\mu\delta}[{}_0R^\mu{}_\nu({}_1R^\nu{}_{\beta/\gamma} + {}_0R^\omega{}_\beta\, {}_1\Delta^\nu{}_{\omega\gamma}) + {}_1R^\mu{}_\nu\, {}_0R^\nu{}_{\beta/\gamma}]$$
$$+ {}_1R^\nu{}_{\beta/\gamma}\, {}_1R_{\delta\nu} + {}_1\Delta^\nu{}_{\omega\gamma}({}_0R^\omega{}_\beta\, {}_1R_{\delta\nu} + {}_1R^\omega{}_\beta\, {}_0R_{\delta\nu})\}. \quad (164)$$

The ${}_1g_{\mu\delta}$ and ${}_1\Delta^\nu{}_{\omega\gamma}$ terms are still unknown. In order to determine them, one has to make use of the formulas

$$R_{\mu\nu} = -L_\mu L_\nu - \Delta^\lambda{}_{\mu\nu/\lambda} + \Delta^\lambda{}_{\lambda\mu/\nu} - \Delta^\lambda{}_{\mu\nu}\Delta^\xi{}_{\xi\lambda} + \Delta^\lambda{}_{\xi\mu}\Delta^\xi{}_{\lambda\nu}, \quad (165)$$

and

$$\Delta^\lambda{}_{\mu\nu} = \tfrac{1}{2} g^{\lambda\xi}(g_{\xi\mu/\nu} + g_{\xi\nu/\mu} - g_{\mu\nu/\xi}), \quad (166)$$

which are easily verified in the coordinate system where ${}_0\Gamma^\lambda{}_{\mu\nu}$ locally vanishes, and must therefore be true in any coordinate system, since they are generally covariant.

In the first order these formulas give

$${}_1\Delta^\lambda{}_{\mu\nu} = \tfrac{1}{2}\, {}_0g^{\lambda\xi}({}_1g_{\xi\mu/\nu} + {}_1g_{\xi\nu/\mu} - {}_1g_{\mu\nu/\xi}), \quad (167)$$

and

$${}_1R_{\mu\nu} = -{}_1\Delta^\lambda{}_{\mu\nu/\lambda} + {}_1\Delta^\lambda{}_{\lambda\mu/\nu}, \quad (168)$$
$$= \tfrac{1}{2}\, {}_0g^{\lambda\xi}({}_1g_{\mu\nu/\lambda\xi} + {}_1g_{\lambda\xi/\mu\nu} - {}_1g_{\mu\lambda/\nu\xi} - {}_1g_{\nu\lambda/\mu\xi}). \quad (169)$$

We now have to solve (169) for ${}_1g_{\mu\nu}$ and then to substitute its value in (167) and (164). Fortunately, such a laborious task is in fact unnecessary, since, as explained in Footnote 4, the same problem also exists in special relativity theory.

Now in special relativity, the terms involving ${}_1g_{\mu\delta}$ and ${}_1\Delta^{\nu}{}_{\omega\delta}$ disappear from (164). The numerator of (140) thus becomes, from (163) and (164):

$$g^{-1/2}g_{\lambda\alpha}\xi^{\alpha\beta\gamma\delta}[(\gamma^2N_\beta)_\gamma L_\delta + \gamma^2N_\delta L_{\beta\gamma} - 2\gamma^2N_\nu N_\beta L^{\nu}{}_\gamma L_\delta + {}_1R_{\nu\beta\gamma}\,{}_1R_\delta{}^{\nu}], \quad (170)$$

where the zero prefix and the stroke have been omitted, for brevity. We further define

$$\xi = 2 \ln \gamma. \quad (171)$$

Taking (130) into account, (170) becomes, after division by (141) and some rearrangement:

$$S'_\lambda = \tfrac{1}{4}g^{-1/2}g_{\lambda\alpha}\epsilon^{\alpha\beta\gamma\delta}(\xi_\gamma N_\beta L_\delta + N_{\beta\gamma}L_\delta + N_\delta L_{\beta\gamma} - 2N_\nu N_\beta L^{\nu}{}_\gamma L_\delta + 2M_\beta \bar{M}_\delta \bar{M}^{\nu} M_{\nu\gamma} - M_\delta \bar{M}_{\beta\gamma} - \bar{M}_\delta M_{\beta\gamma}). \quad (172)$$

We thus see that it is not possible to carry out the program proposed at the beginning of this section: condition b cannot be fulfilled, because of the ξ term in (172). In physical terms, the limit S'_λ of the Rainich vector S_λ essentially de-depends on how the perturbing non-null field tends to zero. Since S_λ is nothing else but the gradient of ϕ (*1*, *3*, *9*), this explains the peculiarities pointed out in Section 5.

It follows that null electromagnetic fields really imply a breakdown of geometrodynamics and thus have a privileged status within the frame of the general theory of relativity.

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REFERENCES

1. G. Y. RAINICH, *Trans. Am. Math. Soc.* **27,** 106 (1925).
2. J. A. WHEELER, *Phys. Rev.* **97,** 511 (1955).
3. C. W. MISNER AND J. A. WHEELER, *Annals of Physics* **2,** 525 (1957).
4. O. BERGMANN, *J. Math. Phys.* **1,** 172 (1960).
5. A. PERES, *Bull. Research Council Israel* **9F,** 129 (1961).
6. A. PERES AND N. ROSEN, *Phys. Rev.* **115,** 1085 (1959).
7. Various preprints by M. CAHEN, R. DEBEVER, J. GOLDBERG, R. KERR, E. NEWMAN, I. ROBINSON AND R. SACHS.
8. V. HLAVATÝ, *J. math. pures appl.* (*Paris*) (to be published).
9. L. WITTEN, *Phys. Rev.* **115,** 206 (1959).
10. H. TAKENO, *Tensor* **7,** 97 (1957); **8,** 59 (1958); **9,** 76 (1959).
11. A. PERES, *Phys. Rev.* **118,** 1105 (1960).
12. F. R. GANTMACHER, "Applications of the Theory of Matrices," p. 13. Interscience, New York, 1959.
13. H. MARGENAU AND G. M. MURPHY, "The Mathematics of Physics and Chemistry," p. 86. Van Nostrand, New York, 1943.
14. E. BLANCHETON, *Compt. Rend.* **248,** 372 (1959).
15. B. BERTOTTI, *Phys. Rev.* **115,** 742 (1959).