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The Method of Images in Geometrostatics*

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Since there exist no nonflat singularity free static solutions of the empty-space Einstein equations we give the name geometrostatics to the study of the time-symmetric (instantaneously static) initial value problem. Using the method of spherical inversion images from electrostatics, we construct the initial values for a time-symmetric solution of the empty space Einstein equations having the topology of an arbitrary number of Einstein–Rosen "bridges." The initial data are analytic and asymptotically flat with positive apparent mass.

I. INTRODUCTION

Physicists have many useful general ideas about the variety of solutions which exist for the Maxwell equations; when applied to a problem this hard won familiarity is usually called physical intuition. A considerable part of it is based on the study of simple situations such as electrostatics or waves in empty space. To obtain a familiarity with the gravitational field as described by the Einstein equations, it is also useful to define and study simple cases. The present paper is devoted to an example with analogies to electrostatics.

[Although it will not be considered in this paper, the gravitational analogue of free electromagnetic waves is not being neglected at the present time. The most important recent studies of gravitational waves are:

1. The Bondi asymptotic solution (*1*) with its generalizations by Sachs (*2*) and by Newman *et al.* (*3*) which describe outgoing gravitational waves in terms of the asymptotic properties of the Riemann tensor along a null ray.

2. The Brill initial conditions (*4*) which show a wave which will be an exact solution of the empty space Einstein equations, free from singularities throughout all space for at least a finite time, and which if chosen sufficiently weak initially would be expected to spread out, getting weaker as predicted by linearized theory, and to remain singularity-free for all time.

3. The Arnowitt, Deser, Misner wave zone analysis (*5*) which shows that coordinate invariant wave amplitudes may be defined for the asymptotic $1/r$

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(radiation) terms in a metric in such a way as to give a complete description of the escaping radiation.]

The obvious analogue of electrostatics for the gravitational field would be the theory of static solutions of the Einstein equations. This analogue, however, is of very limited scope (*6*). In electrostatics, we may imagine distributions of charge which are prevented from moving by forces irrelevant to electromagnetic theory, i.e., by uncharged mechanisms. In gravitation theory, any mechanism we imagine to prevent a mass distribution from moving would itself have mass (or at least stresses) and would also produce a gravitational field. In order to define a highly simplified, yet nontrivial, class of solutions for special study I define *geometrostatics* as the study of instantaneously static solutions of the Einstein equations. This imposes special restrictions which are to hold only on a single spacelike surface $t = 0$, but not necessarily for all time.

Even without special restrictions, the metric on a hypersurface $t = 0$ is not entirely arbitrary, just as in electromagnetism one cannot have arbitrary vector fields **E** and **B**, but only fields subject to the initial value equations $\nabla \cdot \mathbf{E} = 0 = \nabla \cdot \mathbf{B}$. The analogous Einstein equations are $R_\mu{}^0 - \frac{1}{2}\delta_\mu{}^0 R = 0$, which do not contain any second time derivatives of the metric. These equations are usually written in a form independent of the choice of coordinates outside the initial spacelike hypersurface (*7*):

$$^3R + K^2 - K_{ij}K^{ij} = 0 \tag{1.1}$$

$$(K_i{}^j - \delta_i{}^j K)_{|j} = 0. \tag{1.2}$$

Here all notations are three dimensional; ^3R is the scalar curvature of the spacial metric g_{ij}, and K_{ij} is the second fundamental form (*27*) whose eigenvalues are the three principal curvatures of the initial surface as measured out into the 4-space in which it is imbedded. For our purposes it is sufficient to know that by an appropriate choice of coordinates outside the initial surface ($g_{0\mu} = -\delta_\mu{}^0$) one has $K_{ij} = -\frac{1}{2}\partial g_{ij}/\partial t$. Thus we define an instantaneously static solution as one possessing a hypersurface $t = 0$ on which $K_{ij} = 0$. This characterization is geometrical (coordinate independent), but by interpreting K_{ij} as the normal derivative of the metric as above, we see that it implies the existence of a coordinate system in which the mapping $t \to -t$, $x^i \to x^i$ is an isometry. Thus *geometrostatics* is just another name for the study of the time-symmetric initial value problem (*8–10*). In this case the initial value equations (1) and (2) reduce to the single equation

$$^3R = 0. \tag{1.3}$$

As in the case of more general initial data, it is known (*11*) that to any 3-metric satisfying Eq. (3), there corresponds a solution of the full Einstein equations.

The analogue of Eq. (3) in electromagnetism is $\nabla \cdot \mathbf{E} = 0$, with the understanding that the other initial value equation, $\nabla \cdot \mathbf{B} = 0$, is satisfied through the choice $\mathbf{B} = 0$. This does not coincide with the theory of electrostatics, which is a specialization to solutions of $\nabla \cdot \mathbf{E} = 0$ of the form $\mathbf{E} = -\nabla\phi$. An analogy to this condition can also be formulated in general relativity, namely, the specialization of g_{ij} to the form (*8*)

$$dl^2 = \chi^4(dx^2 + dy^2 + dz^2). \tag{1.4}$$

This condition can be chosen on the basis of a canonical formulation of general relativity (*12*). For a particular (arbitrary) way of defining canonical coordinates and moments for the gravitational field, it corresponds to the vanishing of all the canonical variables ("wave modes"). It is in this sense analogous to $\mathbf{E} = -\nabla\phi$ which corresponds to the vanishing of the transverse components of $\mathbf{E}$ which are canonical coordinates in electromagnetic theory. This analogy is not compelling since, in the gravitational case, static solutions will not result from these assumptions on the initial conditions, and a different choice of canonical variables might lead to inequivalent results.

The initial value metric I shall construct in this paper satisfies both Eqs. (3) and (4), and is a generalization of the initial conditions of the Schwarzschild metric to many bodies in a manner first suggested by Einstein and Rosen (*13*). The discussion is given in the framework of Wheeler's (*14*) "geometrodynamics," so we can look for singularity free solutions of Eq. (3) without postulating a source term on the left hand side. The Schwarzschild solution is treated in this manner in Section II.

II. SCHWARZSCHILD INITIAL VALUE METRIC

With the assumption of Eq. (1.4) that the the metric is conformally flat, the instantaneously static initial value equation (1.3) reduces to

$${}^3R(-\tfrac{1}{8}\chi^5) = \nabla^2\chi \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\chi = 0. \tag{2.1}$$

If, further, we assume spherical symmetry, then the solution is

$$\chi = 1 + (m/2r), \tag{2.2}$$

where m is an arbitrary constant, and the boundary condition $\chi \to 1$ as $r \to \infty$ was imposed to give an asymptotically flat metric. (Note $r^2 = x^2 + y^2 + z^2$.) As a solution of Eq. (2.1), the function (2.2) is singular at $r = 0$. However, we shall now see that the metric

$$dl^2 = [1 + (m/2r)]^4(dx^2 + dy^2 + dz^2) \tag{2.3}$$

is *not* singular considered as a *geometry* satisfying ${}^3R = 0$.

Let

$$x^i = (m/2)^2(\xi^i/\rho^2) \qquad (2.4a)$$

where

$$\rho^2 \equiv \xi^i\xi^i \equiv \xi^2 + \eta^2 + \zeta^2 \qquad (2.4b)$$

and substitute in the formula (2.3) to obtain

$$dl^2 = [1 + (m/2\rho)]^4(d\xi^2 + d\eta^2 + d\zeta^2). \qquad (2.5)$$

By comparing (2.5) and (2.3) we see that the geometry described by dl^2 is identical at the points $x^i = a^i$ and $\xi^i = a^i$. Stated differently, the mapping

$$x^i \xrightarrow{\ J\ } (m/2)^2(x^i/r^2) \qquad (2.6)$$

in an isometry. As the sphere $r = m/2$ is invariant under the mapping J, and since J^2 is the identity transformation, we think of J as a reflection in the sphere $r = \frac{1}{2}m$. In Fig. 1 the 2-surface $z = 0$ is shown imbedded in a flat 3-space. The symmetry J is then reflection in the symmetry plane of the imbedding space. The region near $r = 0$ is geometrically equivalent to the region near $r = \infty$, hence we do not consider it singular. In the next section we will construct a solution of ${}^3R = 0$ with the geometry indicated in Fig. 2. In the present

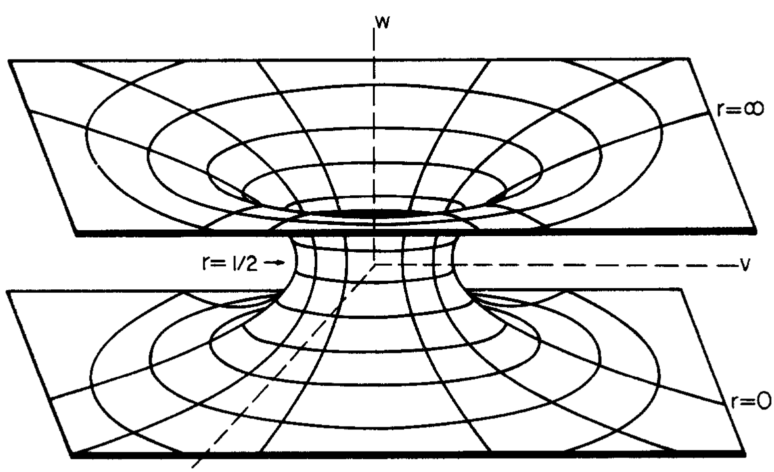

FIG. 1. The two-dimensional section $z = 0$ of the Schwarzschild $t = 0$ space (Eq. (2.3)) shown isometrically imbedded in flat three-space as the paraboloid of revolution $(u^2 + v^2)^{1/2} = \frac{1}{2}w^2 + 2$. The inversion J of Eq. (2.6) is represented by reflection in the plane $w = 0$ of the imbedding space.

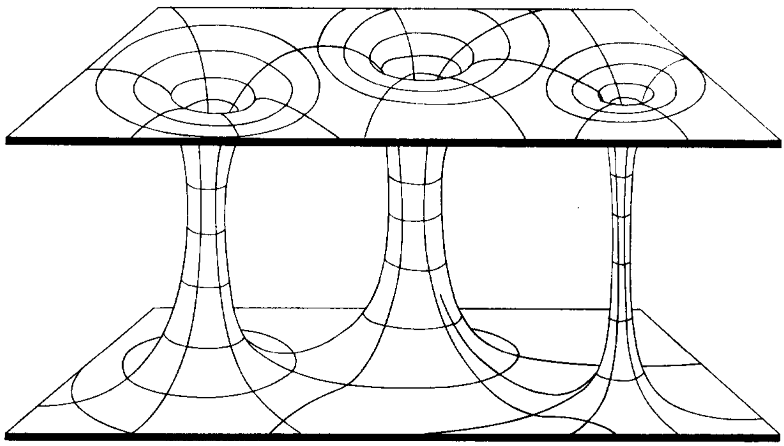

FIG. 2. A sketch of a two-dimensional section through the initial spacelike hypersurface of a space-time containing Einstein–Rosen "bridges," shown imbedded in a higher dimensional flat space in order to suggest the topology and curvature. Note the symmetry corresponding to reflection in a horizontal plane in the imbedding space.

example, we can go on from the initial data $K_{ij} = 0$ and g_{ij} from Eq. (2.3) to the full solution of the Einstein equations. When written as

$$ds^2 = -[1 - (m/2r)]^2[1 + (m/2r)]^{-2}\, dt^2 + dl^2 \tag{2.7}$$

this Schwarzschild metric appears static, but g^{00} become infinite at the Schwarzschild radius $r = m/2$. Since the initial data were nonsingular, this singularity must be spurious. It results from the singular choice of coordinates necessary to make the metric look static. Kruskal (*15*) and Fronsdal (*16*) have given complete representations of the four-dimensional geometry resulting from these initial data. One finds that the sphere in which one has reflection symmetry shrinks from its initial proper circumference $4\pi m$ down to a point in a proper time πm. Thus even the Schwarzschild solution is not static when discussed geometrically as a solution of the source-free Einstein equations.

III. EINSTEIN–ROSEN BRIDGES—ANALYSIS

The logical order in which to present the Riemann manifold suggested by Fig. 2 would be to construct first a differentiable manifold of the appropriate topology and then present on it a metric satisfying the differential equations $^3R = 0$. I shall not follow this order, since it would be impractical to solve a differential equation on such a manifold unless the construction of the manifold were conveniently related to properties of the differential equation. After the construction is complete I will make a formal presentation of the results in Section IV.

As a rough sketch, my procedure is first to assume the conformally flat metric of Eq. (1.4). The initial value equation ${}^3R = 0$ then becomes the Laplace equation $\nabla^2\chi = 0$. This equation is to be solved under appropriate boundary conditions which give the reflection symmetry between upper and lower branches of the manifold sketched in Fig. 2. But a solution somewhat similar to this is already known (*17*), for the geometry corresponding (by Eq. (1.4)) to the solution

$$\chi = 1 + \sum \alpha_i/|\mathbf{x} - \mathbf{c}_i|$$

of $\nabla^2\chi = 0$ is easily seen to correspond to Fig. 3 when the points $\mathbf{c}_i$ are well separated. (The space is clearly asymptotically flat as $|\mathbf{x}| \to \infty$. Near any pole $\mathbf{c}_i$ of χ, say $\mathbf{c}_1$, all terms in χ except $\alpha_1|\mathbf{x} - \mathbf{c}_1|^{-1}$ can be regarded as constant, so in this neighborhood the geometry is the same as near $r = 0$ in the Schwarzschild solution, Section II and Fig. 1. When the $\mathbf{c}_i$ are well separated the region where comparison to Schwarzschild is reasonable includes $r \simeq m$ and hence includes the entire "flange" shown in Fig. 3 corresponding to each pole of χ.) The problem now is how to modify this geometry so the top and bottom parts of Fig. 3 are identical. I attack this problem by attempting to make "flange no. 1," i.e., the region of small $|\mathbf{x} - \mathbf{c}_1|$, look like the top, which clearly means adding to χ a few more poles located near $\mathbf{c}_1$ (i.e., on the bottom sheet)

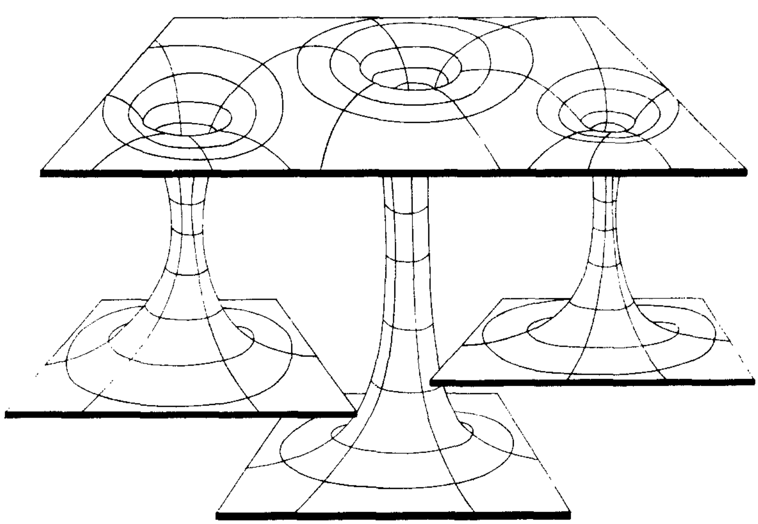

FIG. 3. A sketch of a two-dimensional section of the initial spacelike hypersurface of a space-time containing Schwarzschild-like "flanges." The "flanges" are not connected to each other in the lower half of this diagram, so the corresponding three-space is simply connected.

corresponding to $\mathbf{c}_2$, $\mathbf{c}_3$, $\cdots$ $\mathbf{c}_N$ on the top sheet. It turns out that strengths and locations of these poles in order to make the top and bottom sheets of "flange no. 1" identical are determined by the formulas of spherical inversion images from electrostatics (*18*). This is only a beginning, however, for we must establish also symmetry at "flange no. 2," where not only images of the poles $\mathbf{c}_1$, $\mathbf{c}_3$, $\mathbf{c}_4$, $\cdots$ $\mathbf{c}_N$ must be added, but also images of the new poles previously added near $\mathbf{c}_1$. Doing this, however destroys the symmetry at "flange no. 1," which we restore with new images. This process of bouncing back and forth, adding images of images of images, in fact converges and leads to the geometry of Fig. 2 after we identify all the flanges. Thus in the function χ corresponding to Fig. 2 there are not only the basic poles at $\mathbf{c}_i$ corresponding to each "bridge," but also image poles corresponding to the possibility of approaching each bridge not only by a direct path from $\mathbf{x} = \infty$, but also after a long detour threading through several bridges first.

The key to the construction of the 3-space sketched in Fig. 2 is the reflection operation J discussed in Section II. We begin with a space whose points are labeled $\mathbf{x} = (x, y, z)$ and draw N nonintersecting spheres $|\mathbf{x} - \mathbf{c}_k| = a_k$ where $|\mathbf{x}|^2 = x^2 + y^2 + z^2$. Then we look for a metric on this space which is invariant under reflection in each of these spheres. For definiteness consider the sphere $|\mathbf{x}| = a$. The reflection operation J is defined by

$$\mathbf{x} \rightarrow J\mathbf{x} = \mathbf{x}a^2/|\mathbf{x}|^2 \tag{3.1}$$

The requirement that a metric

$$dl^2 = \chi^4(\mathbf{x})\, dx^i\, dx^i \tag{3.2}$$

be invariant under J is that this *same* metric dl^2 should also be given by

$$\begin{aligned} dl^2 &= \chi^4\left(a^2 \frac{\mathbf{x}}{|\mathbf{x}|^2}\right) d\left(\frac{a^2x^i}{|\mathbf{x}|^2}\right) d\left(\frac{a^2x^i}{|\mathbf{x}|^2}\right) \\ &\equiv \left[\frac{a}{|\mathbf{x}|}\chi\left(\frac{a^2\mathbf{x}}{|\mathbf{x}|^2}\right)\right]^4 dx^i\, dx^i. \end{aligned}$$

This requirement can thus be written

$$J[\chi] = \chi \tag{3.3}$$

where J is defined to operate on a function by the rule

$$J[f](\mathbf{x}) = a|\mathbf{x}|^{-1} f(\mathbf{x}a^2/|\mathbf{x}|^2). \tag{3.4}$$

Note that both as a linear operator (3.4) and as a mapping (3.1), J satisfies

$$J^2 = I = \text{identity}. \tag{3.5}$$

Since we are assuming a conformally flat metric (3.2), the differential equation

${}^3R = 0$ reduces to the simple form $\nabla^2\chi = 0$ with a flat ∇^2 (Eq. (2.1)). But then we have the known result from electrostatics that $\nabla^2 f = 0$ implies $\nabla^2 J[f] = 0$. We will need only two special applications:

$$J\,[1] = a/r, \tag{3.6a}$$

$$J\left[\frac{1}{|\mathbf{r} - \mathbf{d}|}\right] = \frac{a}{|\mathbf{d}|}\,\frac{1}{|\mathbf{r} - J\mathbf{d}|}. \tag{3.6b}$$

The second of these may be stated as follows:

LEMMA 1. *The image of a pole at a point* $\mathbf{p}$ *is a pole at* $J\mathbf{p}$. *If the pole strength at* $\mathbf{p}$ *is* q, *the strength of the image pole is* $(a/d)q$ *where* d *is the (euclidean) distance of* $\mathbf{p}$ *from the inversion center.*

We can now construct the function χ, and hence the metric. We make use of inversion operators $J_k(k = 1, 2, \cdots N)$ for each of the spheres marked out in the coordinates $\mathbf{x}$. Since $\chi \to 1$ as $|x| \to \infty$ as a boundary condition, we see that by inversion symmetry in the kth sphere, χ must also contain a term $J_k[1] = a_n/|\mathbf{x} - \mathbf{c}_n|$. But then, by inversion in the lth sphere, there must also be a term $J_l J_k[1]$. If $l = k$ this term is already accounted for since it is just the constant $J_k^2[1] = I[1] = 1$. Otherwise it is a new pole inside the lth sphere. After n inversions we have a pole $J_{i_1}J_{i_2}\cdots J_{i_n}[1]$ in the sphere i_1. Thus we are led to consider the series

$$S = I + \sum{}' J_{i_1}J_{i_2}\cdots J_{i_n} \tag{3.7}$$

where the sum extends over all series of indices ($i_k = 1, 2, \cdots N$) of all finite lengths $n = 1, 2, \cdots$ subject to the restriction

$$i_{k+1} \neq i_k\,.$$

The function

$$\chi = S[1] \tag{3.8}$$

will satisfy the Laplace equation, since it is a series of poles, and we see that it is invariant under all the reflections as a consequence of

LEMMA 2.

$$J_k S = S. \tag{3.9}$$

The proof of this lemma proceeds by inspecting the series which stand on each side of Eq. (3.9). A typical term $J_{i_1}J_{i_2}\cdots J_{i_n}$ in the series for S on the right is found on the left as $J_k(J_{i_2}J_{i_3}\cdots J_{i_n})$ in case $i_1 = k$, and as

$$J_k(J_kJ_{i_1}J_{i_2}\cdots J_{i_n})$$

in case $i_1 \neq k$. (Recall $J_k^2 = 1$.) This also accounts for all terms in $J_k S$ since again a term's first factor must either by J_k or not.

Actually the formal symmetry (3.9) of the operator S does not solve our problem unless χ defined in Eq. (3.8) actually exists and satisfies the Laplace equation. The existence of χ means here the convergence of the series in Eq. (3.8), while uniform convergence will allow us to differentiate term by term to verify the Laplace equation. The simplest convergence criterion is obtained by recalling from electrostatics that a finite amount of each positive and negative charge distributed in any way leads to a well defined potential. More specifically, let $\sum (q_a/r_a)$ be a sum of poles and let R be a region bounded away from the singularities ($r_a \equiv |\mathbf{x} - \mathbf{p}_a| = 0$), so that in R we have $r_i \geqq \rho > 0$ for all i. Then we have

$$|\sum (q_a/r_a)| \leqq \sum (|q_a|/r_a) \leqq (1/\rho)\sum |q_a|,$$

so absolute and uniform convergence of $\sum (q_k/r_k)$ in R follows from the absolute convergence of $\sum q_i$. Using this convergence criteria the next lemma shows that χ exists provided the spheres defining the inversion operators J_k are not to close to each other.

LEMMA 3. *The series $S[1]$ converges to a function χ which is analytic and satisfies Laplace's equation in an open region R including all points not interior to any sphere, provided $(N - 1)a/d < 1$. Here $a = \max a_k$ and d is the minimum euclidean distance from the center of any of the N spheres to a point in any other sphere.*

The assertions of the lemma follow from uniform convergence in R, and by our previous remarks, then, from the convergence of the series of pole strengths in $S[1]$ together with the absence of poles near R. Let us examine a typical term $J_{i_n} J_{i_{n-1}} \cdots J_{i_1}[1]$ in $S[1]$. Here $J_{i_1}[1] = a_{i_1}/|\mathbf{x} - \mathbf{c}_{i_1}|$ is (cf. Eq. (3.6)) a pole of strength a_{i_1} located at the center of sphere i_1. Then by Lemma 1, $J_{i_2} J_{i_1}[1]$ is a pole inside sphere i_2 of strength $a_{i_2} a_{i_1}/d_{i_1 i_2}$ where $d_{i_1 i_2}$ is the distance from the pole in i_1 to the center of i_2. Similarly $J_{i_n} J_{i_{n-1}} \cdots J_{i_1}[1]$ is a pole in sphere i_n whose strength is $a_{i_1} > 0$ (from J_{i_1}) times a factor $(a/d) > 0$ for each of the $(n - 1)$ subsequent factors J_k. We may estimate it as

$$(\text{strength } J_{i_n} \cdots J_{i_1}[1]) \leqq ((a/d)^{n-1} a)$$

where a and d are defined in the statement of the lemma. Now i_1 is any of the N spheres, and i_k is any of the $(N - 1)$ spheres distinct from i_{k-1}, so there are $N(N - 1)^{n-1}$ terms of the type $J_{i_n} \cdots J_{i_1}$ (cf. Eq. (3.7)). The total pole strength of all terms in $S[1]$ is less than

$$\sum_{n=1}^{\infty} N(N - 1)^{n-1} \frac{a^{n-1}}{d^{n-1}} a = Na \sum_{n=0}^{\infty} \left[\frac{(N - 1)a}{d}\right]^n,$$

which converges when $(N - 1)a/d < 1$. Thus the convergence of the series for χ is assured. To verify that the singularities of χ do not lie arbitrarily close to the surfaces of the spheres $| \mathbf{x} - \mathbf{c}_k | = a_k$ we note that $J_1 J_2 \cdots J_k$ is not only a pole in sphere 1, but it is the image of a pole in sphere 2. Hence it lies inside the image of sphere 2 in sphere 1, and is therefore bounded away from the surface of sphere 1, since the spheres, and therefore their images, do not intersect. See Fig. 4.

For our purposes of defining a metric by Eq. (3.2), analyticity of χ is not a sufficient regularity condition. It is also necessary to require that $\chi(\mathbf{x}) > 0$. Since every term of the series $S[1]$ is positive, we can state

LEMMA 4. *In the region R of Lemma 3, χ satisfies*

$$\chi \equiv S[1] > 1.$$

The construction of the metric corresponding to Fig. 2 and satisfying ${}^3R = 0$ is now complete. The metric is defined by Eq. (3.2) in terms of χ as given in Eq. (3.8). The regularity of this matric in a region R corresponding to slightly more than the top half of Fig. 2 is established. Reflection symmetry suggests that there must be an identical lower half, smoothly matched on. The next section shows precisely how this is true. Note that if we accept the metric just given in the largest domain where $\chi(\mathbf{x})$ is regular, a picture very different from Fig. 2 arises. Each pole in χ corresponds (of Section II) to a distinct asymptotically flat region, and there are infinitely many distinct poles in χ. The domain of regularity of χ is precisely the universal covering of the manifold of Fig. 2.

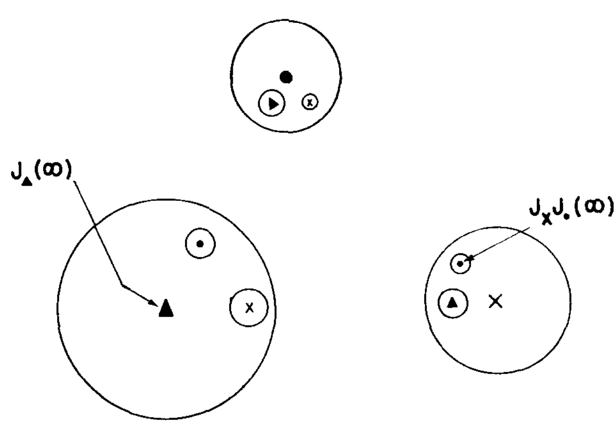

FIG. 4. Three spheres are represented by circles labeled ▲, ×, and ●. The first images of the spheres are shown. Thus the most complicated pole in the series of Eq. (3.7) which is shown here is $J_\times J_\bullet[1]$.

IV. EINSTEIN-ROSEN BRIDGES—TOPOLOGY

Let E_X^3 be a three-dimensional, topologically euclidean manifold; a typical point of E_X^3 will be designated $\mathbf{x}$ in contrast to the typical point y of another topologically euclidean 3-manifold E_Y^3 which enters the discussion. Let $x^i(x)$ be three functions which can be used as standard coordinates on E_X^3. We will frequently regard this set of coordinate values as a column matrix (vector) $\mathbf{x}(x) = \{x^1(x), x^2(x), x^3(x)\}$. Similarly, on E_Y^3 we choose a set of standard coordinates $\mathbf{y}(y)$. Using these coordinates we draw in each space a set of N corresponding spheres of radii a_k with centers at $\mathbf{c}_k$. Then (to correspond to the upper and lower sheets in Fig. 2) we define the subsets

$$\begin{aligned} \bar{U} &= \{x \in E_X^3 \mid \quad |\mathbf{x}(x) - \mathbf{c}_k| \geqq a_k \quad \text{for all } k = 1, 2, \cdots N\} \\ L &= \{y \in E_Y^3 \mid \quad |\mathbf{y}(y) - \mathbf{c}_k| > a_k \quad \text{for all } k = 1, 2, \cdots N\} \end{aligned} \tag{4.1}$$

consisting of the points outside all the spheres. We have included in $\bar{U}$ also the points on the surface of the spheres. The manifold which Fig. 2 is to indicate will be called M, a typical point of M is called z. As a point set, M is defined by

$$M = \{z \mid \text{either } z = x \in \bar{U} \quad \text{or} \quad z = y \in L\}, \tag{4.2}$$

i.e., M consists of $\bar{U}$ and L taken together. To give M the structure of a differentiable manifold[1] it is necessary that each point z of M lie in the interior of at least one coordinate patch in a system of overlapping coordinate systems which cover M. If z is not on the boundary of $\bar{U}$, then this demand is satisfied by using the coordinates $\mathbf{x}(z) \equiv \mathbf{x}(x)$ in case $z = x$ was in U (the interior of $\bar{U}$), or $\mathbf{y}(z) \equiv \mathbf{y}(y)$ in case $z = y$ was in L.

The only remaining case is to assume z is on the boundary of $\bar{U}$ and hence, let us say, on the surface of sphere no. 1. The definition of a coordinate patch around such a boundary point is the crucial step in defining M as a differentiable manifold, since $\bar{U}$ and L have no significant relationship to each other before this is done. Define a new set of coordinates $\mathbf{y}(z) = \mathbf{y}(x)$ for points $x \in \bar{U}$ on or near sphere no. 1 by

$$\mathbf{y}(x) = J_1\mathbf{x}(x) \tag{4.3}$$

where J_1 is the inversion in sphere no. 1 defined (assuming $\mathbf{c}_1 = 0$) by Eq. (3.1). Note that this equation implies that $|\mathbf{y}(x) - \mathbf{c}_1| \leqq a_1$; since $|\mathbf{x}(x) - \mathbf{c}_1| \geqq a_1$; further, when x lies on sphere 1 one has $\mathbf{y}(x) = \mathbf{x}(x)$. These relations allow us to identify the points x in $\bar{U}$ near sphere 1 with points y in E_Y^3 which are inside or on sphere 1_y (hence not in L) by

$$y \equiv x \quad \text{when} \quad \mathbf{y}(y) = \mathbf{y}(x). \tag{4.4}$$

[1] See ref. *14*, p. 555, or ref. *19*.

The regularity of these new coordinate functions $y^i(x)$ in the interior, U, of $\bar{U}$ where we have already picked one set of regular coordinates, $x^i(x)$, is determined by the regularity of the coordinate transformations $y^i(x^j)$ and $x^i(y^j)$, i.e., of $\mathbf{y} = J_1\mathbf{x}$ and $\mathbf{x} = J_1^{-1}\mathbf{y} = J_1\mathbf{y}$. But $J_1\mathbf{x}$ is an analytic function of its arguments $\mathbf{x} = \{x^i\}$ except at $\mathbf{x} = \mathbf{c}_1$, so we may consider $\mathbf{y}(x)$ as analytic coordinates on U. Now consider $\mathbf{y}(z)$ as coordinates defined over a region of M including not only L where $\mathbf{y}(z) = \mathbf{y}(y)$ but a neighborhood of sphere 1 in $\bar{U}$ (where $\mathbf{y}(z) = \mathbf{y}(x)$ as given in Eq. (4.3)). Of the two conditions (*19*) which a set of functions $\mathbf{y}(z)$ must satisfy in order to be a coordinate system, we have verified one, namely, that they are related regularly to any other coordinate system where the patches overlap. The other requirement on $\mathbf{y}(z)$ is that $\mathbf{y}(z)$ map the points z in the $\mathbf{y}(z)$ coordinate patch in a one to one way onto an *open* set of the euclidean 3-space of the column matrices $\mathbf{y}$. The difficulty in achieving this is that we wish to include the points of sphere 1, which are on the *boundary* of $\bar{U}$ as *interior* points of the $\mathbf{y}(z)$ coordinate patch. As defined here, $\mathbf{y}(z)$ satisfies this condition if its domain of definition (coordinate patch) is taken to include, in addition to L, those points of $\bar{U}$ satisfying

$$a_1 - \epsilon < |\mathbf{y}(x) - \mathbf{c}_1| \leqq a_1 \tag{4.5}$$

for some $\epsilon > 0$. Then, according to (4.4) the set L_ϵ of all corresponding matrices $\mathbf{y}(z)$ can be thought of as consisting of L (in E_Y^3) modified by reducing the radius of sphere 1 by ϵ, and this is an open set of E_Y^3, as is L. The preceding sentence is precisely true only if all the points x of E_X^3 which satisfy (4.5) also belong to $\bar{U}$, i.e., if $\mathbf{y}(x)$ cannot be in the J_1 image of any other sphere. If the spheres intersect this is impossible to achieve; we assume no intersections and then take ϵ sufficiently small.

Clearly the definition of the $\mathbf{y}(z)$ coordinates can be similarly extended (using $\mathbf{y} = J_k\mathbf{x}$) to include ϵ neighborhoods (4.5) of all the other spheres in $\bar{U}$. For sufficiently small ϵ, these neighborhoods do not overlap and it is unambiguous which J_k to take in defining $\mathbf{y}(x)$. The structure of M as an analytic manifold is then defined by the two coordinate patches, $\mathbf{x}(z)$ on $\bar{U}$, and $\mathbf{y}(z)$ on L plus the ϵ neighborhoods of spheres in $\bar{U}$ (i.e., $\mathbf{y}(z)$ on the region R of Lemmas 3 and 4). An analytic Riemannian metric on M is defined by taking

$$dl^2 = \chi^4(\mathbf{y})(dy_1^2 + dy_2^2 + dy_3^2) \tag{4.6y}$$

on the $\mathbf{y}(z)$ coordinate patch, and

$$dl^2 = \chi^4(\mathbf{x})(dx_1^2 + dx_2^2 + dx_3^2) \tag{4.6x}$$

on the $\mathbf{x}(z)$ coordinate patch, where χ as a function of three real variables is defined by Eq. (3.8). According to Lemma 3, χ (and hence dl^2) is analytic; according to Lemma 4, χ is never zero, so dl^2 always has the proper (elliptic) signature. Further, since χ satisfies the flat space Laplace equation, it follows from Eq. (2.1) that dl^2 has vanishing scalar curvature R. The final statement we need is that, where the coordinate patches overlap, Eqs. (4.6) do not give inconsistent definitions of dl^2, i.e., the right hand sides must be related by the coordinate transformation law $y^i(x^j)$. This coordinate transformation is $\mathbf{y} = J_k\mathbf{x}$, choosing the appropriate J_k in each connected overlap region. Substituting this transformation law in Eqs. (4.6) yields as the consistency requirement just Eq. (3.3). That is, χ must have those symmetries which were established in Lemma 2.

V. DISCUSSION

Two questions will be discussed here: some special cases and possible generalizations of the result stated in the previous section, and the question of associating a conserved "intrinsic mass" parameter with the topological features of this manifold.

One special case of the solution of the initial value problem given in Section IV is of course the Schwarzschild initial values discussed in Section II. Another special case, where there are only two Einstein–Rosen bridges, has been obtained previously (*20*) by other methods. If the two spheres which define the solution mentioned above have the same radius, $a_1 = a_2$, then the solution may be interpreted (*20*) as a "wormhole" rather than as a pair of identical Einstein–Rosen bridges. The methods of the present paper do not seem to allow a "wormhole" interpretation of the solution unless a plane of reflection symmetry is introduced; it does not appear sufficient merely to make the radii of the spheres equal in pairs, $a_{2n} = a_{2n+1}$.

The idea of using strange topologies was introduced by Wheeler (*21*) in order to allow for charged objects without introducing a charge density, or a charged field, into the theory. A manifold of the type shown in Fig. 3 has been given (*17*), where the flux of electric field through each "neck" is arbitrary. However, I had not been able to find any manifolds resembling Fig. 2 which satisfied the equations of geometrostatics including an electric field, namely,[2]

$$g^{1/2}R = \tfrac{1}{2}g^{-1/2}g_{ij}\mathcal{E}^i\mathcal{E}^j, \tag{5.1g}$$

$$\mathcal{E}^i_{,i} = 0. \tag{5.1e}$$

This problem has been solved by Lindquist (*22*) who presents solutions of Eqs.

[2] In these initial value equations we have set $B^i = 0$ so that we might continue to assume $K_{ij} = 0$. The notations are all three dimensional, $\mathcal{E}^i$ is a vector *density* in the 3-manifold so the metric does not appear in Eq. (5.1e). Heaviside (rationalized) electromagnetic units are used, and we set $c = 1 = 16\pi\gamma c^{-4}$ where γ is the Newtonian gravitational constant. Cf. ref. (*12*) and ref. (*7*).

(5.1) with the topology of Einstein-Rosen bridges[3] as in Fig. 2. In the neutral case again, I have found solutions (*9*) which have the appearance of Fig. 3 on a small scale, but on a larger scale (where the "flanges" go unnoticed) have the appearance of a spherical universe instead of an asymptotically flat space. An analogous modification of the solution given in the present paper would give a closed universe satisfying the empty-space Einstein equations, but I have not succeeded in finding such a solution. In any case solutions of the empty-space initial value problem, Eq. (1.3), are known to exist in closed spaces (*4*) where the effective matter density is supplied by "gravitons" (gravitational waves) rather than by "topologicons" (Einstein–Rosen bridges, etc.).

To what extent will the "bridges" in Fig. 2 maintain their identity as the metric evolves in time? Assume the time dependent solution is obtained under the coordinate conditions[4] $g^{00} = -1$, $g_{0i} = 0$. Evidently the same computations are involved in obtaining the solution using spacial coordinates $\mathbf{x}(z_+)$ near a point $z_+ = x$ in U as using the coordinates $\mathbf{y}(z_-)$ near the corresponding point $[\mathbf{x}(z_+) = \mathbf{y}(z_-)]$, $z_- = y$ in L. The coordinate conditions $g_{0\mu} = -\delta_{0\mu}$ are preserved by time independent coordinate transformations of the type $\mathbf{y} = J_k\mathbf{x}$, so all the symmetries of the initial conditions are preserved in the time development. In particular the preferred spheres in terms of which the initial conditions were defined have a geometrical significance for the dynamic 4-space as the set of points invariant under the symmetry $z \to J_k z$. Thus the "bridges" do remain recognizable, and the time trajectory of the "neck" can even be located.

Since the "bridges" can be identified, the next question is whether they have any simple properties by which they can be characterized. In the charged case (*22*) the charge of each bridge is a well defined constant of the motion (*14*); in the present, neutral case the most natural thing to look for is some sort of "intrinsic mass" parameter. This quantity is most easily found for metrics of the type indicated by Fig. 3. There one may introduce on any "flange" a set of coordinates in which the metric components approach the flat rectangular

[3] Note that Einstein and Rosen (*13*) proposed changing the sign of the gravitational constant, i.e. of the right hand side of Eq. (5.1g) in order to avoid the excessively high coulomb self-energies which result from the small gravitational (topological) cutoff. This would have a disastrous effect on the equations of motion as is apparent from the fact that the Papapetron solution (*23*) where Newtonian attraction and coulomb repulsion are exactly balanced by taking $m_k = e_k$ would no longer be static with the Einstein-Rosen choice of sign. The self-energy of physical particles in any case must certainly involve quantum effects. Cf. ref. *14*, Eq. (244), and ref. *12*.

[4] It is known that in these coordinates the metric tensor components will develop a singularity within a finite proper time (*24*). This could be merely a singularity of the coordinate system (*25*) but in many cases it turns out to be a real geometrical singularity as in the Schwarzschild solution (R. W. Lindquist, private communication) or the Friedman cosmological models.

metric, $g_{ij} \sim \delta_{ij}$, in the infinite regions out on the flange. Then, using any standard surface integral formula for total energy, one defines the "intrinsic mass" of flange k as the apparent total mass of the systems as viewed from the asymptotic reaches of flange k. For instance, take (*26*)

$$m_k = \oint_k (g_{ij,j} - g_{jj,i})\, dS_i \tag{5.2}$$

where the integral is over an arbitrarily large sphere in an asymptotically rectangular coordinate system for flange k chosen (Lorentz transformations) so that the corresponding integrals for total momentum vanish. Because of the topological peculiarities, the time independence of an integral like (5.2) does not follow by converting it to a volume integral of a pseudotensor t_0^0 and applying a conservation law $t_{\nu,\mu}^{\mu} = 0$. However a direct computation (26) of the time derivative of (5.2) using the field equations only in the asymptotic region shows that m_k is in fact a constant. One might attempt to extend this argument to the case of Fig. 2 as follows: Since the metric given in Section IV is analytic, one can in principle analytically continue it from any small neighborhood to the entire covering space consisting of all the regular points of the functions χ defined in Section III. This is then a manifold with infinitely many flanges (one for each pole in χ), and a constant of the motion m_a is associated with each. The m_a corresponding to poles inside sphere k (of Fig. 4) would then be characteristic properties of sphere k. This argument fails when one remembers that all the poles in χ are images of each other under the symmetries $J_{i_1} J_{i_2} \cdots J_{i_n}$ of the metric; consequently all the m_a are equal and must be interpreted as the total mass of the interacting system.

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