PURE MAGNETIC AND ELECTRIC GEONS *

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Wheeler [1] has proposed the investigation of possible persistent structures of sourceless electromagnetic fields contained by the curvature of space-time associated with their own energy density. Such concentrated persistent structure have been called "geons". Though the investigations (and the term) have hitherto referred only to dynamic (wave) structures, it seemed worthwhile to try the simpler case of a static magnetic or electric field. This note is a report of an investigation in which there was found a rigorous static cylindrically-symmetric solution of the combined sourceless Einstein-Maxwell system showing the persistent local energy-stress concentration which may be taken as the defining characteristic of a geon. With the one added idea that stress-energy acts as gravitating mass, we can describe this solution in classical-mechanical language as follows: *the solution represents a parallel bundle of magnetic or electric flux held together by its own gravitational pull*. The existence of such a simple rigorous solution to the combined gravitational-electromagnetic equations is of considerable physical interest. It can serve as a starting point for a number of further investigations.

The analysis may be generalized in several ways. First the solution does not depend specifically upon the assumption that the seat of the energy-stress distribution is an electromagnetic field. Any polar-cylindrical system of stresses in which there is a tension T_{22} along the principal axis (longitudinal) direction x^2, and a pressure $T_{11} = -T_{22}$ of equal magnitude along the perpendicular radial direction x^1, will have a similar form of solution for the distribution of gravitational potentials and of stress, provided the stress T_{33} along the third-azimuthal-direction $x^3 = \varphi$ is related to the energy density T_{44} by the equation

$$T_{44} = (p-1)T_{33} \quad (p \text{ a constant}) .$$

We can think of space with such a distribution of stresses in it as a medium - whether or not it contains "matter" made of particles. To give it a specific name we call it a "plasm of index p". For such a plasm there will be an equilibrium solution which reduces to that of the static magnetic or electric geon when the index p is set equal to 2.

Second one may consider more general static distributions with longitudinal as well as radial variation. Third, one may keep the full cylinder symmetry, but allow a time variation as well as a radial variation. Such a treatment will allow us to study the evolution of an initially static geon when it is perturbed away from equilibrium.

Besides the value of having a simple rigorous equilibrium solution to study, there is another timely motivation. In the last two or three years there have been discovered among the galaxies [2] what appear to be enormous energy sources - several hundred times larger than any hitherto known. Each such source radiates energies of the order of one earth mass per second. No one has been able to suggest any adequate source of this energy except gravitational collapse, and it may well be that the phenomena are within the collapse regime predicted by general relativity for several different types of systems. Gravitational instability is thus at this time a great issue, and one wants examples to study. It can be expected - as pointed out by J. A. Wheeler in a discussion - that the static solution to be described here is unstable. It provides a simple case in which the dynamics of gravitational collapse may be studied explicitly.

The basic non-linear equations of the Einstein-Maxwell system can be reduced in the case of cylindrical symmetry to a sufficiently simple form to be solved for equilibrium distributions. This is done by following with respect to the gravitational terms the procedure of Weyl [3] and Levi-Civita [4] for static gravitational fields with polar [5] † symmetry. In this procedure, the only assumption additional to that of symmetry is that the stress tensor densities $\mathscr{T}_\nu{}^\mu$ satisfy $\mathscr{T}_1{}^1 + \mathscr{T}_2{}^2 = 0$; fortunately by the well-known balances within the system of Faraday-Maxwell stresses this assumption is auto-

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† Our space-symmetry terminology follows that indicated in ref. [5]. The use of the adjective "axial" as in Weyl and Bach is not consistent with the standard nomenclature for vectors and tensors.

matically valid for a static electric or magnetic field, a fact which oddly enough does not seem to have been noticed or used by Weyl and all those who followed him in exploiting the assumption.

Whatever be the original radial and longitudinal coordinates, canonical ones can be chosen so that only two gravitational potentials have to be determined. We designate these canonical radial and longitudinal coordinates $x^1 = r$, and $x^2 = z$, the aximuthal coordinate $x^3 = \varphi$, and the time coordinate $x^4 = t$. Under these conditions the line element takes the form

$$\mathrm{d}s^2 = \mathrm{e}^{2\psi}\,\mathrm{d}t^2 - \mathrm{e}^{-2\psi}\left(\mathrm{e}^{2\gamma}\,\mathrm{d}r^2 + \mathrm{e}^{2\gamma}\,\mathrm{d}z^2 + r^2\,\mathrm{d}\varphi^2\right). \tag{1}$$

The potential ψ here is formally equal to the Newtonian potential$/c^2$, and satisfies Poisson's equation with the sum $1/c^2(\mathscr{T}_4{}^4 - \mathscr{T}_3{}^3)/r$ - essentially the energy density plus the azimuthal pressure - acting as gravitational mass density. (The contributions of the radial pressure $\mathscr{T}_1{}^1/r$, and of the longitudinal tension, $\mathscr{T}_2{}^2/r$, to the mass just cancel each other):

$$\nabla^2\psi = \frac{k}{2r}\left(\mathscr{T}^4_4 - \mathscr{T}^3_3\right), \tag{2}$$

(k = Einstein's gravitational constant). The "accelerations of gravity", $c^2\psi_r$ and $c^2\psi_z$, in the radial and longitudinal directions respectively, obey the equations

$$\left(\mathscr{T}^4_4 - \mathscr{T}^3_3\right)\psi_r = -\mathscr{T}^2_{2/1} + \mathscr{T}^2_{1/2} - \frac{1}{r}\mathscr{T}^3_3 \qquad (\psi_r = \partial\psi/\partial r)\,, \tag{3}$$

$$\left(\mathscr{T}^4_4 - \mathscr{T}^3_3\right)\psi_z = \mathscr{T}^2_{1/1} - \mathscr{T}^2_{2/2} \qquad (\psi_z = \partial\psi/\partial z)\,, \tag{3'}$$

where, as indicated, differentiation with respect to a variable is labeled either by a *subscript letter* without a preceding slash or by a slash followed by the appropriate number index. These correspond to the stress-balance equations of Newtons mechanics, with $(1/c^2)(\mathscr{T}_4{}^4 - \mathscr{T}_3{}^3)/r$ again representing the mass density. Once ψ has been determined, the second potential γ is determined by quadratures as soon as the stress densities $\mathscr{T}_2{}^2$ and $\mathscr{T}_4{}^1$ are known:

$$\gamma_r = r(\psi_r^2 - \psi_z^2) - \mathscr{T}^2_2\,, \tag{4}$$

$$\gamma_z = 2r\psi_r\psi_z - \mathscr{T}^1_2\,. \tag{4'}$$

Actually the stresses are to be obtained from the fields satisfying Maxwell's equations in the curved spacetime, therefore themselves depending upon the potentials ψ and γ. In principle, various static equilibrium solutions for the Einstein-Maxwell equations coupled in this way can be calculated, with certain resulting field distributions.

The simplest interesting equilibrium solution is that of a cylindrically symmetric pure magnetic field (or, alternatively, pure electric field) pointing along the z direction and a function only of the radial variable r. The Einstein-Maxwell equations in this case reduce to

$$\frac{1}{r}\frac{\mathrm{d}(r\psi_r)}{\mathrm{d}r} = k\mathscr{T}/r \tag{5}$$

$$(2\mathscr{T}/r)\frac{\mathrm{d}\psi}{\mathrm{d}r} = -\frac{\mathrm{d}(\mathscr{T}/r)}{\mathrm{d}r}\,, \tag{6}$$

$$\gamma_r = r\psi_r^2 + k\mathscr{T}. \tag{7}$$

Eq. (6), which has the simple significance of an equation of hydrostatic equilibrium, yields

$$\mathscr{T}/r = \tfrac{1}{2}B_0^2\,\mathrm{e}^{-2\psi} \tag{6'}$$

(B_0 an arbitrary constant). At this point we do not yet have to go into details as to how the stress or energy density $\mathscr{T}$ in eq. (6'), in e.g., the case of magnetic geon, is partitioned into a B (magnetic flux field) part and an H (current potential) part. In this simple case one can think in terms of a Newtonian model which - with the one supplementary idea that mass (gravitating) = energy/c^2 + pressure/c^2 - employs only the equations of hydrostatic equilibrium, eq. (6) and Poisson's equation, eq. (5). These determine an equation for ψ - in ordinary cylindrical coordinates - of exactly the same form as that given by general relativity in canonical coordinates.

This key equation, found by combining eqs. (5) and (6'), can be written

$$\frac{1}{\rho}\frac{\mathrm{d}(\rho\psi_\rho)}{\mathrm{d}\rho} = 4\mathrm{e}^{-2\psi} \tag{8}$$

where

$$\rho = \frac{r}{\bar{a}}, \quad \bar{a} = \frac{1}{B_0}\left(\frac{8}{k}\right)^{\frac{1}{2}} = \frac{1}{B_0}\frac{c^2}{(\pi G)^{\frac{1}{2}}} = \frac{1.108\times10^{24}}{B_0}\ \mathrm{cm}\,.$$

Here $\bar{a}$ is a length parameter governing the range (in r-coordinate measure) of the distribution in the radial direction. Introducing the new independent variable s and the new dependent variable θ by the consecutive definitions

$$\rho = \mathrm{e}^s, \qquad \theta = \psi - s$$

the differential equation takes the form

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = 4\mathrm{e}^{-2\theta} \tag{8'}$$

and this has the first integral

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 = q^2 - 4\mathrm{e}^{-2\theta}\,, \tag{8''}$$

where

$$q^2 = 4e^{-2\psi_1} + (\psi_{\rho 1} - 1)^2, \qquad \psi_1 \equiv \psi|_{\rho=1},$$
$$\psi_{\rho 1} \equiv \psi_\rho|_{\rho=1} \quad (s=') \,. \tag{8'''}$$

Substitution of the integration variable $\xi = \frac{1}{2}ke^{\theta}$, quadrature and inversion, yields the solution

$$\psi = \ln \{\tfrac{\rho}{q}[(\beta\rho)^q + \frac{1}{(\beta\rho)^q}]\} \,, \tag{9}$$

where β is a constant of integration. Now if we require only that ψ be regular at $\rho = 0$, this determines q to be equal to 1 and, it is then evident, β is a trivial scale factor which can be set equal to 1. Thus eq. (8), subject only to the condition that ψ be regular at $\rho = 0$, has the unique solution

$$\psi(\rho) = \ln (1 + \rho^2) \,. \tag{9'}$$

Eq. (7) then yields

$$\gamma(\rho) = 2 \ln (1 + \rho^2) \,. \tag{10}$$

The line element, eq. (1), then becomes

$$ds^2 = (1+\rho^2)^2 dt^2 - (1+\rho^2)^2 (dr^2 + dz^2) - \frac{r^2}{(1+\rho^2)^2} d\varphi^2 \,. \tag{11}$$

The radial *proper distance* out to a given value of ρ is

$$l_\rho = \bar{a} \int_0^\rho (1 + \rho^2)\, d\rho = (\rho + \frac{\rho^3}{3})\, \bar{a} \,, \tag{12}$$

and the circumference of a circle with radius given by ρ and perpendicular to the axis is

$$\text{circumference}_\rho = \frac{2\pi\rho}{1+\rho^2}\, \bar{a} \,.$$

This has a maximum at $\rho = 1$ and goes to zero hyperbolically for large ρ, so that as one goes out radially the geometry obtained is like what one would find if one moved along the stem of a wineglass towards the narrowing end.

From eq. (9') we see that the significance of the parameter $\bar{a}$ is that it is the r-coordinate interval out to the radius $\rho = 1$ where ψ has a point of inflexion, i.e., where the acceleration of gravity ψ_ρ is a maximum - what one may call the "surface" of the static geon. The corresponding proper distance is $\frac{4}{3}\, \bar{a}$, and the proper circumference at the surface $= \pi\, \bar{a}$.

We turn now to the specific interpretation of the static geon as a magnetic (alternatively electric) structure. The tensor density of stress or energy density is given in the most general (metric-independent) form of Maxwell theory for this case by

$$\sqrt{g}\, T = \mathscr{T} = -\mathscr{T}^1_1 = \mathscr{T}^2_2 = -\mathscr{T}^3_3 = \mathscr{T}^4_4 = \tfrac{1}{2} H^{31} B_{31} \,. \tag{13}$$

The general significance of the $H^{\alpha\beta} \equiv (\boldsymbol{H}, \boldsymbol{D})$ and $B_{\mu\nu} \equiv (\boldsymbol{B}, \boldsymbol{E})$ is as follows: Electromagnetism can be founded on the two basic principles of conservation of charge-current j^α, and conservation of magnetoelectric flux $B_{\mu\nu}$. From charge-current conservation it follows that j^α is representable as the divergence of a contravariant bivector density $H^{\alpha\beta}$

$$j^\alpha = H^{\alpha\beta}/\beta \,, \tag{14}$$

in analogy to the manner in which from flux conservation it follows that $B_{\mu\nu}$ is representable as the curl of a vector potential. The $H^{\alpha\beta}$ plays the role of a "charge-current potential".

How the expression for $\mathscr{T}$ in eq. (6') is to be partitioned into H and B parts, is given immediately by the relevant Maxwell equation

$$(H^{31})/1 = 0 \,, \tag{14'}$$

or *H is everywhere a constant*, which we choose equal to B_0. Thus from eq. (6') there follows:

$$B_{31} = B_0\, r\, e^{-2\psi} \,, \tag{15}$$

so that the *physical component B* of the magnetic flux distribution, which is given by the covariant B_{31} divided by the x^3 and x^1 scale factors, is

$$B = (g_{33} g_{11})^{-\frac{1}{2}} \quad B_{31} = B_0\, e^{-\gamma(r)} \,. \tag{15'}$$

We note that *it is only because of the nonuniformity of the metric that B, and also the stress energy density, shows a concentrated distribution.*

Eq. (15) is just the same as would be obtained from the basic postulate of the theory of electromagnetism in gravitational fields, according to general relativity:

$$H^{\alpha\beta} = \sqrt{g}\; B^{\alpha\beta} \equiv \sqrt{g}\, g^{\alpha\mu} g^{\beta\nu} B_{\mu\nu} \,, \tag{16}$$

which may be described as defining the "permeability" and "dielectric constant" of space in the presence of gravitation. When applied to eq. (14') this gives

$$(H^{31})_r = (\sqrt{g}\, g^{33} g^{11}\, B_{31})_r = (\frac{e^{2\psi}}{r} B_{31})_r = 0 \,. \tag{14''}$$

Thus, in this simple case, one need not make explicitly the general relativity postulate for the electromagnetic field equations in curved space-time. The flux distribution follows directly from the equation for the balance of Maxwell stress against the force of gravity, together with the general metric-independent equations of electromagnetism *.

* The Newtonian model, in which gravitation is an extra entity added onto a flat-space background, comprises an alternative description of the phenomena to that of general relativity in which gravitation is an intrinsic aspect of the curved geometry. In either case we should think of the space as filled with a medium - the "magnetic

From eqs. (15') and (10) we have

$$B = \frac{B_0}{(1+\rho^2)^2} \,. \tag{17}$$

On the axis the flux field strength is B_0; at the "surface" $r = \bar{a}$ or $\rho = 1$ it falls to $\frac{1}{4}$ and, at $\rho = 2$, to $\frac{1}{25}$ its central value. We note that with $B_0 = 10^{-5}$ gauss, the order of magnitude of magnetic fields found in interstellar regions [6], the range $\bar{a}$ of a static magnetic geon is 3.5×10^{29} cm. (When gaussian units are used, we must replace B_0 in the Heaviside-Lorentz units which we have been using hitherto by $B_0(\mathrm{gauss})/\sqrt{4\pi}$.) With $B_0 = 34\,000$ gauss corresponding to the very large magnetic fields observed in extreme magnetic stars [7], the range $\bar{a}$ equals 1.2×10^{20} cm, or about four million times the diameter of the earth's orbit about the sun.

The total magnetic flux Φ is obtained by integrating the physical component of B over the entire physical area perpendicular to the z axis:

$$\Phi_{\mathrm{equil.}} = 2\pi B_0 \int_0^\infty \frac{1}{(1+\rho^2)^2} \frac{\bar{a}\rho}{1+\rho^2} (1+\rho^2)\, \bar{a}\, \mathrm{d}\rho = \pi\bar{a}^2 B_0 = \sqrt{\pi}\frac{c^2}{G^{\frac{1}{2}}}\bar{a}. \tag{18}$$

We define the "effective electromagnetic energy" E_{em} per unit physical length in the z direction by the integral of the physical magnetic energy density

$$T_{44}/g_{44} = T_4^4 = \mathcal{T}_4^4/\sqrt{|g|} = \frac{1}{2}\,\frac{B_0^2}{(1+\rho^2)^4}$$

over the volume spanned by the entire physical area perpendicular to the z-axis and by the physical distance corresponding to the z-interval 0 to 1. The integrand differs from that of eq. (18) only by the additional factor $B_0/(1+\rho^2)\,\mathrm{d}z$ and we obtain

$$E_{\mathrm{em}} = \frac{\pi\bar{a}^2 B_0^2}{4} = \frac{\Phi B_0}{4} = \frac{\Phi^2}{4\pi\bar{a}^2} = \frac{c^4}{4G}. \tag{19}$$

Thus we find the "electromagnetic mass per unit length" to be

$$m_{\mathrm{em}} = \frac{c^2}{4G} = 3.369 \times 10^{27} \text{ g/cm} \,. \tag{20}$$

The stability of the system and the bearing on the gravitational collapse problem will be discussed in a subsequent communication.

If we go from the magnetic (or electric) case to the more general case of an index-p plasm geon we find that we can integrate the dynamical equations in the same way as before. Only slight generalisations of the forms for $\mathcal{T}/r$, $\bar{a}$, and ψ arise:

$$\mathcal{T}/r = \tfrac{1}{2}B_0^2\, \mathrm{e}^{-p\psi}$$

$$\bar{a} = \frac{1}{B_0}\left(\frac{16/p}{k}\right)^{\frac{1}{2}}$$

$$\psi = \frac{2}{p} \ln\left[\sqrt{\tfrac{k}{2}}(1+\rho^2)\right]$$

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plasm". If we want to extend the flat-space model to include the description of the magnetic field distribution we must arbitrarily endow the plasm with permeability $e^{2\psi}/r$, as can be seen from eq. (14").

References

1) J. A. Wheeler, Phys. Rev. 97 (1955) 511.
See also F. J. Ernst, Phys. Rev. 105 (1957) 1665.
2) G. R. Burbidge, Paris Symp. on Radio Astronomy, Stanford University (1959);
C. K. Seyfert, Astrophys. J. 97 (1963) 28,
F. Hoyle and W. A. Fowler, Nature 197 (1963) 533
3) H. Weyl, Ann. Physik 54 (1917) 117; 59 (1919) 185;
R. Bach, Math. Z. 13 (1922) 134.
4) T. Levi-Civita, Rend. Acc. Lincei 28 (1919) 101.
5) M. A Melvin, Revs. Modern Phys. 32 (1960) 477 fig. 5; 28 (1956) 18 fig. 1.
6) R. D. Davies et al., Nature 187 (1960) 1088.
7) H. W. Babcock, Astrophys. J. 132 (1960) 521.

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EVIDENCE FOR ROTATIONAL STATES UP TO 14+ IN Dy^{160}

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Rotational states of the nuclide Dy^{160} up to an angular momentum $10\,\hbar$ have recently been studied by Morinaga and Gugelot [1] by measuring the gamma rays following the $(\alpha, 4n)$ reaction on Gd^{160}. Following their work an experiment has been carried out to search for states with higher angular momentum in order to check the prediction of Mottelson and Valatin [2] about the upper limit of

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