On the analogy between electromagnetism and turbulent

hydrodynamics

Haralambos Marmanis

Department of Theoretical and Applied Mechanics University of Illinois at Urbana-Champaign Urbana, Illinois 61801-2935, USA.

February 1, 2008

Abstract

In this note, we propose an exegesis of the Maxwell equations for electromagnetism. We begin with an analogy between the homogeneous Maxwell equations and the equations needed to describe the vorticity field of an incompressible inviscid fluid. We suggest that the inhomogeneous equations are analogous to two equations valid in turbulent hydrodynamics. Once the analogy is completed we give the mechanical analogue of the Poynting vector and we explain the influence of a long solenoid on the motion of a charged particle.

1 Introduction

The nature of the electromagnetic field has been a puzzle since the foundations of the subject by Maxwell and even earlier (Whittaker, 1951). Scientists have tried to imagine the electric and magnetic field in terms of other physical circumstances where insight is provided by the phenomenon itself. In order to do this people had postulated the subsistence of a very fine medium, the so-called 'aether'. This seemed to be a legitimate act, because the electromagnetic theory based on Maxwell's equations predicts the existence of waves. However, no experimental proof that the aether really exists could be found; the negative result of the Michelson-Morley experiment has been verified many times with various modifications. Thus the concept of aether has been abandoned. Nevertheless the problem of what is the nature of the electromagnetic field cannot be said to have been dealt with completely. What has happened is that physicists simply assumed the existence of the fields and postponed the resolution of the problem. Feynman (1964) devotes a whole section of his lectures on this matter.

In the sequel we propose an analogy between the vector fields of electromagnetism and the hydrodynamical fields of a turbulent fluid flow. In this analogy the field quantities are envisaged as the fundamental entities whereas the charges and currents are byproducts of the former. In that sense this work is a reminiscence of Faraday's theory were electric charges were to be regarded as epiphenomena having no independent or substantial existence. The electric current, accordingly, was to be viewed not as manifesting the flow of actually existing electrical fluids but rather as constituting an "axis of power", reflecting the dynamics of "a certain condition and relation of electrical forces." (Siegel, 1991). Einstein suggested that special kinds of non-linear fields might exist, having modes of motion in which there would be pulse-like concentrations of fields, which would stick together stably, and would act almost like small moving bodies. Heisenberg had a similar point of view. Misner (1956) proposed that electromagnetism was a property of curved empty space. Rainich (1925) already long before had shown under what conditions a curvature of space-time can be regarded as due to electromagnetic field. Finally Misner and Wheeler (1957) presented gravitation, electromagnetism, unquantized charge, and mass as properties of curved empty space. Furthermore, the particle theory developed by M.Vigier and his collaborators (Hillion, Halbwachs, Lochak), is also along the same line of thought but is focused primarily on the quantum mechanical phenomena. It is the purpose of this paper to propound that the space-time manifold can be represented as the manifold of a turbulent fluid flow. The latter is possible to initiate profound changes about the interpretation of electromagnetism and quantum mechanics.

The analogical approach presented here, it will provide us with a mechanical representation, which will be helpfull in the task of understanding the nature of electromagnetism by using more familiar terms. In section 2, we present the Maxwell equations and the equations of hydrodynamics for an inviscid incompressible fluid, in terms of the velocity and the Bernoulli energy function, and in terms of the vorticity and the Lamb vector. The resemblance between these equations will be the starting point of our analogy. In section 3 a correspondence between electromagnetic and hydrodynamical quantities is established, where the vector and scalar potential of the electromagnetic field appear to be as 'real' as the magnetic and the electric field. We argue that all the electromagnetic variables can be interpreted as hydrodynamical variables of a turbulent flow field. In section 4, we discuss the origin of the Poynting vector and the effect of a long solenoid on the motion of a charged particle passing nearby, according to the mechanical analogue.

2 The equations of Maxwell and of hydrodynamics

The Maxwell equations have different coefficients according to the system of units that is chosen. If we choose the electrostatic (esu) system, the Maxwell equations for sources in vacuum can be written as follows (Jackson, 1975)

$$\nabla \cdot \mathbf{B} = 0, \qquad (1)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \qquad (2)$$

$$\nabla \cdot \mathbf{E} = 4\pi \,\rho \,, \tag{3}$$

$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \nabla \times \mathbf{B} - 4\pi \mathbf{J}, \qquad (4)$$

where $\mathbf{B}(\mathbf{x}, t)$ is the magnetic field, $\mathbf{E}(\mathbf{x}, t)$ is the electric field, $\rho(\mathbf{x}, t)$ is the charge density, and $\mathbf{J}(\mathbf{x}, t)$ is the current. Moreover, the fields \mathbf{B} and \mathbf{E} form a six-component system, but not all of these components are entirely independent; this is implicitly expressed in equations (1) and (2). In other words, we can find a more economical description of the fields with fewer components. This is done by introducing the vector potential $\mathbf{A}(\mathbf{x}, t)$ and the scalar potantial $\phi(\mathbf{x}, t)$. It is easy to see that the substitutions

$$\mathbf{B} = \nabla \wedge \mathbf{A}, \qquad (5)$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \tag{6}$$

yield (1) and (2) as identities.

On the other hand, in hydrodynamics the flow of an incompressible inviscid fluid, of constant density ρ , is governed by the Euler equations. The latter can be written in the following form

$$\frac{\partial \mathbf{u}}{\partial t} = -\left(\mathbf{w} \wedge \mathbf{u}\right) - \nabla\left(\frac{p}{\rho} + \frac{u^2}{2}\right),\tag{7}$$

where $\mathbf{u}(\mathbf{x},t)$ is the velocity field, $\mathbf{w}(\mathbf{x},t)$ is the vorticity field, and $p(\mathbf{x},t)$ is the pressure field. The vector product of the vorticity with the velocity is called the Lamb vector, and it will be denoted as

$$\mathbf{l}(\mathbf{x},t) = \mathbf{w} \wedge \mathbf{u}.$$

The quantity in the parenthesis of the second term in the r.h.s. of (7) is called the Bernoulli energy function or total 'head', and will be denoted as

$$\Phi(\mathbf{x},t) = \frac{p}{\rho} + \frac{u^2}{2}.$$

The Euler equations are accompanied by the continuity equation which, for an incompressible fluid, is reduced to

$$\nabla \cdot \mathbf{u} = 0. \tag{8}$$

The vorticity field, defined as

$$\mathbf{w} = \nabla \wedge \mathbf{u},\tag{9}$$

obeys the following two equations

$$\nabla \cdot \mathbf{w} = 0, \tag{10}$$

$$\frac{\partial \mathbf{w}}{\partial t} = -\nabla \wedge \mathbf{l}. \tag{11}$$

Equation (11) can be easily derived by taking the *curl* of (7).

3 The hydrodynamical analogy

Now, by comparing equations (1) and (2) with equations (10) and (11), respectively, we observe that are the same if **B** corresponds to **w**, and **E** corresponds to **l**. In addition, by comparing equations (5) and (6) with (9) and (7), respectively, we see that the analogy can be extended, so that it includes the potentials as well. In particular, the comparison suggests that the vector potential **A** corresponds to the velocity field **u**, and the scalar potential ϕ to the Bernoulli energy function Φ . The complete correspondence, between the electromagnetic fields and their hydrodynamical analogues, can be summarized as follows

Electromagnetism	Hydrodynamics
$\mathbf{A}(\mathbf{x},t)$	$\mathbf{u}(\mathbf{x},t)$
$\phi(\mathbf{x},t)$	$\Phi(\mathbf{x},t)$
$\mathbf{B}(\mathbf{x},t)$	$\mathbf{w}(\mathbf{x},t)$
$\mathbf{E}(\mathbf{x},t)$	$\mathbf{l}(\mathbf{x},t)$

From the above analogy, the magnetic field appears to have a rotatory character. This was known to Maxwell himself, since the very begining of his investigations. He was aware of the fact that magnetism produces, at least, one rotatory effect, i.e. the rotation of the plane of polarized light when transmitted along the magnetic lines (Faraday rotation). Even though later he seemed to decline his theory of molecular vortices, he never did give up the belief that there was a real rotation going on in the magnetic field. By applying the divergence operator on both sides of (7), we get

$$\nabla \cdot \mathbf{l}(\mathbf{x}, t) = -\nabla^2 \Phi.$$
(12)

One can also express the divergence of the Lamb vector in the form

$$\nabla \cdot \mathbf{l}(\mathbf{x}, t) = \mathbf{w} \cdot \mathbf{w} + \mathbf{u} \cdot \nabla^2 \mathbf{u}$$
(13)

$$= \mathbf{w} \cdot \mathbf{w} - \mathbf{u} \cdot \nabla \wedge \mathbf{w} . \tag{14}$$

In general, the Laplacian of Φ will be a function of position and time, so let us call this function the *turbulent charge density* and denote it as $n(\mathbf{x}, t)$, then we can write

$$\nabla \cdot \mathbf{l}(\mathbf{x},t) = -\nabla^2 \Phi \equiv 4\pi \, n(\mathbf{x},t) \,, \tag{15}$$

where the 4π proportionality factor is introduced for later convenience. It is clear that the function $n(\mathbf{x}, t)$ will be significantly greater in a turbulent flow than in a laminar flow. For, in a turbulent flow, the enstrophy is larger, mainly due to the stretching of vortex filaments, the velocity is larger and the flexion vector is also larger. Thus the designation of $n(\mathbf{x}, t)$ as *turbulent* is justified. On the other hand equation (15) reminds the Poisson equation of electromagnetism, which connects the electric potential $\phi(\mathbf{x}, t)$ and the electric charge density $\rho(\mathbf{x}, t)$ as follows

$$\nabla^2 \phi = -4\pi \,\rho(\mathbf{x},t)\,. \tag{16}$$

Because of this resemblance we envision $n(\mathbf{x}, t)$ as a *charge* density.

Let us note that there are two points of view about the interpretation of the above equation which disclose two points of view about the nature of physics, in general. According to the first interpretation the space-time continuum is not inherently connected with the existence of the fields and the particles. These entities have a right of existence on their own. The description of physical phenomena requires the space-time arena, the fields and the particles. The second interpretation according to Misner & Wheeler (1957) can be described as follows: "There is nothing in the world except empty curved space. Matter, charge, electromagnetism, and other fields are only manifestations of the bending of space. Physics is geometry". Herein we adopt the later point of view, thus in equation (16) the electric potential is the cause and the electric charge the effect. This is plain for equation (14) where we introduced the notion of turbulent charge density as a result of the curvature of the Bernoulli energy function.

At this point, one could say that the nature of the electromagnetic field is that of an incompressible inviscid turbulent fluid flow, which in addition to the known equations described above, obeys also the following equation

$$\frac{\partial \mathbf{l}}{\partial t} = c^2 \, \nabla \wedge \mathbf{w} - 4\pi \, \mathbf{I} \,, \tag{17}$$

where $\mathbf{I}(\mathbf{x}, t)$ is given as a function of $n(\mathbf{x}, t)$, through the relation

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{I} = 0.$$
(18)

J. J. Thomson (1931) showed that in the case of a homogeneous and isotropic flow with a large number of vortex filaments, the Lamb vector obeys equation (17) with $\mathbf{I} = 0$; as required by homogeneity. The idea of delineating turbulent flows as a large irrotational region which occupies most of the flow field and narrow regions of concentrated vorticity came almost two decades later (Onsager, 1949; Kida, 1975). Numerical evidence supporting the above picture, especially for the small scales, has been presented recently (Siggia, 1981; Kerr, 1985; Vincent & Meneguzzi, 1991; Jiménez *et al.*, 1993; Kida, 1993).

4 Two old questions

One special characteristic, that didtinguishes our analogy from previous attempts, is the mechanical exegesis of a few important questions in electromagnetism. We will illustrate this by two concrete examples. The first case is the derivation of the Poynting vector

$$\mathbf{S} = c^2 \mathbf{E} \wedge \mathbf{B} \,. \tag{19}$$

The latter gives the rate at which the field energy moves around in space.

In general, the conservation of energy for electromagnetism is written as

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{S} = \mathbf{E} \cdot \mathbf{J}, \qquad (20)$$

where U is given by

$$U = \frac{1}{2} \mathbf{E} \cdot \mathbf{E} + \frac{c^2}{2} \mathbf{B} \cdot \mathbf{B}.$$
(21)

The above equation corresponds to the evolution equation of the enstrophy in a fluid containing a large number of vortex filaments. Let us examine the case of fields in vacuum with no charges, i.e. the homogeneous case considered by Thomson, then $\rho = 0$ and $\mathbf{J} = 0$ in (3) and (4) respectively. By multiplying equation (11) with \mathbf{w} , we get

$$\frac{\partial}{\partial t} \frac{\mathbf{w}^2}{2} = -\mathbf{w} \cdot \nabla \wedge \mathbf{l}.$$
(22)

However

$$-\mathbf{w}\cdot\nabla\wedge\mathbf{l} = -\nabla\cdot(\mathbf{l}\wedge\mathbf{w}) + \mathbf{l}\cdot\nabla\wedge\mathbf{w}, \qquad (23)$$

therefore (23) can be written as

$$\frac{\partial}{\partial t} \frac{\mathbf{w}^2}{2} + \nabla \cdot (\mathbf{l} \wedge \mathbf{w}) = \mathbf{l} \cdot \nabla \wedge \mathbf{w}.$$
(24)

The term on the r.h.s. has been called the turbulence creating term and its significance to turbulence theory has been investigated by Theodorsen (1952). Substituting the equation (19) derived by Thomson (1931) and rederived by Marmanis (1996), we have

$$\frac{\partial}{\partial t}\frac{\mathbf{w}^2}{2} + \frac{1}{c^2}\frac{\partial}{\partial t}\frac{\mathbf{l}^2}{2} + \nabla\cdot(\mathbf{l}\wedge\mathbf{w}) = 0$$
(25)

or

$$\frac{\partial}{\partial t} \left(\frac{c^2 \mathbf{w}^2}{2} + \frac{\mathbf{l}^2}{2} \right) + c^2 \nabla \cdot (\mathbf{l} \wedge \mathbf{w}) = 0, \qquad (26)$$

which according to our analogy is exactly (21) if the Poynting vector corresponds to

$$\mathbf{S} = c^2 \mathbf{l} \wedge \mathbf{w} \,. \tag{27}$$

Notice that if this is true the Poynting vector is aligned with the vector potential. It is also interesting that the electrostatic energy has as mechanical analogue a special combination of the three most important quantities in any turbulent flow, namely the kinetic energy, the enstrophy and the magnitude of the helicity density. In particular, we have

$$\mathbf{l}^2 = \mathbf{w}^2 \, \mathbf{u}^2 - (\mathbf{u} \cdot \mathbf{w})^2 \,. \tag{28}$$

Furthermore the Poynting vector is given also by these important quantities as

$$\mathbf{l} \wedge \mathbf{w} = (\mathbf{w}^2)\mathbf{u} - (\mathbf{u} \cdot \mathbf{w})\mathbf{w} \,. \tag{29}$$

The above equation tells us a little more about the energy flow according to our analogy. It seems that the energy flow increases in the direction of the vector potential when the magnetic energy increases and decreases in the direction of the magnetic field when the electromagnetic helicity decreases. A special case is the two-dimensional flows. In the latter case the helicity is zero and the energy flow is given solely in terms of the enstrophy and the velocity. However, the enstrophy in a two-dimensional inviscid flow is conserved, therefore any change of the energy flow will be due to the velocity.

Our second example is the influence of a long solenoid on the motion of charged particles (Feynman, 1964). Classically, i.e. not quantum mechanically, the force depends only on **B**. That is, in order to know that the solenoid is carrying current, the particle must go through it. Quantum mechanics predicts an influence on the motion given in terms of the magnetic change in phase. According to the picture proposed herein, the influence of the field is explained as follows. As the particle approaches the solenoid, it will be exposed to the flow field created by it. The latter can be approximated by that of a vortex tube. Therefore the influence will be more or less significant according to the value which describes the strength of the vortex tube. This quantity is just the integral of the velocity on a path surrounding the vortex tube, that is

$$\oint \mathbf{u} \cdot d\mathbf{s} , \qquad (30)$$

where $d\mathbf{s}$ is the differential arc element along the path. This result is in accordance with the result of quantum mechanics, i.e.

$$\delta = \frac{q}{\hbar} \oint \mathbf{A} \cdot d\mathbf{s} , \qquad (31)$$

where δ is the magnetic change in phase of the wave function attributed to the particle, q is the charge and h the Plank's constant. It is also remarkable that the vector potential of a long solenoid behaves in exact the same way as the velocity field of a vortex filament, i.e. they both decrease as r^{-1} with increasing r.

5 Discussion

We decsribed the analogy between the equations of electromagnetism and the equations of turbulent hydrodynamics. There is a one-to one correspondence between quantities in the two cases. This leads us to interpret classical electromagnetism as a turbulent flow field. Of course, such a statement has consequences proportional to its generality and its vagueness. For example, it has long been recognized that turbulent fluid flows have an intermittent character, especially at the small scales (Batchelor & Townsend, 1949). The separation of the various sources of intermittency is insufficiently recognized in the literature (Kraichnan, 1991). For reasons that we explain elsewhere (Marmanis, 1993), we claim that the intermittency of turbulent fluid flows should be accredited to intermittency effects intrinsic to the dissipation range. Therefore it is tempting to propose a new set of equations for electromagnetism. The modified Maxwell equations will read

$$\nabla \cdot \mathbf{B} = 0, \qquad (32)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} + \hbar \nabla^2 \mathbf{B}, \qquad (33)$$

$$\nabla \cdot \mathbf{E} = 4\pi \,\rho \,, \tag{34}$$

$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \nabla \times \mathbf{B} - 4\pi \mathbf{J} + \hbar \nabla^2 \mathbf{E}, \qquad (35)$$

where \hbar is Plank's constant per unit mass.

In conclusion, an analogy between electromagnetism and hydrodynamics is presented. The analogy by Thomson (1931) has been very similar to ours as far as the construction of equation (4) is concerned. However, the interpretation given herein is different than Thomson's interpretation. The early ideas of Faraday, Maxwell, Rowland and others about the nature of electromagnetism are now illuminated more than ever; under the new perspective of turbulent hydrodynamics.

Acknowledgements

The author wants to acknowledge the valuable communication with Dr. R.H. Kraichnan and Prof. A. Leggett, as well as the helpful discussions with Prof. R. Adrian, Prof. H. Aref, Prof. G.I. Barenblatt, Dr. Meleshko, and Prof. R. Moser.

References

- Batchelor, G.K. & Townsend, A.A. 1949 The nature of turbulent motion at large wave numbers, Proc. R. Soc. Lond A 199, 238-255.
- [2] Feynman, R.P. 1964 The Feynman Lectures on Physics, vol.II, Addison-Wesley
- [3] Jiménez, J., Wray, A. A., Saffman, P. G. & Rogallo, R.S. 1993 The structure of intense vorticity in homogeneous isotropic turbulence, J. Fluid. Mech. 255, 65-90
- [4] Kerr, R.M. 1985 Higher-order derivative correlation and the alignment of small-scale structures in isotropic turbulence, J. Fluid. Mech. 153, 31-58
- [5] Kida, S. 1975 Statistics of a system of line vortices, J. Phys. Soc. Jpn. 39, 1395-1404
- [6] Kida, S. 1993 Tube-like structures in turbulence, Lecture Notes in Numerical Applied Analysis 12, 137-159
- [7] Kraichnan, R.H. 1991 Turbulent cacsade and intermittency growth, Proc. R. Soc. Lond.
 A 434 65-78

- [8] Misner, C.W. & Wheeler, J.A. 1957 Gravitation, Electromagnetism, Unquantized charge, and mass as properties of Curved Empty space Ann. of Phys. 2, 525-660
- [9] Onsager, L. 1949 Statistical hydrodynamics Nuovo Cimento, suppl. to vol. 6, 279-287
- [10] Rainich, G.Y. 1925 Trans. Am. Math. Soc., 27, 106
- [11] Siegel, D.M. 1991 Innovation in Maxwell's electromagnetic theory, Cambridge: Cambridge University press
- [12] Siggia, E.D. 1981 Numerical study of small scale intermittency in three-dimensional turbulence, J. Fluid. Mech. 107, 375-406
- [13] J. J. Thomson 1931 On the analogy between the electromagnetic field and a fluid containing a large number of vortex filaments. *Phil. Mag.* S.7. Vol.12 No.80 Nov., 1057-1063
- [14] Vincent, A. & Meneguzzi, M. 1991 The spatial structure and statistical properties of homogeneous turbulence, J. Fluid. Mech. 225, 1-25