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The binding of cosmological structures by massless topological defects

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ABSTRACT

Assuming spherical symmetry and weak field, it is shown that if one solves the Poisson equation or the Einstein field equations sourced by a topological defect, i.e. a singularity of a very specific form, the result is a localized gravitational field capable of driving flat rotation (i.e. Keplerian circular orbits at a constant speed for all radii) of test masses on a thin spherical shell without any underlying mass. Moreover, a large-scale structure which exploits this solution by assembling concentrically a number of such topological defects can establish a flat stellar or galactic rotation curve, and can also deflect light in the same manner as an equipotential (isothermal) sphere. Thus, the need for dark matter or modified gravity theory is mitigated, at least in part.

Key words: (cosmology:) dark matter – galaxies: clusters: intracluster medium – galaxies: structure – gravitational lensing: weak.

1 INTRODUCTION

The nature of dark matter (DM), defined specifically in this letter as an unknown component of the cosmic substratum responsible for the extra gravitational field that binds galaxies and clusters of galaxies, has been an enigma for more than a century since the pioneering papers of Kapteyn (1922) and Oort (1932), Spitzer observation (Morales-Salgado et al. 2022), and *Gaia* observation (Battaglia & Nipoti 2022) on galactic scales, and Zwicky (1933) and Chaurasiya et al (2024) on the scale of clusters. Although the laboratory search for DM is an ongoing effort, see e.g. Aalbers et al (2023) and Cebrian (2023), alternative theories of gravity were also proposed to enable a considerably smaller amount of matter than is required by Newtonian gravity (and General Relativity in the weak field limit) to produce the same gravitational field strength, see Nojiri et al. (2017) and references therein.

The purpose of this paper is to revisit the logical steps that connect DM existence to the standard theory of gravity. It turns out that, in the weak field limit where Newtonian theory suffices, there are two types of impulsive source term $\rho(r)$ for the gravitational Poisson equation, which can lead to an attractive central force as the solution. The first has magnitude $\propto 1/r^2$ and is associated with an underlying spherically symmetric mass distribution, while the second has magnitude $\propto 1/r$ and is associated with no underlying mass if space is isotropic. The first is evidently Newton's law of universal gravitation, but our interest here is in the second solution because not only has it been ignored, but it can also yield a flat stellar rotation curve in an equipotential environment (apart from a logarithmic increase of Φ with radius r) in the absence of mass.

Of course, the availability of a second solution, even if it is highly suggestive, is not by itself sufficient to discredit the DM hypothesis – it could be an interesting mathematical exercise at best. One naturally queries (a) the physical meaning of the singular sources of the type which give rise to the solution; (b) the stability of the solution; (c) the robustness of the model in accounting for a variety of flat rotation curve of galaxies and velocity dispersion in clusters of galaxies; and (d) observational evidence (beyond the virialized motion of stars in a galaxy and galaxies and hot gas in clusters) of such sources.

We will address (a) in the next section, and (b) in Section 3 where we will demonstrate stability by solving the Einstein field equations to show that the metric tensor for the new solution is time-independent. Next, (c) and (d) are the subjects of Sections 6 and 7 where we shall argue that the properties of the massless singular shell sources capable of driving an attractive central force field are specified by several parameters having values to be determined observationally; moreover, the increasing frequency of sightings of ring and shell like formation of galaxies in the Universe lends evidence to the type of source being proposed here. Beyond that, we will also show in Section 5 that, in respect of (c), the proposed model can reproduce the bending of light, hence the gravitational lensing[1] by a DM isothermal sphere without necessarily enlisting DM.

2 BIRKHOFF THEOREM FROM POISSON EQUATION: GRAVITY ON A MASSLESS SPHERICAL SHELL IN EMPTY SPACE

We begin with the gravitational Poisson equation as applied to an isotropic environment

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\rho(r). \qquad (1)$$

[1]Gravitational lensing probes the metric tensor in a manner not shared by the observation of motion of material bodies.

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It is readily seen that for a *continuum* mass distribution a radially directed force per unit mass of the form $-\hat{r}\mathrm{d}\Phi/\mathrm{d}r = -Gm(r)\hat{r}/r^2$ is implied by equation (1), where $m(r)$ is the underlying mass, or the integral of $\rho(r)$ over the spherical region of radius r. Specifically, the solution of equation (1) for an inverse-square density source $\rho(r) = A/r^2$ is $\Phi(r) = 4\pi GA\log r$, where the constant $A = m(r)/(4\pi r)$ with $m(r) \propto r$ being the enclosed mass within radius r, which leads to the requisite force per unit mass of a flat rotation curve $\mathbf{F} = -\hat{r}\mathrm{d}\Phi/\mathrm{d}r = -4\pi GA\hat{r}/r$. The amount of source mass $m(r)$ underlying a Keplerian circular orbit of radius r and velocity v is then given by the equation $v^2 = 4\pi GA = Gm(r)/r$, which for a typical observed orbital velocity $v = 300$ km s^{-1} at $r = 3$ kpc is a colossal $m = 1.2 \times 10^{44}$ g, almost an order of magnitude larger than the mass of all the observable baryons within such a galactic scale (Bland-Hawthorn & Gerhard 2022; Chan 2019); hence, the existence of DM seems to be a justifiable conjecture. Such a conclusion is inevitable because both the test mass acceleration v^2/r and the mass encumbering beneath the orbital radius are proportional to A, hence to each other.

While attempting to look for a way out, consider a discrete density source in the form of a thin spherical shell centred at the origin of total mass M, which is a *constant*, radius $R > 0$ (also a constant) and density

$$\rho(r) = \frac{M\delta(R-r)}{4\pi r^2}, \qquad (2)$$

where $\delta(x)$ is the 1D Dirac delta function. The total mass enclosed by a spherical surface of radius r is

$$m(r) = 0,\ \ r < R;\ \text{and}\ M,\ \ r > R. \qquad (3)$$

The general solution of equation (1) has two constants of integration. By choosing them to ensure there is no singularity of Φ at the origin and $\Phi \to 0$ as $r \to \infty$, one obtains

$$\Phi(r) = -\frac{GM}{r}\ \ r \geq R,\ \ \text{and}\ \ -\frac{GM}{R}\ \ r < R. \qquad (4)$$

This gives, in essence, the Birkhoff theorem in the weak field limit where General Relativity reduces to equation (1), namely the force field $-\mathrm{d}\Phi(r)/\mathrm{d}r$ vanishes in the region $r < R$ inside the shell (analogous to Newton's hollow sphere) and is radially attractive outside the shell, being given by the inverse square law with M as the source of gravitational mass.

At this point, we depart from the conventional approach. Apart from equation (2), there is at least one other type of shell singularity, or more precisely, an isotropic impulsive source term in the form of a topological defect, capable also of producing a central force field. Consider replacing $\rho(r)$ in equation (2) by a linear combination of $\delta(R-r)$ and $\delta'(R-r)$, namely

$$\rho(r) = \frac{c^2}{8\pi G}\left[\frac{2\alpha s}{r^2}\delta(R-r) + \frac{2\alpha s}{r}\delta'(R-r)\right], \qquad (5)$$

where $\alpha > 0$ is dimensionless and $s \to 0^+$ is a length, such that

$$s\delta(R-r) = 1\ \text{at}\ r = R. \qquad (6)$$

In this formulation, the mass within a sphere of radius $r > R$ is

$$\begin{aligned} m(r) &= \int_0^r 4\pi r'^2\rho(r')\mathrm{d}r' \\ &= \frac{\alpha s c^2}{G}\int_0^r [\delta(R-r') + r\delta'(R-r')]dr' \\ &= \frac{\alpha s c^2}{G}\left[r'\delta(R-r')\right]_0^r \\ &= M\left[r'\delta(R-r')\right]_0^r \\ &= 0, \end{aligned} \qquad (7)$$

where $M = \alpha sc^2/G$.

Consequently, the singular shell structure contributes nothing to the total mass of the spherical cavity within which it resides. Thus, the mass of the shell itself must vanish.

To analyse further the mass distribution, note that part of the source function represents the topological defect of a dipole shell because the derivative of a Dirac delta function at $r = R$ is the limiting case of a half Gaussian for $r < R$ joining with an inverted half Gaussian for $r > R$, with the former corresponding to a thin positive density shell and the latter negative of the same magnitude. Since the outer shell has a slightly larger radius, the total mass of both shells ought to be negative, had it not been for the presence of another term in the density function (5), which is proportional to $\delta(r-R)$. When both terms in equation (5) are included, the total enclosed mass for any radius $r > R$ vanishes exactly.

Moreover, as the width of the two Gaussians tends to zero, there is *no* finite spherically symmetric region (be it cavity or shell) over which the integral of $\rho(r)$ yields a resolvably negative mass $m(r)$ and its potentially undesirable consequences. Thus, if one computes the mass of the inner half of the shell where the density is positive by applying (6) and (7) with $r = R$, one obtains[2] $m(r = R) = \alpha Rc^2/G > 0$. But if one lets r exceed R by any finite amount, equation (7) would give $m(r) = 0$ because one has now included the irresolvable outer half of the shell where the density is positive, together with any hollow region which may lie beyond it. Further discussions on equations (5) and (7) will take place below.

The potential gradient, on the other hand, is obtained by integrating equation (1) once, with ρ as in equation (5), with the result is an inward acceleration of (or an attractive radial force on a test particle of unit mass),

$$F = -\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\alpha sc^2}{r}\delta(R-r) = -\frac{GM}{r}\delta(R-r). \qquad (8)$$

Thus, on the singular shell itself, there is a gravitational inward pull of magnitude $\propto 1/r$, the basic characteristic of a force field capable of driving a colour red of stellar velocities, even though the shell is massless. With a sufficient number of such unresolvably closely spaced singular shells, the potential Φ is effectively a continuous function. One could then envisage a galaxy or cluster of galaxies, both being populated by discrete clumps of baryons, namely the stars and galaxies, respectively, as well as thermal gas as a virialized ensemble with the force responsible for binding them given by equation (8).

Since the shells themselves are massless, the ensuing total mass of each large-scale structure is simply the mass of the baryons, and the need for enlisting DM is mitigated, at least in part. The shells can be spaced 0.1 au apart in galaxies and 100 pc apart in clusters (see

[2]This strictly speaking is the total mass of the central and hollow region $0 \leq r < R$ where both the density and mass vanishes, and the inner half of the spherical shell where both are positive.

Section 6 for details). For telescopes with resolving power ≈1 arcsec, stars which are 1 pc or further away from the observer will not reveal the shell spacings, nor will galaxies in a cluster located 1 Gpc or further from the observer reveal the shells in clusters.

3 STABILITY OF THE SECOND SOLUTION

In order to demonstrate the stability of the model, one must find a time-independent metric tensor solution to the Einstein field equations with the stress-energy tensor being that of pressureless matter of density $\rho(r)$ as given by equation (5). If such a solution exists and has the same physical properties as the previous section in the non-relativistic and weak field limit, i.e. with the metric tensor reducing to the static potential function $\Phi(r)$, it would mean the model is stable.

For steady state and isotropic space, a line element is expressible in the form

$$ds^2 = c^2 g^2(r)dt^2 - f^2(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (9)$$

The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}R/2$ of this metric is

$$r^2 G_{00} = 1 - \frac{1}{f^2(r)} + \frac{2}{f^3(r)} r f'(r); \quad (10)$$

and

$$r^2 G_{11} = 1 - f^2(r) + \frac{2rg'(r)}{g(r)}; \quad (11)$$

and

$$G_{22} = \frac{r\{f(r)[rg''(r) + g'(r)] - g(r)f'(r) - rf'(r)g'(r)\}}{f^3(r)g(r)}; \quad (12)$$

and

$$G_{33} = G_{22}\sin^2\theta. \quad (13)$$

We shall consider the limiting case in which the functions $f(r)$ and $g(r)$ are expressible in the form $f(r) = 1 + \delta_1$ and $g(r) = 1 + \delta_2$, where $\delta_{1,2}$ are generally r-dependent perturbations obeying $|\delta_{1,2}| \ll 1$ at least for the range of r of interest. This depicts the weak field limit where the line element (9) tends to Minkowski as $\delta_{1,2} \to 0$.

Then the simplest way of constructing a flat rotation curve environment, valid to order $\delta_{1,2}$ is by writing

$$f(r) = 1 + \alpha; \quad g(r) = 1 + \gamma \log(r), \quad (14)$$

where α and $\gamma \log(r)$ are $\ll 1$, the latter at least within a certain range of r relevant to observations. When (14) is substituted into (10) through (12) to calculate the Einstein tensor one finds, to order α and γ,

$$r^2 G_{00} = 2\alpha; \quad r^2 G_{11} = 2(\gamma - \alpha); \quad G_{22} = G_{33} = 0. \quad (15)$$

In the case of pressureless and non-relativistic matter, one can ignore fluid pressure P and bulk flow velocities[3] u_i (namely, $P \ll \rho c^2$, where ρ is the mass density of the fluid, and $u_i u^i \ll c^2$); the corresponding stress energy tensor $T_{\mu\nu} = (\rho + P/c^2)u_\mu u_\nu - P g_{\mu\nu}$ has T_{00} as its only non-negligible component.

Thus to lowest order in α and γ, equations (9) and (14) form a solution of the Einstein field equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} = \frac{8\pi G}{c^4} \frac{u_\mu u_\nu}{c^2} \rho \quad (16)$$

if

$$G_{00} = \frac{8\pi G \rho}{c^2}; \text{ and } G_{11} = G_{22} = G_{33} = 0, \quad (17)$$

to this order. From equations (15) and (5), one finds that

$$\alpha = \gamma, \quad (18)$$

in which case

$$G_{00} = \frac{2\alpha}{r^2}. \quad (19)$$

When the angular momentum of a test particle is finite, the time homogeneity and azimuthal symmetry of equation (9) allows it to be recast in a form (assuming an orbital plane at $\theta = \pi/2$) involving the constants of the motion $\epsilon = g^2 dt/d\tau$ and $\ell = r^2 d\phi/d\tau$ with $c d\tau = ds$, as

$$\frac{f^2 g^2}{c^2}\left(\frac{dr}{d\tau}\right)^2 + \frac{\ell^2 g^2}{c^2 r^2} + g^2 - 1 = \epsilon^2 - 1. \quad (20)$$

In particular, for circular orbits where r is constant and the first term vanishes, differentiating the remaining terms in equation (20) w.r.t. r yields

$$\frac{g'}{g} - \frac{\ell^2}{c^2 r^3} = 0 \Rightarrow \alpha c^2 = \frac{\ell^2}{r^2} \Rightarrow v = \sqrt{\alpha} c \quad (21)$$

in the weak field and low-velocity limit, where $v = \ell/r$ is the (constant) tangential speed of the test particle. Thus, one finds a centrifugal acceleration

$$\frac{v^2}{r} = \frac{\alpha c^2}{r}. \quad (22)$$

In fact, the constancy of v in equation (21) accounts for the flat rotation curve of galaxies.

For a typical speed of $v \approx$ 300 km^{-1}, one requires $\alpha \approx 10^{-6}$. This also means the G_{11} tensor component, which is only finite to $O(\alpha^2)$, lies below G_{00} by a factor $\approx 10^6$, consistent with $G_{11}/G_{00} \approx u_r^2/u_0^2 \approx u_r^2/c^2 \approx 10^{-6}$. This reality check can also be applied to G_{22} and G_{33} leading to the same conclusion.

Is the centrifugal acceleration (22) driven by an amount of underlying mass consistent with Newton's law of universal gravitation, as is to be expected given that $v \ll c$? To check that, we may use the density $\rho(r)$ of the non-relativistic and pressureless cosmic substratum, as inferred from equation (5) and the value of G_{00} in equation (19), to compute the equivalent Newtonian mass[4] $m(r)$ within a circular orbit of radius r, namely

$$m(r) = \int^r 4\pi r'^2 \rho(r')dr' = \frac{\alpha c^2 r}{G} \quad (23)$$

after ignoring any factor $f(r)$ in the integral which is O(1) by equation (14). Thus, the Newtonian acceleration is $-Gm(r)/r^2 = -c^2\alpha/r$, which is in agreement with equations (22) and (21). Moreover, comparing to the Poisson equation approach to the same problem, one finds that the two treatments yield the same amount of underlying Newtonian mass equivalent because γ relates to the constant A in the first paragraph of the last section by the equation $\alpha = 4\pi GA/c^2$. The development so far simply spells out the obvious, i.e. General

[3] $u_i = g_{i\nu}dx^i/d\tau \approx -dx^i/dt = -u^i$ because the lowest order approximation to $f(r)$ and $g(r)$ is $f(r) = g(r) = 1$.

[4] The term refers to the equivalent total mass of an *underlying* (or *enclosed*) spherically symmetric matter-energy distribution which produces the same attractive central force (by Newton's theory of universal gravitation) as a shell. As proven earlier, the actual total mass of the shell itself vanishes exactly, even though the attractive force it exerts on a test particle riding the shell is finite and given by equation (8).

Relativity reduces to Newtonian Physics in the weak field and non-relativistic limit. There is no explicit time dependence in the metric; and hence, both the solution and the source are stable.

4 FLAT ROTATION CURVE

What sort of line element in General Relativity would correspond in the weak field limit to the Newtonian (Poisson equation) approach to the $1/r$ attractive force on the massless singular spherical shell described in Section 2? The reader could easily verify that a line element with exactly the same characteristics is given by equation (9) having

$$f(r) = 1 + \alpha s\delta(R - r);\;\; g(r) = 1 + \frac{\alpha s\theta(r - R)}{R} + \Phi_0, \qquad (24)$$

where $\theta(r)$ is the Heaviside step function. In equation (24), $g(r)$ differs infinitesimally from unity because $s \to 0$, equation (25). By letting s be the width of the Dirac delta function, i.e. $s \to 0$ such that

$$s\delta(R - r) = 1 \;\text{ at } r = R. \qquad (25)$$

Further, by letting the dimensionless parameter α satisfy $\alpha \ll 1$ and restricting oneself to lowest order terms in α it can readily be shown, using equations (10) through (13), that

$$G_{00} = \frac{8\pi G\rho(r)}{c^2} \;\; G_{11} = G_{22} = G_{33} = 0, \qquad (26)$$

where $\rho(r)$ is given by equation (5). Thus, equation (26) is consistent with equation (17). In particular, the total mass the shell contributes to a concentric spherical region of radius $r > R$, as given by equation (7), which is just the integral of $\rho(r)$ from the origin to $r > R$, vanishes. According to equation (22), the centrifugal acceleration of a test particle anywhere on the shell itself, namely $r = R$, has magnitude

$$\frac{g'c^2}{g} = \frac{f^2 - 1}{2r}c^2 = \frac{\alpha s c^2}{r}\delta(R - r) = \begin{cases} \frac{\alpha c^2}{r}, & r = R \\ 0, & r \neq R \end{cases} \qquad (27)$$

where the last equality is enforced by equation (25). Note that equation (27) is consistent with the equipotential sphere result, namely equations (22) and (8). The speed of the test particle on the shell is $v = \sqrt{\alpha}c$, in agreement with equation (21). In this way, if there are N concentric singular shells with density satisfying (5) within a galaxy or cluster,

$$f(r) = \left[1 + \alpha s \sum_{n=1}^{n_L} \delta(nR - r)\right];$$
$$g(r) = 1 + \alpha s \sum_{n=1}^{n_L} \left[\frac{\theta(r - nR)}{nR}\right] + \Phi_0, \qquad (28)$$

where in equation (28) Φ_0 is a constant but the shells need not be evenly spaced, i.e. $N \le n_L$ and the $<$ sign is because in the summation sign of equation (28) n can change in steps $\delta n > 1$.

The total mass of the galaxy (cluster) still vanishes[5]; but its baryonic contents, namely stars (galaxies) and gas, are primarily moving in bound orbits or random walk (i.e. in thermal virialized form) on the shells, giving the impression of a large amount of DM hidden within each spherical shell. The magnitude of the inwardly directed radial acceleration a test particle experiences on-shell is independent of R, and according to equation (27) is given by $\alpha c^2/r$ provided equation (25) holds. The value of R is arbitrary, but for a galaxy is likely to be below the mean spacing between stars.

5 GRAVITATIONAL BENDING OF LIGHT

Since evidence of DM in galaxies, groups, and clusters come not only from stellar rotation curves and viralized thermal gas but also gravitational lensing (Massey 2010; Hoekstra et al 2013; Vegetti 2023; Zürcher 2023), we examine in this section how the currently proposed configuration of concentric singular spherical shells could possibly bend light in the same manner as an equipotential sphere.

In general, for the line element of equation (9), the gravitational inward deflection of a light ray propagating along the z-direction and skirting a large-scale structure at impact parameter a as measured in the y-direction is, in the weak field limit,

$$\theta = \frac{2a}{c^2}\int_a^\infty \frac{1}{\sqrt{r^2 - a^2}}\left(\frac{g'}{g} - \frac{f'}{f}\right)\mathrm{d}r. \qquad (29)$$

For a point mass M at the origin where $f = 1 + GM/r$, $g = 1 - GM/r$, equation (29) yields the standard result $\theta = 4GM(c^2 a)$. For the line element of equation (14), one obtains instead

$$\theta = \alpha\pi, \qquad (30)$$

which is the constant deflection scenario of an equipotential (isothermal) sphere.[6]

The question, therefore, is whether the line element of concentric singular shells, equation (28) could also lead to a constant inward deflection like equation (30). Applying equations (28) and (29), one finds, after accounting for the oddness of $\delta'(x)$,

$$\theta = 2\alpha s\left\{\sum_{n=n_S}^{n_L}\int_a^\infty \left[\frac{a\delta(nR - r)}{r\sqrt{r^2 - a^2}} + \frac{a}{\sqrt{r^2 - a^2}}\delta'(r - nR)\right]\mathrm{d}r\right\}, \qquad (31)$$

where the smallest shell is indexed n_S and satisfies $n_S R - a \approx R \ll a$; and the largest shell is indexed n_L which satisfies $n_L R \gg a$.

The first integral corresponds to the $\mathrm{d}g/\mathrm{d}r$ term in equation (29) and equals $\theta_1 = \alpha\pi s/R$. The second comes from $-\mathrm{d}f/\mathrm{d}r$ term in equation (29) and may be written after integration by parts as

$$\theta_2 = 2\alpha s \sum_{n=n_S}^{n_L}\left\{\left[\frac{a\delta(r - nR)}{\sqrt{r^2 - a^2}}\right]_a^\infty + \left[\frac{anR\theta(r - nR)}{(n^2R^2 - a^2)^{3/2}}\right]_a^\infty\right\}, \qquad (32)$$

where $a \le nR$. The first term of equation (32) vanishes unless the delta function is satisfied at either one of the two limits (it is not satisfied at the upper limit, which lies beyond the large-scale structure), which is improbable for a random sightline because the shells are thin (see equation 25). Thus, this term is ignored.[7] The second term is finite only at the upper limit, and equals $2\alpha s\sum_n anR/(n^2R^2 - a^2)^{3/2}$. The summation over n is dominated by the term at which $n^2R^2 - a^2$ is minimized, namely at the lower n limit (for the given a) where $nR \gtrsim a$, or more precisely $nR - a \approx \Delta$, where Δ is the average distance

[5]The enclosed mass is finite if the spherical surface coincides exactly with a shell, but the probability of coincidence is zero because the shells are infinitely thin. Rather, such singular shells manifest themselves by the orbital motion of stars and galaxies (and thermal motion of virialized baryons) riding on them, or the deflection of light passing through them.

[6]Ignoring the logarithmic increase in $g(r)$ in equation (14).

[7]Even though at the lower limit $r = a$ the denominator vanishes, resulting under the $r \neq nR$ scenario in an indeterminate of the form 0/0, this is resolved by applying L'Hospital rule once to obtain the form $\delta'(r - nR)\sqrt{r^2 - a^2}$ which clearly vanishes at $r = a \neq nR$.

the light path at the $r = a$ position is from the nearest shell. Thus the result is

$$\theta \approx \theta_2 = \frac{\alpha s}{\Delta}\left(\frac{a}{2\Delta}\right)^{1/2}, \tag{33}$$

which is $\gg\theta_1$ since $a \gg R$.

In order for the deflection angle $\theta \approx \theta_2$ to be the constant of the equivalent equipotential sphere (30) one must have, by equation (33),

$$\Delta = 2^{-1/3}\pi^{-2/3}s^{2/3}a^{1/3}, \tag{34}$$

where it is understood that $s \ll R \lesssim \Delta \ll a$. A gradual change in the s/Δ ratio may be interpreted as a mathematical way of enforcing uneven spacing in the sum over shells of equation (28), *i.e.* both s and R may themselves be constants, but the spacing Δ between adjacent shells becomes wider towards larger radii because not every integer n corresponds to a shell. This does not affect the acceleration of stars and galaxies on each shell, as it is determined only by the coefficient α of the delta function in equation (27).

Before leaving this section about light deflection, it should be noted from the discussion immediately preceding (33) that although the quantity $nR - a$ in equation (32) is finite for most line of sights, being given by equation (34) on average, there will inevitably be a small fraction of them for which $nR - a \to 0$, resulting in a large θ_2. This offers a possible discrimination between the current model and standard DM theory; namely, according to the former, the bending of light from background sources by intervening large-scale structures is not totally smooth despite one restricting one's observation to the weak lensing scenario of large impact parameters which take the ray through the structure's outskirts. That in turn is because if this outermost ray touches one of the shells, it will still be deflected substantially, causing an unusually large tangential shear of the image of the background source, even possibly magnification (i.e. strong lensing). DM models do not predict such a possible behaviour. Detailed mapping of background sources behind the outer parts of a cluster or galaxy, in search of large convergence fluctuations there, could offer a scrutizing test.

6 ROLE OF THE BARYONS

How may one expect actual rotation curves, as predicted by the shell model, to look like when baryons are taken into account? On galactic scales, the baryon to DM mass ratio is below the cosmological average of ≈20 per cent (Fukugita et al. 1998), and most of the baryons are in stars if one considers only the galactic disc by ignoring the halo (Cautun 2020). Since the model adopted here is spherically symmetric, with applications also to clusters of galaxies,[8] the halo gas has to be taken into account. For a galaxy like the Milky Way, the inclusion of halo gas would double the stellar mass, and leads to a total baryonic mass of ≈10 per cent of the DM mass (Cautun 2020).

In this section, we estimate the effect of baryons on galaxy rotation curves in the context of the shell model, by assuming the baryons as a gas in hydrostatic equilibrium in the potential field of the massless shells, with both the gas distribution and the shells being spherically symmetric. Under this scenario, two equations govern the two unknowns $\rho_b(r)$ and $T(r)$, respectively, the density and temperature profile of the baryons (Bulbul et al. 2010). They are

$$\frac{1}{\mu m_p n_e(r)}\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r};\; n_e(r) = n_{eS}\left(\frac{T}{T_S}\right)^p, \tag{35}$$

where n_e is the number density of electrons in the halo gas, which is treated here as a plasma of mean molecular mass μ, $P = P(r) = n_e(r)kT(r)$, m_p is the proton mass, $p > 0$ is the polytropic index of the plasma, and the subscripts S and L refers, respectively, to the smallest and largest radii within which the hierarchical shell structure exists (i.e. in the notation of equation (28) we have $r_S = n_S R$ and $r_L = n_L R$).

Now in equation (28) one may in the limit of many shells, approximate the summation by an integral, namely

$$g(r) = 1 + \Phi(r), \tag{36}$$

with

$$\begin{aligned}\Phi(r) &= 2^{1/3}\pi^{2/3}s^{1/3}\alpha c^2\int_{r_S}^{r}\xi^{-4/3}\mathrm{d}\xi + \Phi_0\\ &= -3\times 2^{1/3}\pi^{2/3}\alpha c^2 s^{1/3}r^{-1/3}.\end{aligned} \tag{37}$$

In equation (37) $\xi = nR$, and use was made of equation (34). Specifically, due to the change in the shell index n becoming $\delta n > 1$ towards larger radii, fewer shells are summed compared to the regular spacing scenario of $\delta n = 1$. This means, starting with the continuum representation of the regular spacing case,

$$\Phi(r) = \alpha s c^2\sum_{n=n_S,1}^{n_L}\frac{\theta(r-nR)}{nR} + \Phi_0, \tag{38}$$

one must now replace $\mathrm{d}\xi$ by $R\mathrm{d}\xi/\Delta < \mathrm{d}\xi$, which is what we did (equation 37). Moreover, in arriving at the last step of equation (37), we fixed the constant Φ_0 such that $\Phi(r) \to 0$ as $r \to \infty$. Note that the shells must share a common acceleration parameter α to yield a flat rotation curve for stars in a galaxy. The continuum approximation in equation (37) means the baryonic gas is able to occupy the space in between shells by collision and subsequent virialization, while stars are bound to the shells and follow circular orbits.

The three equations (28), (36), and (35) may then be solved to yield

$$T(r) = -\frac{1}{1+p}\frac{\mu m_p}{k}\Phi(r) = \frac{3\times 2^{1/3}\pi^{2/3}\alpha c^2 s^{1/3}}{1+p}\frac{\mu m_p}{k}\frac{1}{r^{1/3}}; \tag{39}$$

and

$$M_b(r) = \frac{12\pi\mu n_{eS} m_p r_S^{p/3}}{9-p}\left[r_L^{(9-p)/3} - r_S^{(9-p)/3}\right],\; p \neq 9. \tag{40}$$

Such an amount of baryons would provide an extra acceleration $-GM_b(r)\hat{r}/r^2$ which is to be added to the shell acceleration of equation (27) to obtain the total circular rotation speed of a star, due to the shell it is on and the baryons underlying it. Results for various polytropic indices p are shown in Fig. 1, where it can be seen that a variety of observed rotating curves (see fig. 1 of Zhang 2019) are reproduced.

Concerning the actual values of the shell thickness s and spacing R, one could conjecture for galaxies that $s = 4 \times 10^9$ cm and $R = 3 \times 10^{12}$ cm. This would satisfy equation (34) with $n \approx 10^9$ out to a radius $a \approx nR \approx 1$ kpc. The actual shell separation at 1 kpc radius is, by equation (34), $\Delta \approx 1.34 \times 10^{13}$ cm, which is 4 times larger than R. Thus, at this radius, there is one shell every $\Delta n \approx 4$. For $p = 3/2$ and $\mu = 1$, the temperature of the gas from equation (39) is $T_S \approx 4 \times 10^3$ K, and the electron density is obtainable from the second of equation (35) as $\approx n_{eS} \approx 0.127$ cm^{-3}, both evaluated at $r_S = 0.5$ kpc and with the latter being the value which yields the cosmological

[8]In clusters, the baryon fraction is quite close to the cosmological value (Fukugita et al. 1998).

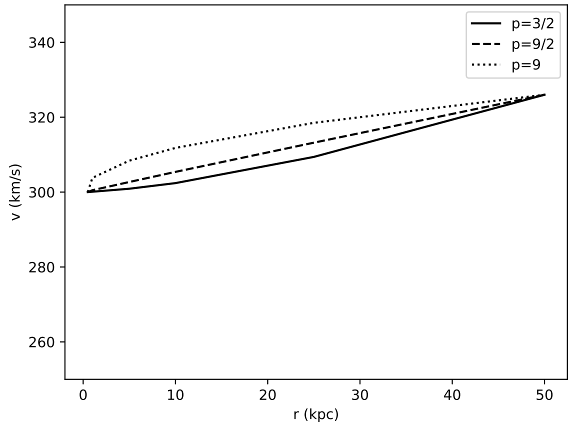

Figure 1. Circular velocity of stars in a galaxy, taking into account the extra centrifugal acceleration provided by the gravitational pull of the underlying baryons, as well as the singular shell. Here, p denotes the polytropic gas index of the baryons, and the central electron density n_{eS} is chosen to enable the baryons within 50 kpc radius to equal 18 per cent of the DM thereby mass, consistent with the cosmological value of this ratio. Note that this way of fixing n_{eS} overestimates the baryons in clusters (see the text for elaboration); hence, the extra acceleration due to the baryons. Nevertheless, the rotation curve is still sufficiently flat, indicating that the baryons alone do not play much role in providing the necessary centrifugal acceleration.

baryon to DM mass ratio of 18 per cent at the radius of 50 kpc. Such values of T_S and n_{eS} are consistent with our understanding of the warm diffuse H II component of the interstellar gas (Cox 2005). In the case of clusters of galaxies, a pair of galactic rather than solar system scale lengths, such as $s = 3 \times 10^{19}$ cm (or 10 pc) and $R = 2 \times 10^{20}$ cm, would by equation (34) imply ≈1, 500 shells out to the radius $a \approx nR \approx 100$ kpc, at which point the actual separation between consecutive shells is $\Delta = 2.4 \times 10^{20}$ cm $\approx R$ by equation (34). Moreover, for $p = 3/2$ a temperature $T_S \approx 1.75 \times 10^7$ K for the baryons at $r_S = 100$ kpc radius, by equation (35) with[9] $\alpha = 10^{-5}$. Moreover, the baryonic mass between $r_S = 100$ kpc and $r_L = 1$ Mpc is, by equation (40) with $n_{eS} = 10^{-3}$ cm^{-3}, $M_b = 3.58 \times 10^{13}$ M$_\odot$ for $p = 3/2$. This is close to the cosmological baryon mass budget of ≈18 per cent of the DM mass where, by equation (23), $M_{DM} = \alpha c^2 r_L/G$ with $\alpha \approx 10^{-5}$ corresponding, via equation (21), to a velocity dispersion of $v \approx 10^3$ km s^{-1} which is typical of clusters, as are n_{eS} and T_S (see e.g. Kravtsov & Borgani 2012).

Lastly, as already pointed out in Section 2, neither the shell thickness s nor the spacing Δ between adjacent shells is resolvable by current telescopes. Moreover, the value of s is sufficiently small that stars in a galaxy only experience an infinitely thin shell, same for galaxies in a cluster. In both cases, they cannot respond to the fine details of the double layer (of positive and negative mass) of a shell.

7 EVIDENCE FOR THE EXISTENCE OF THE PROPOSED TOPOLOGICAL DEFECTS

Although topological effects, or singular distribution of mass-energy in space, have by now been proposed in many and varied forms since they were first envisaged by Kibble (1976), so long as they are stable, not all of them are understood in terms of their origin. An example is a thin sheet of negative mass wrapping around a wormhole to maintain cylindrical symmetry – the system is shown to be stable – irrespective of origin (Eiroa et al. 2016).

As far as the monopole–dipole shell combination in this work is concerned, although its origin is uncertain (despite being deemed stable in Section 3), there are scalar field solutions which can give rise to it (see Hosotani et al. 2002), and there is a body of evidence indicative of its existence, namely the discovery of a giant arc of proper size ≈1 Gpc at $z \approx 0.8$ which adds to an already existing set of cosmologically sized distribution of galaxies in the form of walls and rings (see Lopez et al. 2022 and table 1 therein). These galaxy arcs, rings, and walls appear to be particularly singling out our proposed model because, on Gpc scales, there is simply no way of attributing to DM overdensity as the source of any centrifugal force which could bind together structures of such enormity, i.e. the Universe is too homogeneous by then. Thus, the specific topological defect proposed in this paper is reasonably corroborated by the unusually large number of observations of organized shell-like or (as a projection effect) ring-like manifestations of stars and galaxies on kpc scales and beyond. Such defects can in principle provide the necessary centrifugal force without enlisting 'missing' mass.

8 CONCLUSION

In addition to the well-known Green function solution of the gravitational Poisson equation, which gives rise to no net force inside a thin spherically symmetric shell of mass and an attractive inverse-square force outside, the existence of another singular shell solution which sources no mass but drives an attractive $1/r$ force on the shell itself is demonstrated in this letter. When a galaxy (cluster) comprises many concentric singular shells of this type, stars (galaxies) orbit, and diffuse baryonic gas can be virialized, on each of the shells. The result is a flat rotation curve with large Keplerian velocities in the case of orbital motion and high temperature thermal motion in the case of visualized gases; both phenomena are symptomatic of DM even though such matter may not exist or exist in an amount smaller than is required for binding a galaxy or cluster.

The observation of giant arcs and rings (e.g. Lopez et al. 2022) could lend further support to the proposed alternative to the DM model. It should be pointed out that the thin shells advocated here do not have to cover an entire spherical surface to be effective, i.e. the result of Section 2 is readily shown to apply also to a topological defect which occupies part of a spherical surface, although it cannot account for elliptical orbits of a test mass. In this paper, we also discussed how gaseous baryons can be in hydrostatic equilibrium with the potential established by the shells, leading to temperature and density in agreement with observations. Moreover, their 'feedback' effect on the acceleration provided by each shell was estimated and was shown to be slight, namely it is only a small perturbation of the shell acceleration of stars and galaxies. Hence, the flat rotation curve of stars is not distorted by much.

Apart from rotational curves and virialization, this paper also addresses the other major piece of evidence for DM in large-scale structures – gravitational lensing. As it turns out, the proposed concentric singular shells can deflect the light path to a distant point source by a small and constant angle independently of the impact parameter in the weak lensing limit, provided the spacing Δ between consecutive shells widens with radius r as $\Delta \propto r^{1/3}$. Thus, again, the need to enlist DM is not inevitable.

Nevertheless, this paper does not attempt to tackle the problem of structure *formation*; nor does it argue against the role of DM

[9]Clusters have about 10 times higher α than galaxies because their member galaxies have a velocity dispersion ≈1000 km s^{-1} which is ≈3 times higher than galaxies (see Kravtsov & Borgani 2012). Hence, $\alpha = 10^{-5}$ by equation (21).

in accounting for the ratio of the first two acoustic peaks of the cosmic microwave background (Roos 2012), and the presence of trace Deuterium[10] from Big Bang Nucleosynthesis (Kawasaki et al. 2015; Sibiryakov et al. 2020). Rather, we focussed our attention on the evidence as provided by galaxies and clusters of galaxies.

DATA AVAILABILITY STATEMENT

No new data were generated or analysed in support of this research.

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[10]Safe to say that, like baryons and DM, an early Universe comprising only baryons and shell singularities of the type presented here would probably not provide enough collisions to annihilate all deuterium.

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