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Bootstrap Gravitational Geons*

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It is shown that there exist solutions of the vacuum Einstein field equations with the property that exterior to the Schwarzschild radius, $R=2m$, the solution appears to be that of a static spherically symmetric particle of mass m (that is, strictly Schwarzschild), whereas interior to the Schwarzschild radius the topology remains Euclidean and the solutions have the property of a bundle of gravitational radiation so intense that the mutual gravitational attraction of the various parts of the bundle prevent the radiation from spreading beyond the Schwarzschild radius. It is not known whether there exist solutions of this type which remain nonsingular for all times; however, no singularity can ever be observed exterior to the Schwarzschild radius. The Cauchy data for one such solution are explicity exhibited.

I. INTRODUCTION

THE rigorous solution for the gravitation field of a spherically symmetric massive particle in general relativity, the Schwarzschild solution

$$ds^2=dR^2/(1-2m/R)+R^2(d\theta^2+\sin^2\theta d\phi^2)-(1-2m/R)dT^2 \quad (1.1)$$

has long stirred interest because of the puzzling singularity which occurs at $R=2m$, when the solution is written in terms of the particularly natural set of coordinates adapted to the full symmetry of the solution. Kruskal[1] has considerably clarified the situation by obtaining, by means of a coordinate transformation, the maximal analytic extension of the Schwarzschild solution. For our subsequent use it will be convenient to exhibit the Kruskal solution in slightly different coordinates than that of the original paper. Thus, if we perform the coordinate transformation

$$r=2m+2(2m)^{1/2}(R-2m)^{1/2}\times\exp[(R-2m)/4m]\cosh(T/4m)\,, \quad (1.2)$$

$$t=2(2m)^{1/2}(R-2m)^{1/2}\times\exp[(R-2m)/4m]\sinh(T/4m)\,,$$

the metric of Eq. (1.1) takes the form

$$ds^2=2m\exp\{-[(R(r,t)-2m)/2m]\}\times(dr^2-dt^2)/R(r,t)+R^2(r,t)(d\theta^2+\sin^2\theta d\phi^2)\,, \quad (1.3)$$

where the function, $R(r,t)$ is obtained by inverting the relation

$$(r-2m)^2-t^2=8m(R-2m)\exp[(R-2m)/2m]. \quad (1.4)$$

The Kruskal metric, Eq. (1.3), exhibits no singularity at $R=2m$. However, other anomalies appear in its stead. For example, the space-like surface $t=0$ no longer has Euclidean topology. As the coordinate r varies from $+\infty$ to $-\infty$, the Schwarzschild luminosity distance, R, varies from $+\infty$ to a minimal value of $2m$ (attained when $r=2m$), and then back to $+\infty$. The surface $t=0$, as well as the immediately neighboring space-like hypersurfaces, are seen to have a double-sheet topology connected by a spherical "worm-hole." Another anomaly of the Kruskal metric is that the true singularity at $R=0$ cannot occur on the space-like surface $t=0$, or on the surfaces in its immediate neighborhood. This singularity is seen to occur at a finite temporal distance from the $t=0$ surface. It is therefore not a singularity of the sort one customarily would associate with a source of the field. The "source" of the gravitational field would appear to be the "worm-hole."

Of particular interest to us is the appearance of the formerly singular surface $R=2m$. From Eq. (1.4), we find that in our present system of coordinates this surface satisfies the equation

$$(r-2m)^2-t^2=0 \quad (1.5)$$

which we recognize to be the equation of a light cone. This surface may be characterized in a unique invariant fashion as the only null orbit of the four parameter Lie group which the Schwarzschild solution admits.

Although the Kruskal metric is the unique *analytic* continuation of the Schwarzschild solution across $R=2m$, the fact that the latter surface is null indicates that we may drop the requirement of analyticity and obtain valid solutions of the vacuum field equations. (The nonanalytic behavior of the metric across a null surface could be interpreted physically as a pulse of gravitational radiation traveling along the null cone.) We can now pose the following two questions: (1) Must the spatial topology of all those solutions which coincide with the Schwarzschild solution for $R\geq 2m$, be non-Euclidean, or is it possible to find a source-free (nonanalytic) extension of Schwarzschild which has Euclidean topology? (2) Are there (nonanalytic) continuations of the Schwarzschild solution across $R=2m$ which are nonsingular for all times, either with or without Euclidean topology?

In this paper we shall treat only the first question. We shall show that there exist solutions of the vacuum Einstein field equations such that the initial space-like hypersurface has Euclidean topology and such that for $R\geq 2m$ the solution is strictly Schwarzschild. That the nonanalytic behavior of the metric is always confined

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[1] M. D. Kruskal, Phys. Rev. **119**, 1743 (1960).

to the interior of the sphere $R=2m$, may be given the physical interpretation that the source of the gravitational field is gravitational radiation whose intensity is so great that the null cones along which the radiation travels are kept bound to the interior of the sphere by the mutual gravitational attraction. Hence, the title of this paper. (The term *geon* was originally introduced by Wheeler[2] to characterize a bundle of *electromagnetic* radiation so intense that the mutual gravitational attraction kept the system bound. We believe that the present usage is a legitimate extension of the term.)

II. JUMP CONDITIONS ON CAUCHY DATA

Papapetrou and Treder[3] have shown that it is admissible to consider solutions of the Einstein field equations for which the metrics are of class C^0 provided the discontinuities in the derivatives of the metric satisfy certain conditions across characteristic surfaces which assure that there are no surface sources introduced by the discontinuities. Since it is our intention to exhibit the Cauchy data which determine a solution of the desired type, we shall now develop an equivelant set of jump conditions purely in terms of the Cauchy data themselves. We shall at this point employ the notation of Bruhat-Choquet.[4]

The constraint equations for the vacuum field equations may be written

$$\bar{\nabla}_j(P^{ij}-g^{ij}P)=0\,, \tag{2.1}$$

$$\bar{R}+P_{ij}P^{ij}-P_i{}^{i2}=0\,, \tag{2.2}$$

where $\bar{\nabla}_j$ denotes covariant differentiation with respect to the spatial metric g_{ij}, $\bar{R}$ denotes the scalar curvature of the spatial metric, and the second fundamental form of the initial surface P_{ij} is defined by

$$P_{ij}=\tfrac{1}{2}|g^{00}|^{-1/2}(\partial_0 g_{ij}-\bar{\nabla}_i g_{0j}-\bar{\nabla}_j g_{0i})\,, \tag{2.3}$$

and Latin indices are raised and lowered by means of the spatial metric. The fundamental theorem[4] which we shall appeal to is that for every set of Cauchy data on the initial surface ($t=0$), which satisfies the constraint equations (2.1) and (2.2), there exists a solution of the vacuum Einstein field equations in a finite four-dimensional neighborhood of the initial surface. The solution is unique up to coordinate transformations which preserve both the Cauchy data and the equation of the initial surface. For the nonanalytic Cauchy problem, the solution at a finite temporal distance from the initial surface is determined exclusively by the portion of the data in the relativistic past of the point under consideration. It is this latter statement which assures us that if at $t=0$ we assign the Cauchy data appropriate to the Schwarzschild solution in the region $R\geq 2m$, the solution will remain Schwarzschild for $R\geq 2m$ for all time, T.

If the metric under consideration is of class C^0, the second fundamental form, P_{ij}, will in general be of class C^{-1}, that is, there may be finite discontinuities. Both sets of constraint equations, (2.1) and (2.2), containing as they do second derivatives of the components of the metric tensor, may have terms of class C^{-2}, that is, Dirac delta functions. The presence of such singular terms would correspond to the existence of localized material sources of the field. We must, therefore, require that such terms do not occur. The simplest way of assuring that C^{-2} terms do not occur is to integrate the constraint equations across the (two-dimensional) surface of discontinuity and require that the resulting expressions vanish. In this fashion, we obtain the jump conditions

$$[P^{is}]n_s=n^i[P] \tag{2.4}$$

and

$$g^{mn}n_s[\Gamma^s{}_{mn}]=n^s[\Gamma_s]\,, \tag{2.5}$$

where n_s is a vector normal to the surface across which the discontinuity occurs, and the square brackets denote the jump of the enclosed quantity across the surface. The C^0 property of the spatial metric enables us to introduce the tensor h_{ij} via the relation

$$[g_{ij,k}]\equiv h_{ij}n_k\,. \tag{2.6}$$

If, in addition, we define the tensor k_{ij},

$$[P_{ij}]\equiv k_{ij}\,, \tag{2.7}$$

the two jump conditions may be written, respectively,

$$k_s{}^i n^s=n^i k_s{}^s \tag{2.8}$$

and

$$h_{st}n^s n^t=h_s{}^s n^t n_t\,. \tag{2.9}$$

Not all discontinuities are intrinsic, for it is possible to perform coordinate transformations of class C^1 which leave the metric tensor continuous but introduce discontinuities in its derivatives. Such a coordinate transformation must necessarily have the properties

$$[\partial^2\bar{X}^i/\partial X^j\partial X^k]=a^i n_j n_k \tag{2.10}$$

and

$$[\partial^2\bar{X}^0/\partial X^j\partial X^k]=b n_j n_k\,. \tag{2.11}$$

We can, in addition, select the transformation such that on the surface of discontinuity S,

$$\partial\bar{X}^i/\partial X^j|_s=\delta_j{}^i\,. \tag{2.12}$$

Employing these relations we find that the tensors h_{ij} and k_{ij} transform as follows:

$$\bar{h}_{ij}=h_{ij}-a_i n_j-a_j n_i \tag{2.13}$$

and

$$\bar{k}_{ij}=k_{ij}+(g^{00})^{-3/2}b n_i n_j\,. \tag{2.14}$$

It is readily observed that Eqs. (2.13) and (2.14) leave invariant the jump conditions, Eqs. (2.8) and (2.9).

[2] J. A. Wheeler, Phys. Rev. **97**, 511 (1955).
[3] A. Papapetrou and H. Treder, Math. Nachr. **23**, 371 (1962).
[4] Y. Bruhat, in *The Cauchy Problem; Gravitation*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), p. 130.

This, of course, is an essential requirement, since the jump conditions assure the absence of material sources, and it would not be reasonable to be able to introduce (or remove) material sources by means of C^1 coordinate transformations. We may employ Eqs. (2.13) and (2.14) to impose auxiliary conditions on the discontinuities. Thus, if we take

$$a_i=h_{is}n^s(n^pn_p)^{-1}-\tfrac{1}{2}n_ih_{st}n^sn^t(n^pn_p)^{-2}, \quad (2.15)$$

we find

$$\bar{h}_{is}n^s=0. \quad (2.16)$$

Similary, selecting b such that

$$b=-(g^{00})^{\frac{1}{2}}k_{st}n^sn^t(n^pn_p)^{-1} \quad (2.17)$$

yields

$$\bar{k}_{st}n^sn^t=0. \quad (2.18)$$

Combining these four auxiliary conditions, Eqs. (2.16) and (2.18), with the four jump conditions, Eqs. (2.8) and (2.9), we obtain

$$k_s{}^in^s=k_s{}^s=0 \quad (2.19)$$

and

$$h_s{}^in^s=h_s{}^s=0. \quad (2.20)$$

We thus find that the intrinsic discontinuities can be made transverse and trace-free exhibiting the two modes of polarization available to gravitational radiation: the "electric mode," where the transverse components of h_{ij} are purely diagonal, and the "magnetic mode," where the transverse components of h_{ij} are purely off-diagonal. The k_{ij} give the rate of change of these two modes. As the solution propagates in time off the initial surface, discontinuities in derivatives of the metric can occur across surfaces which are not null; however, this can always be transformed away by means of C^1 coordinate transformation. Even along null surfaces, discontinuities in the longitudinal and time-like components of the derivative of the metric tensor can always be transformed away. However, discontinuities in the transverse, spatial components will be intrinsic and may be identified physically with the presence of gravitational radiation.

III. CONSTRUCTION OF THE GEON

In order to exhibit the existence of a solution of the vacuum gravitational field equations of the type discussed in the first section, we must construct a solution of the constraint equations (2.1) and (2.2) on a three-dimensional, topologically Euclidean surface, such that for $r\geq 2m$, it coincides with the Cauchy data for the Schwarzschild metric, and such that the piecing of this data to the data for $r\leq 2m$ is consistent with the jump conditions, Eqs. (2.8) and (2.9). To assure that there is no singularity or cusp at the origin we shall choose the solution to be strictly flat in the range $0\leq r\leq 0.1m$.

From Eqs. (1.3) and (1.4), we find that the Cauchy data for the Schwarzschild-Kruskal solution on the space-like hypersurface $t=0$ may be given thus:

$$P_{ij}=0 \quad (3.1)$$

and

$$\begin{aligned} &g_{11}=2m\exp\{-[R(r)-2m]/2m\}/R(r),\\ &g_{22}=[R(r)]^2,\\ &g_{33}=[R(r)]^2\sin^2\theta,\\ &g_{st}=0,\ s\neq t, \end{aligned} \quad (3.2)$$

where the function $R(r)$ is determined by the inversion of the expression

$$r=2m+2(2m)^{1/2}(R-2m)^{1/2}\exp[(R-2m)/4m]. \quad (3.3)$$

[In the range of variables $R\geq 2m$ (and therefore $r\geq 2m$) this inversion is unique although it cannot be expressed by elementary functions.] For the purpose of piecing Cauchy data at $r=2m$, it is important to note that at $R=2m$ the Kruskal metric as given in Eq. (1.3) has the properties

$$\partial g_{st}/\partial r\,|_{r=2m}=0 \quad (3.4)$$

and

$$g_{11}\,|_{r=2m}=1. \quad (3.5)$$

Since in this paper we are not concerned with the most general solution of the constraint equations for the region $0\leq r\leq 2m$, but merely in the existence of a solution, we shall assume that Eq. (3.1) holds everywhere. This is trivially consistent with the jump conditions, namely, there are no discontinuities in the second fundamental form. The constraint equations (2.1) and (2.2) thereby reduce to the single equation

$$\bar{R}=0. \quad (3.6)$$

We have at our disposal six functions g_{ij} with which to satisfy this single equation, and although only three of these functions may be independent, (in view of the freedom we have to perform coordinate transformations) it seems rather evident that there should be a considerable number of solutions consistent with the boundary conditions. Despite this fact the actual construction of an explicit solution proved to be rather intricate.

For simplicity we shall only consider discontinuities of the "electric mode" type. That is, in the region $0\leq r\leq 2m$ we shall assume that the spatial metric has the form

$$de^2=A(r)dr^2+B(r)d\theta^2+C(r)\sin^2\theta d\phi^2. \quad (3.7)$$

Furthermore, it is rather clear that for this metric we can perform a coordinate transformation on the radial variable in order to obtain Eq. (3.5) everywhere in the domain $0\leq r\leq 2m$. That is

$$A=1 \quad \text{for} \quad 0\leq r\leq 2m. \quad (3.8)$$

As we have already mentioned, we shall require the metric to be strictly flat in a finite neighborhood of the origin. Thus

$$B(r)=C(r)=r^2 \quad \text{for} \quad 0\le r\le 0.1m. \tag{3.9}$$

(We cannot hope to have $B=C$ everywhere for then the solution would be spherically symmetric in which case we know that only the analytic Kruskal metric can occur.[5] This is essentially due to the fact that gravitational radiation has spin 2, and therefore cannot be emitted as S waves.)

For the metric of Eqs. (3.7) and (3.8), the only jump condition remaining to be satisfied is the second of Eqs. (2.20). This now reads

$$0=(1/B)[dB/dr]+(1/C)[dC/dr] =[d\ln BC/dr]. \tag{3.10}$$

It will therefore be convenient to introduce as new sets of variables

$$F=BC, \quad G=B/C. \tag{3.11}$$

The jump conditions will be satisfied by every solution of Eq. (3.6) such that F is of class C^1 and G is of class C^0. (The deviation of the function G from the value 1 measures the deviation from spherical symmetry. For $G<1$ the surfaces $r=\text{const}$ are oblate spheroids; for $G>1$ they are prolate spheroids.)

In terms of the functions F and G, the constraint equation (3.6) may be written

$$G'^2=G^2(16F''^{-1/2}G^{-1/2}+5F'^2F^{-2}-8F''F^{-1}), \tag{3.12}$$

where primes denote differentiation with respect to r. Assuming for the moment $F(r)$ given, this equation is of the form

$$G'^2=D(G,r). \tag{3.13}$$

In every domain in which we are assured that $D\ge 0$, (for example, $F''\le 0$ in our present case) we can always find a C^0 solution of Eq. (3.13) which varies over an arbitrarily narrow range of values about an arbitrarily chosen initial value. This is easily seen by taking the square root of Eq. (3.13). We obtain two differential equations, namely,

$$G'=\pm D^{1/2}. \tag{3.14}$$

As we reach the upper bound of the imposed interval, we merely tack on in a continuous fashion the solution with negative slope, and *vice versa*. Specifying the initial value of G, the initial choice of the sign of the slope, and the upper and lower bounds of permitted variation of G we obtain a unique solution of Eq. (3.13) in the class of functions C^0, which we may call a "zig-zag solution." Provided $0\le D<\infty$, it is evident that such a solution can span an arbitrarily large interval. We shall denote such a solution of Eq. (3.13) by the symbol $Z_D^{\pm}(r;r_1,\alpha;\epsilon_1,\epsilon_2)$, where the $+$ or $-$ denotes the initial choice of sign in Eq. (3.14); (r_1,α) denotes that at the initial point r_1 the function assumes the value α; $\epsilon_1\ge 0$ denotes that the upper bound is $\alpha+\epsilon_1$, and $\epsilon_2\ge 0$ denotes that the lower bound is $\alpha-\epsilon_2$. (We note that either, but in general not both, of ϵ_1 and ϵ_2 may be taken to be 0.) If we now define the function

$$z_D(r;r_1,r_2;\alpha;\epsilon) \equiv \mathrm{Max}[Z_D^-(r;r_1,\alpha;0,\epsilon),Z_D^+(r;r_2,\alpha;0.\epsilon)]. \tag{3.15}$$

where $r_1<r_2$, we obtain a C^0 solution of Eq. (3.13) whose upper bound is α, whose lower bound is $\alpha-\epsilon$, and which attains the value α at precisely r_1 and r_2.

We are now in a position to exhibit a specific solution of the constraint Eq. (3.12). In domains in which the function F has the property $F''\le 0$, it will be convenient (although not always essential) to employ the "zig-zag function" of Eq. (3.15), where the symbol D shall hereafter denote the right-hand side of Eq. (3.12). For those domains where $F''>0$ we shall explicitly exhibit the form of the functions F and G which solves the equation. The solution, which in itself is not very illuminating, is presented by intervals. The reader may readily confirm that F is of class C^1 and G of class C^0.

Domain I: $0\le r\le 0.1m$.

$$A=1;\ G=1;\ F=r^4.$$

(Note: In this interval the solution is simply Euclidean flat space.)

Domain II: $0.1m\le r\le 0.3m$.

$$A=1;\quad G=4(3\exp[(10r-m)/m] -\exp\{-[10r-m]/m\})^{-2};$$

$$F=\tfrac{1}{4}m^4 10^{-4}(3\exp[(10r-m)/m] -\exp\{-[10r-m]/m\})^2.$$

Domain III: $0.3m\le r\le 0.32m$.

$$A=1;\quad G=z_D(r;0.3m,0.32m;\alpha^{-1};10^{-4});$$

$$F=10^{-4}[\alpha m^4+\beta m^3(10r-3m)+\gamma m^2(10r-3m)^2 +\delta m(10r-3m)^3+\mu(10r-3m)^2(10r-3.2m)^2];$$

where

$$\alpha\equiv\tfrac{1}{4}(3e^2-e^{-2})^2\sim 121.35,$$
$$\beta\equiv\tfrac{1}{2}(9e^4-e^{-4})\sim 245.69,$$
$$\gamma\equiv 75(3.6)^4-75\alpha-10\beta-20(3.6)^2\sim 105.7,$$
$$\delta\equiv 250\alpha+25\beta-250(3.6)^4+100(3.6)^3\sim 844.7,$$
$$\mu\equiv -\delta^2/200\sim -3568.$$

[Note: μ is so chosen to assure that $F''(r)<0$ on the entire interval.]

Domain IV: $0.32m\le r\le m/25 \times(-1+9\exp[\arccos\alpha^{-1/4}])\sim 1.2m$.

$$A=1;\quad G=\cos^4[\arccos\alpha^{-1/4}-\ln\{(25r+m)/9m\}];$$
$$F=(r+m/25)^4.$$

[5] P. G. Bergmann, M. Cahen, and A. B. Komar, J. Math. Phys. **6**, 1 (1965).

Domain V: $m/25(-1+9\exp[\arccos\alpha^{-1/4}] \leq r \leq 1.95m$.

$$A=1;\quad G=1;\quad F=(r+m/25)^4.$$

(Note: The solution in this interval is locally flat, but if extrapolated to the origin would give rise to a conical cusp.)

Domain VI: $1.95m \leq r \leq 1.975m$.

$$A=1\,(G\text{ defined below});$$

$$F=10^{-4}[\xi m^4+\eta m^3(10r-19.5m)+\lambda m^2(10r-19.5m)^2+\sigma m(10r-19.5m)^3];$$

where

$$\xi \equiv (19.9)^4,$$

$$\eta \equiv 4(19.9)^3,$$

$$\lambda \equiv 8[6(20)^4-6\xi-\eta] \sim -99712,$$

$$\sigma \equiv 16[8\xi+\eta-8(20)^4] \sim +97792.$$

[Note: On the entire interval $F''(r)<0$].

Domain VII: $1.975m \leq r \leq 2m$.

$$A=1\,(G\text{ defined below});$$

$$F=(2m)^4.$$

(Note: On the entire interval we evidently have $F''=0$.)

Domains VI and VII: $1.95m \leq r \leq 2m$.

$$G=z_D(r;1.95m,2m;1,10^{-4}).$$

Domain VIII: $2m \leq r \leq \infty$.

$$A=2m/R(r)\exp\{-[R(r)-2m]/2m\};\quad G=1;$$

$$F=[R(r)]^4;$$

where $R(r)$ is the function defined by the inversion of Eq. (3.3). (Note: On this interval we are simply specifying the Schwarzschild-Kruskal metric.)

On the entire interval $0<r<\infty$ both F and G are found to be finite, nonvanishing and positive. We can therefore readily extract the components, B and C, of the spatial metric which satisfy the correct boundary and jump conditions:

$$B=(FG)^{1/2},\quad C=(F/G)^{1/2}. \tag{3.16}$$

The vanishing of B and C at the origin, as well as their divergence at $r=\infty$, is simply due to the use of polar coordinates. The reader may be puzzled by the vanishing of A at $r=\infty$. This is only due to an anomaly of the Kruskal coordinate system which we could readily have avoided had we, for example, chosen to present the Schwarzschild line element in the interval $2m \leq R \leq \infty$ in isotropic spherical coordinates.

V. CONCLUSION

We have demonstrated the existence of a solution of the initial value problem for the vacuum Einstein field equations which has the property that the initial spacelike hypersurface has Euclidean topology, and such that exterior to the Schwarzschild radius $R \geq 2m$ the solution remains indistinguishable from that of Schwarzschild for all "times" T. There is much arbitrariness in our construction. For example, we never made an effort to employ "magnetically" polarized gravitational radiation. However, all such solutions appear from the outside to be stable, spherically symmetric particles of mass m, although there are no singularities, "wormholes" or material sources for the gravitational field. The only "source" seems to be the nonlinear structure of the theory itself, which permits gravitational radiation to interact directly with itself and thereby keep itself bound.

In general, we should expect that as the various solutions propagate in time off from the initial surface, singularities will develop at finite temporal distances from the initial surface (although necessarily at a luminosity distance less than the Schwarzschild radius). In this situation we are of course no worse off than we were with the analytic Kruskal metric. With our vastly increased degrees of freedom there is even a hope that for some choice of Cauchy data a solution which remains nonsingular for all times could be found. Should this eventually prove to be impossible starting from an initial surface of Euclidean topology, we continue to have at our disposal the possibility of doctoring at will of the initial topology interior to the Schwarzschild radius. None of these latter possibilities have as yet been investigated.

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