Klein-Gordon Geon*

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A study of the spherically symmetric eigenstates of the Klein-Gordon Einstein equations (Klein-Gordon geons) reveals that these geons have properties that are uniquely different from other gravitating systems that have been studied. The equilibrium states of these geons seem analogous to other gravitating systems; but when the question of stability is considered from a thermodynamical viewpoint, it is shown that, in contrast with other systems, adiabatic perturbations are forbidden. The reason is that the equations of state for the thermodynamical variables are not algebraic equations, but instead are differential equations. Consequently, the usual concept of an equation of state breaks down when Klein-Gordon geons are considered. When the question of stability is reconsidered in terms of infinitesimal perturbations of the basic fields, it is then found that Klein-Gordon geons will not undergo spherically symmetric gravitational collapse. Thus, Klein-Gordon geons are counterexamples to the conjecture that gravitational collapse is inevitable.

I. INTRODUCTION

ALTHOUGH Einstein's theory of general relativity is more than fifty years old, it is still impossible to fully comprehend and estimate the richness and the implications of this theory. The reason for this is not a lack of research in this area, but is rather the general complexity and the nonlinearity of the equations. Also, since there are no experimental results available for strong gravitational fields, and since almost all our experience has been limited to Newtonian effects, it is difficult to know how to extrapolate theoretical results into unknown regions.

A prime example of this is the question of the stability of a gravitating system.

Geons, as we are using the term, are gravitating systems which are held together by gravitational forces and are composed of fundamental, classical fields. Thermal and electromagnetic geons of a statistical nature have been investigated,[1] and such systems are, more or less, unstable. More familiar gravitating systems are neutron stars and gravitating fluids, which are also statistical in nature. The results of the investigations on the stability of these latter two systems are perhaps best summarized in Ref. 2. The major result is that there are always spherically symmetric equilibrium states that are unstable, and if these states are perturbed in a certain manner, they will then undergo gravitational collapse into the Schwarzschild singularity. This result is independent of the equation of state as long as it is a local equation of state.[3]

These results had led Wheeler and others[2] to conjecture that perhaps gravitational collapse of a gravitating system is inevitable, once it has become sufficiently massive. However, investigations on the stability of pure electromagnetic, cylindrical geons, first described by Melvin,[4] show that this is not necessarily so. Not only are these geons stable under infinitesimal radial perturbations, but they are also stable for arbitrarily large radial perturbations[5] and will never undergo radial gravitational collapse. Other types of cylindrical gravitating systems including the perfect fluid have been investigated by Thorne.[6] For a cylindrically symmetric gravitating fluid, the stability properties are again different from those of a spherically symmetric gravitating fluid. With spherical symmetry, once the central density rises above a certain value, instability sets in; however, for cylindrical symmetry, the system is *stable* for this same range of central densities. Although the cylindrically symmetric system is unstable for a small range of central densities well below this value, one would not expect gravitational collapse to occur in this region.[6] Instead, the system should eventually reach another equilibrium state.

The general relativistic Klein-Gordon equation has properties which are quite different from the flat-space equation. In analogy with the hydrogen atom, one would expect that the Klein-Gordon equation would possess bound states in the presence of a point mass. At least this is so in the Newtonian limit. But if one represents the point mass by the Schwarzschild metric, no normalizable eigenstates exist. This was first shown by Peres.[7] Everson and Brill[8] have further investigated this problem and have shown that not only are these states unnormalizable, but also, no matter how weak the Klein-Gordon field is, it will significantly affect the metric near the Schwarzschild singularity. Thus it is necessary to consider the effects of the Klein-Gordon field on the metric. In other words, the complete Klein-Gordon Einstein equations must be considered.

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[1] Edwin A. Power and John A. Wheeler, Rev. Mod. Phys. **29**, 480 (1957).
[2] B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitational Theory and Gravitational Collapse* (University of Chicago Press, Chicago, 1965).
[3] C. W. Misner and H. S. Zapolsky, Phys. Rev. Letters **12**, 635 (1964).
[4] M. A. Melvin, Phys. Letters **8**, 65 (1964).
[5] K. S. Thorne, Phys. Rev. **139**, B244 (1965).
[6] K. S. Thorne, Ph.D. thesis, Princeton University, 1965 (unpublished).
[7] A. Peres, Phys. Rev. **120**, 1044 (1960).
[8] Bjarne L. Everson and Dieter R. Brill, Bull Am. Phys. Soc. **12**, 578 (1967).

However, when this is done, Everson and Brill[8] showed that because of the nonlinearity of the equations, the Schwarzschild event horizon will not occur. And, if the Schwarzschild event horizon does not occur, then we can no longer interpret the system as containing a point particle. Rather, because of the absence of any singularities, the system is now more analogous to a pure classical field bound by its own gravitational self-interaction. It is this latter interpretation that we will use in this paper, and by the term "Klein-Gordon geon" we will refer to an eigenstate of the Klein-Gordon Einstein equations. In addition to discussing the equilibrium solutions of these equations, we will also consider the stability of this gravitating system.

Although the problem considered here is strictly academic in nature, the unique properties of this system do merit investigation. As will be shown later, the spherically symmetric Klein-Gordon geon has eigenstates that are analogous to certain types of gravitating systems. However, at this point, the analogy ceases. When the stability question is considered, the properties of this geon are completely different from those of a normal spherically symmetric system, because the equations of state exhibit a *nonlocal* behavior and because the reasonable assumption, that adiabatic perturbations exist, is no longer valid. These two properties, which make this system uniquely different from other systems that have been studied, appear to be the major reason why the Klein-Gordon geon is resistant to gravitational collapse.

As mentioned before, in the classical sense that we are using, the Klein-Gordon field is not to be considered as a quantum field describing Bose-Einstein particles, but is rather to be considered as a pure classical field, free from singularities. In this sense, the equations are treated as in a unitary field theory,[9] wherein particles are not singularities in the fields, but instead are the localized regions of space in which the fields are concentrated.

It is to be emphasized that by no means is a new unitary field theory of elementary particles to be presented, because, as is well known, gravitational effects can certainly be neglected in high-energy physics.

In Sec. II, we present and discuss our conventions and notation, and in Sec. III, the eigenstates of the Klein-Gordon Einstein equations.

Starting with Sec. IV, the stability of the Klein-Gordon geon is treated, and the general equations required for the stability analysis is developed. Section V is concerned with the stability problem from a thermodynamical viewpoint. It is shown here that, unlike a normal system, the equations of state for the geon are nonlocal, and also that adiabatic perturbations are forbidden. Thus the Klein-Gordon geon is an example of a thermodynamic system with nonlocal properties, and these nonlocal properties appear to alter very seriously the stability properties, as will be seen in Sec. VI, where the general stability problem for radial perturbations will be treated. The result, contrary to the results for local systems,[2] will be that the Klein-Gordon geon is resistant to gravitational collapse.

[9] R. Finkelstein, R. Le Levier, and M. Ruderman, Phys. Rev. **83**, 326 (1951).

II. NOTATIONS AND CONVENTIONS

Since there is no general, definite convention which is used by everyone working in general relativity, we shall first specify our convention, which follows closely that of Tolman.[10] We take the signature of the metric to be $(-,-,-,+)$, where x^k $(k=1,2,3)$ are spacelike coordinates and $x^4=t$ is the timelike coordinate. Subscripts and superscripts designated by i, j, or k will be restricted to the values of 1, 2, or 3, while Greek ones can take on the full range of 1, 2, 3, or 4. The sign convention for the Riemann tensor is given by

$$A^{\alpha}{}_{;\mu\nu}-A^{\alpha}{}_{;\nu\mu}=-A^{\beta}R^{\alpha}{}_{\beta\mu\nu}\,, \tag{2.1}$$

where the semicolon indicates covarient differentiation. We define the Ricci tensor by

$$R_{\mu\nu}=R^{\alpha}{}_{\mu\nu\alpha}\,. \tag{2.2}$$

With these conventions, the Einstein equations are

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-8\pi GT_{\mu\nu}\,, \tag{2.3}$$

where G is the Newtonian gravitational constant and $T^4{}_4$ is positive and is the "energy density."

We take our units to be those where

$$c=8\pi G=1\,, \tag{2.4}$$

and c is the velocity of light. This leaves the unit of length still arbitrary, but in Sec. III we shall take it to be equal to the Compton wavelength of the bare mass of the Klein-Gordon field. In this case

$$\lambda\!\!\!^{-}_c=\hbar/M=1\,, \tag{2.5}$$

so that in these units the bare mass M is equal to Planck's constant divided by 2π.

Since we use the phrases "Klein-Gordon" and "Klein-Gordon Einstein" quite frequently, we shall hereafter refer to them by KG and KGE, respectively.

III. KLEIN-GORDON GEON

A. KGE Equations

The KGE equations that we are using are, of course, identical to other KGE equations found in the literature,[11] the only difference being notation. These equations can be derived from a variational principle if we take the total Lagrangian to be the sum of the two individual Lagrangians.

$$L=R+g^{\alpha\beta}\Phi^*{}_{;\alpha}\Phi_{;\beta}-(M/\hbar)^2\Phi^*\Phi\,, \tag{3.1}$$

[10] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Clarendon Press, Oxford, 1934).

[11] A. Das, J. Math. Phys. **4**, 45 (1963).

where R is the contraction of the Ricci tensor and Φ is the complex KG field. As mentioned in Sec. II, we shall take our unit of length equal to $\hbar/M$, so that the quantity $(M/\hbar)^2$ appearing in Eq. (3.1) will be unity.

Varying Φ^* in (3.1) will give the general relativistic KG equation, which is

$$\Phi^{;\alpha}{}_{;\alpha}+\Phi=0. \tag{3.2}$$

And, varying $g^{\alpha\beta}$ will give the Einstein equations, which are

$$G_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-T_{\mu\nu}, \tag{3.3}$$

with the KG stress-energy tensor being given by

$$T_{\mu\nu}=\tfrac{1}{2}[\Phi^*{}_{;\mu}\Phi_{;\nu}+\Phi^*{}_{;\nu}\Phi_{;\mu}+g_{\mu\nu}(\Phi^*\Phi-\Phi^*{}_{;\alpha}\Phi^{;\alpha})]. \tag{3.4}$$

When Φ is not required to be real, then the KG field will possess a conserved vector current

$$J_\nu=\tfrac{1}{2}i(\Phi^*{}_{;\nu}\Phi-\Phi_{;\nu}\Phi^*), \tag{3.5}$$

which, owing to (3.2), has a vanishing divergence.

$$J^\nu{}_{;\nu}=0. \tag{3.6}$$

Normally, $eJ^\nu/\hbar$ would be interpreted to be the electromagnetic charge current, and then Eq. (3.6) would express charge conservation. However, since the KGE equations do not include the electromagnetic fields, we shall instead interpret J^ν to be a "particle-number" current. Then the electromagnetic charge of KG geons will be zero and Eq. (3.6) will express conservation of "particle number."

B. Spherically Symmetric, Time-Invariant, KG Geon

We now restrict our attention to a particular set of eigenstates of the KGE equations: those states that have a particlelike nature in that they are localized to some finite region of space. Also, we shall demand all fields to be nonsingular, the topology of the space to be simply connected and homomorphic to Minkowski 4-space, and the physically measurable quantities of the eigenstates to be spherically symmetric.

As is well known, we may take the metric to be the Schwarzschild metric

$$g_{\mu\nu}=\mathrm{diag}[-a^2,\,-r^2,\,-r^2\sin^2\theta,\,b^2], \tag{3.7}$$

where $x^1=r$, $x^2=\theta$, $x^3=\phi$, and $x^4=t$. For the KGE equations, Everson and Brill[8] have shown that the Schwarzschild singularity will never occur, so that a^2 in Eq. (3.7) will always remain finite. Then, from the algebraic properties of the KG stress-energy tensor and the assumption that the KG field will be localized, one can show[12] that $a^2\geq 1$, and that b^2 is always nonzero, finite, and is a monotonically increasing function of the radial coordinate r. Also, $a^2(0)=1=a^2(\infty)$, and since a^2 and b^2 will never cross zero, we may take both a and b to be positive. Since the KG field is to be localized, our KGE metric will approach the empty-space Schwarzschild metric as $r\to\infty$, where $b(\infty)=1$. Then, since b is monotonic, the value of b at $r=0$ is restricted by $0<b(0)<b(\infty)=1$. We now consider the effect of this metric on the KG equation, to see if localized solutions can exist.

Although all physically measurable quantities are to be time-independent, this does not imply that the KG field must be time-independent. Rather, as in the case of ordinary quantum mechanics, since all physically measurable quantities are bilinear combinations of Φ, we may take

$$\Phi(r,t)=e^{iEt}\psi(r), \tag{3.8}$$

where E is a real constant and ψ is a real function of r. Then, from (3.2), (3.7), and (3.9), the KG equation becomes

$$\psi''+(2/r+\nu'-\lambda')\psi'+a^2(E^2/b^2-1)\psi=0, \tag{3.9}$$

where the primes indicate differentiation by r, and where λ and ν are defined by

$$a=e^\lambda, \tag{3.10a}$$

$$b=e^\nu. \tag{3.10b}$$

Note that λ and ν, as defined by (3.10), differ by a factor of 2 from their usual definitions.

The eigenvalue nature of (3.9) can be seen more clearly if we transform to a new radial coordinate, where

$$s\equiv\int_0^r \frac{a}{b}dr. \tag{3.11}$$

In terms of this coordinate, the KG equation is

$$\frac{d^2\psi}{ds^2}+\frac{2}{r}\frac{b}{a}\frac{d\psi}{ds}+(E^2-b^2)\psi=0. \tag{3.12}$$

Except for the second term, Eq. (3.12) is exactly the radial part of the Schrödinger equation for the potential b^2, and with an eigenvalue E^2. As $r\to\infty$, the potential b^2 is very similar to the potential for the hydrogen atom in that it contains a $1/r$ term and is monotonically increasing. From this fact we may expect that there will be a countable infinity of localized solutions for ψ, corresponding to the different number of nodes that ψ may have. (Since, in the Newtonian limit, only the zero-node state appears to be stable,[12] we shall consider only this state in this paper.) Finally, note from (3.12) that if localized solutions are to exist, then $E^2<b^2(\infty)=1$. And in this case ψ will then vanish exponentially if E^2 is an eigenvalue. Also, since the KGE equations only depend on E^2 and are independent of the sign of E, we shall take E to be positive. Of course, if E is an eigenvalue, then $-E$ is also an eigenvalue corresponding to the same eigensolution for the metric, but where Φ is replaced by Φ^* [see Eq. (3.8)].

Finally, note that although the KG equation is linear

[12] David J. Kaup, Ph.D. thesis, University of Maryland, 1967 (unpublished).

in ψ, the Einstein equations are not. Thus the Einstein equations will be very strongly dependent on the value of ψ at $r=0$, while the KG equation will be only indirectly dependent on it. Therefore, all our equilibrium quantities will be dependent on this variable, which we shall call β and define it by

$$\beta \equiv \psi(0). \tag{3.13}$$

Since the phase of Φ is arbitrary due to the bilinear forms of $T^{\mu\nu}$ and J^ν, we shall take β to be positive. Then, if we restrict ourselves to the zero-node solutions, the various eigenstates of the KGE equations may be identified by the value of β.

Before giving the results of numerical calculations, we define four equilibrium quantities in addition to the eigenvalue E.

First, since the divergence of J^ν vanishes, we may define a particle number, since the integral of J^4 will be a constant of the motion and is conserved quantity. We define this conserved quantity N by

$$N=\int_0^\infty r^2abJ^4dr. \tag{3.14}$$

Then from (3.5) and (3.8)

$$N=\int_0^\infty r^2a\frac{E}{b}\psi^2dr. \tag{3.15}$$

The definition of the mass energy m is taken to be equivalent to the integral of $T^4{}_4$, in that we shall define m from the asymptotic form of the metric. For spherically symmetric motion, this m is also a constant of the motion, and like N is then a conserved quantity. From the Einstein equations, if ψ is localized, as $r\rightarrow\infty$, the metric is given by

$$a^2=(1-m/r)^{-1}+O(\psi^2), \tag{3.16a}$$

$$b^2=a^{-2}+O(\psi^2), \tag{3.16b}$$

where m is the mass energy of the eigenstate.

From (3.9) and (3.16) one may now obtain the asymptotic form of ψ:

$$\psi\simeq Ae^{-\kappa r}r^\alpha[1+O(1/r)], \tag{3.16c}$$

where A is some constant and

$$\kappa=(1-E^2)^{1/2}, \tag{3.16d}$$

$$\alpha=-1-(m/2\kappa)(1-2\kappa^2). \tag{3.16e}$$

As one can see, κ is an indication of the size of the geon, since it governs how rapidly ψ will vanish. Thus we will refer to $1/\kappa$ as the size of the geon.

The last item of interest is a quantity that we shall call the "binding energy" of the state. Consider an arbitrary eigenstate of N "particles" of mass energy m. It can be shown[12] that in the Newtonian limit[13] both N and m approach zero, but their ratio approaches unity. Thus the mass energy of N particles, infinitely separated from each other, is simply N. When these N particles are brought together to form an eigenstate, then the mass energy will be m. Thus, the "binding energy per particle," designated by B, is

$$B=(N-m)/N. \tag{3.17}$$

C. Numerical Results and Discussion

Table I is a summary of the numerical results for KG geons with zero nodes and at various values of β, while Fig. 1 shows the solution for ψ at $\beta=0.4$.

One of the most striking features of these results is the similarity to the results for gravitating fluids or neutron stars. In Fig. 2, we have plotted N and m versus β, and one can see that the relationship of N and m is similar to that obtained for a perfect fluid (see Ref. 3). First, m and N rise with m just below N, until they both achieve a maximum. Then they both fall off, but the m curve crosses the N curve and then the system enters into a region of "energy excess," where the binding energy is negative. Finally, N and m start to oscillate and approach certain limit points.

These are just the properties of neutron stars as treated by Harrison *et al.*,[2] except for the minor difference that they use the central pressure p_c instead of the parameter β. However, when we discuss the KG stress-energy tensor in Sec. V, we shall define a p_c as well as a central density ρ_c. Both of these parameters are given in Table I, and, as one can see, there is a one-to-one relation between β and p_c.

The one remaining, unexplained item in Table I is $1/b(0)$, which is simply the fractional amount that the energy would be red-shifted if a photon were emitted from the center of the geon.

Although at first it might seem surprising that the KGE equations possess these localized solutions, later on, when we discuss the thermodynamic properties of the KG geon in Sec. V, we shall show that the $T^{\mu\nu}$ of the geon is very similar to that of other gravitating systems. Since for gravitating fluids, the m and N curves are relatively insensitive to the equations of state,[3] it is not surprising that the m and N curves for the geon are similar to others. Also, since the values of m, N, and κ for gravitating fluids (κ^{-1} is roughly the radius of the geon) approach limiting values,[2] one would also expect that these parameters for the KG geon would also approach limiting values as shown in Table I.

In concluding this section, we want to point out the relations between m, M, and E, and this can best be done by quantizing[9] the eigenstates in the following manner. In quantizing the electromagnetic four-current $eJ^\nu/\hbar$, where J^ν is given by (3.5), one requires the total

[13] The Newtonian limit is where the variation of the metric from the flat-space values becomes vanishingly small. This is given by letting $\beta\rightarrow 0$, but never letting β reach zero.

TABLE I. Equilibrium quantities for Klein-Gordon geons.

β	E	κ	m	N	B	$1/b(0)$	p_c	ρ_c
10^{-4}	0.999951	0.00989	0.0347	0.0347	$+1.6314\times10^{-5}$	1.0000949	4.593×10^{-13}	1.0000×10^{-8}
10^{-2}	0.9951	0.0985	0.3418	0.3423	$+1.6085\times10^{-3}$	1.00954	4.639×10^{-7}	1.0046×10^{-4}
0.1	0.9540	0.2997	0.9494	0.9629	$+1.4047\times10^{-2}$	1.1003	5.094×10^{-4}	1.051×10^{-2}
0.2	0.9137	0.4065	1.1681	1.1965	+0.0237	1.2126	4.551×10^{-3}	4.455×10^{-2}
0.3	0.8785	0.4777	1.2499	1.2875	+0.0292	1.3392	1.729×10^{-2}	1.073×10^{-1}
0.4	0.8484	0.5294	1.2655	1.3054	+0.0306	1.4825	4.654×10^{-2}	2.065×10^{-1}
0.5	0.8230	0.5681	1.2440	1.2796	+0.0278	1.6457	1.043×10^{-1}	3.543×10^{-1}
0.6	0.8023	0.5970	1.2009	1.2265	+0.0209	1.8327	2.091×10^{-1}	5.691×10^{-1}
0.8	0.7751	0.6318	1.0817	1.0749	−0.0063	2.2981	6.954×10^{-1}	1.335×10^{0}
1.0	0.7677	0.6408	0.9478	0.9009	−0.0520	2.9331	2.035×10^{0}	3.035×10^{0}
1.2	0.7807	0.6250	0.8216	0.7376	−0.1139	3.8276	5.708×10^{0}	7.148×10^{0}
1.4	0.8099	0.5865	0.7247	0.6155	−0.1776	5.1421	1.602×10^{1}	1.798×10^{1}
1.5	0.8266	0.5628	0.6952	0.5793	−0.2000	6.0433	2.695×10^{1}	2.920×10^{1}
1.6	0.8410	0.5410	0.6818	0.5632	−0.2105	7.1813	4.541×10^{1}	4.797×10^{1}
1.8	0.8557	0.5176	0.6994	0.5839	−0.1979	1.0514×10^{1}	1.295×10^{2}	1.327×10^{2}
2.0	0.8530	0.5219	0.7398	0.6311	−0.1723	1.6116×10^{1}	3.760×10^{2}	3.800×10^{2}
2.2	0.8451	0.5346	0.7629	0.6582	−0.1590	2.5676×10^{1}	1.137×10^{3}	1.142×10^{3}
2.4	0.8403	0.5422	0.7634	0.6588	−0.1587	4.2340×10^{1}	3.642×10^{3}	3.648×10^{3}
2.7	0.8407	0.5415	0.7512	0.6444	−0.1659	9.5769×10^{1}	2.363×10^{4}	2.363×10^{4}
3.0	0.8426	0.5385	0.7490	0.6417	−0.1672	2.3595×10^{2}	1.779×10^{5}	1.779×10^{5}
3.5	0.8422	0.5392	0.7513	0.6445	−0.1658	1.2875×10^{3}	7.201×10^{6}	7.201×10^{6}
4.0	0.8422	0.5391	0.7510	0.6440	−0.1660	8.9415×10^{3}	4.537×10^{8}	4.537×10^{8}
6.0	0.8422	0.5391	0.7510	0.6440	−0.1660	2.3959×10^{8}	7.330×10^{17}	7.330×10^{17}
8.0	0.8422	0.5391	0.7510	0.6440	−0.1661	3.3192×10^{14}	2.501×10^{30}	2.501×10^{30}

charge of the system to be equal to an integer times e. Taking this integer to be unity, and then letting $e\to 0$, we have

$$\frac{1}{\hbar}\oint J^4 dS_4 = 1\,,$$

so that by Eq. (3.14)[14]

$$4\pi N = \hbar\,. \tag{3.18}$$

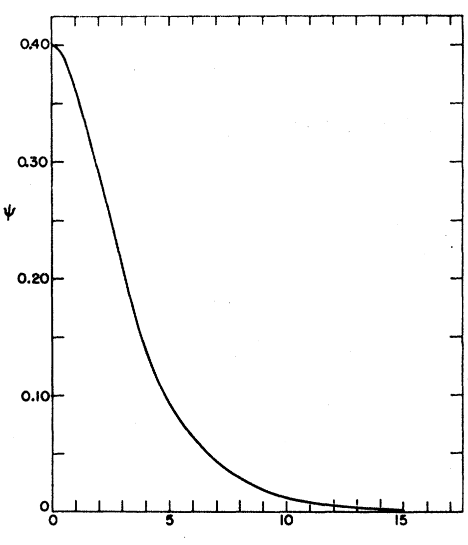

FIG. 1. Eigensolution for $\beta=0.4$. The coordinate r is dimensionless.

[14] Note that the factor of 4π in Eq. (3.18) is correct and is due to the following conventions. First, we have taken $8\pi G=1$ instead of $G=1$, and second, we have omitted the usual factor of 2 in Eq. (3.61a). Because of this, m is given by $m=\int_0^\infty r^2 T^4{}_4 dr$, which differs by a factor of 4π from the usual expression. Since it is desirable to have $(N/m)\to 1$ in the Newtonian limit, then N must be given by (3.14), which also differs from the usual expression by a factor of 4π.

In cgs units, Eq. (3.18) is

$$N=\frac{\hbar}{4\pi}\frac{(8\pi G)}{c^3}\left(\frac{cM}{\hbar}\right)^2 = M(2GM/\hbar c)\,. \tag{3.19}$$

From the asymptotic form of the metric, one can show[12] that the gravitational mass in grams, designated by m_g, is related to the unitless parameter m by

$$m_g = (\hbar c/2GM)m\,. \tag{3.20}$$

Then from (3.19) and (3.20)

$$m_g = (m/N)M = (1-B)M\,. \tag{3.21}$$

Interpreting m_g as the total mass energy of the geon, we

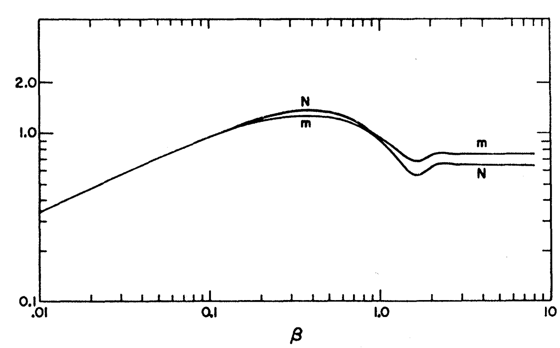

FIG. 2. Plot of m and N versus β.

then see that m_g is equal to the bare mass, reduced by a factor corresponding to the binding energy.

In ordinary relativistic quantum mechanics of point particles, the eigenvalue E is interpreted to be the total mass energy of the system; however, this is not so for extended particles, like KG geons. It can be shown[12] that in the Newtonian limit the ratio m/N is given by

$$\frac{m}{N}\simeq E\int_0^\infty r^2\left(\frac{1}{ab}\right)\left(\frac{a}{b}\right)\psi^2dr\bigg/\int_0^\infty r^2\left(\frac{a}{b}\right)\psi^2dr. \quad (3.22)$$

Then, *if* the product ab is equal to unity, m/N would be equal to E, so that, from (3.21), m_g would equal EM. And, *in this case*, E could also be interpreted to be the total mass energy of the system. However, $ab=1$ only in *empty* space, so that, for an extended particle where gravitational interactions are important, m/N will not in general be equal to E. In fact, one can show,[12] in the Newtonian limit, that for KG geons

$$B=1-m/N\simeq\tfrac{1}{3}(1-E) \quad (3.23a)$$

or

$$E\simeq1-3B. \quad (3.23b)$$

Thus the mass associated with E (which is $\hbar E=ME$) is *less* than the gravitational mass in the Newtonian limit.

There is one final comment to be made concerning Eq. (3.19). Since Table I shows that there is an upper limit on N, Eq. (3.19) places an upper limit on the value of M, which is

$$M=1.76\times10^{-5}\,\text{g}. \quad (3.24)$$

If quantized states for KG geons are to exist, then their bare mass must be less than or equal to this value. For a geon with this value of a bare mass, the Schwarzschild radius is almost equal to the radius of the geon, which is about 2×10^{-33} cm.

IV. PERTURBED KGE EQUATIONS FOR SPHERICAL SYMMETRY

Before starting the discussion on the stability of the KG geon, we first derive and present the perturbed equations for spherical symmetry.

Upon being perturbed, the metric can still be given by the Schwarzschild form, if we allow g_{11} and g_{44} to become time-dependent. Choosing this gauge, we obtain for the perturbed metric

$$\delta g_{\mu\nu}=\text{diag}[-a^2 2\delta\lambda,\,0,\,0,\,2b^2\delta\nu], \quad (4.1)$$

where $\delta\lambda$ and $\delta\nu$ are functions of both r and t. With (4.1), the perturbed Einstein tensor is

$$\delta G^1{}_1=\frac{2}{a^2}\left(\frac{\delta\nu'}{r}-2\frac{\nu'}{r}\delta\lambda-\frac{\delta\lambda}{r^2}\right), \quad (4.2a)$$

$$\delta G^1{}_4=(2/a^2)\delta\dot\lambda/r, \quad (4.2b)$$

$$\delta G^4{}_4=\frac{2}{a^2}\left(2\frac{\lambda'}{r}\delta\lambda-\frac{1}{r}\delta\lambda'-\frac{\delta\lambda}{r^2}\right), \quad (4.2c)$$

where the dots denote time differentiation. We have omitted the 22 component in the above equations since it is linearly dependent on the other three components.

The unperturbed KG field is given by (3.8). We choose to express the perturbed KG field in the following manner:

$$\delta\Phi(r,t)=e^{iEt}[R(r,t)+iI(r,t)], \quad (4.3)$$

where R and I are real functions of r and t.

First, we perturb the KG equation [Eq. (3.2)], and upon separating the real and imaginary parts we have the two following equations:

$$R''+\left(\frac{2}{r}+\nu'-\lambda'\right)R'+a^2\left(\frac{E^2}{b^2}-1\right)R+2E\frac{a^2}{b^2}\dot I-\frac{a^2}{b^2}\ddot R+\psi'(\delta\nu'-\delta\lambda')+2a^2\psi\left(\frac{E^2}{b^2}-1\right)\delta\lambda-2a^2\psi\frac{E^2}{b^2}\delta\nu=0, \quad (4.4a)$$

$$I''+\left(\frac{2}{r}+\nu'-\lambda'\right)I'+a^2\left(\frac{E^2}{b^2}-1\right)I-\frac{a^2}{b^2}\ddot I-2E\frac{a^2}{b^2}\dot R+a^2(E/b^2)\psi(\delta\dot\nu-\delta\dot\lambda)=0. \quad (4.4b)$$

Similarly, for the KG stress-energy tensor, given by (3.4), we find

$$\delta T^1{}_1=-R\psi\left(\frac{E^2}{b^2}-1\right)-\frac{1}{a^2}\psi'R'-\frac{E}{b^2}\psi\dot I+(\delta\nu/b^2)E^2\psi^2+a^{-2}\psi'^2\delta\lambda, \quad (4.5a)$$

$$\delta T^1{}_4=-\frac{1}{a^2}\dot R\psi'+\frac{E}{a^2}(\psi'I-I'\psi), \quad (4.5b)$$

$$\delta T^4{}_4=R\psi\left(\frac{E^2}{b^2}+1\right)+\frac{1}{a^2}\psi'R'+\frac{E}{b^2}\psi\dot I-(\delta\nu/b^2)E^2\psi^2-a^{-2}\psi'^2\delta\lambda, \quad (4.5c)$$

while the unperturbed Einstein equations are equivalent to

$$\frac{2}{r}(\lambda'+\nu')=a^2\frac{E^2}{b^2}\psi^2+\psi'^2, \quad (4.6a)$$

$$\frac{a^2-1}{r^2}+\frac{\lambda'-\nu'}{r}=\tfrac{1}{2}a^2\psi^2. \quad (4.6b)$$

If the KG equation,

$$\psi''+(2/r+\nu'-\lambda')\psi'+a^2(E^2/b^2-1)\psi=0, \quad (4.7)$$

is satisfied, one may readily show that (4.8) is a solution of the perturbed KGE equations:

$$\delta\nu = \dot{G}(t)\,, \tag{4.8a}$$

$$I = E\psi(r)G(t)\,, \tag{4.8b}$$

$$R = 0\,, \tag{4.8c}$$

$$\delta\lambda = 0\,, \tag{4.8d}$$

where G is an arbitrary function of time. This is a gauge solution, which corresponds to a relabeling of the t coordinate,[12] and thus it is an unphysical solution.

We now expand our infinitesimal quantities R, I, $\delta\lambda$, and $\delta\nu$ in eigenstates of $e^{i\omega t}$. And, in order to eliminate the unphysical gauge solution given by (4.8), we define two new functions K and $\delta\mu$ by the following equations:

$$I = \psi \int_0^r \frac{a}{b}\frac{\dot{K}}{\psi}dr\,, \tag{4.9a}$$

$$\delta\nu = \delta\mu + \frac{1}{E}\int_0^r \frac{a}{b}\frac{\ddot{K}}{\psi}dr. \tag{4.9b}$$

Then the perturbed KG equations become

$$R'' + \left(\frac{2}{r} + \nu' - \lambda'\right)R' + a^2\left(\frac{E^2+\omega^2}{b^2} - 1\right)R - \frac{\omega^2}{E}\frac{a}{b}\frac{\psi'}{\psi}K$$
$$+\psi'(\delta\mu' - \delta\lambda') + 2a^2\psi\left(\frac{E^2}{b^2} - 1\right)\delta\lambda$$
$$-2\frac{a^2E^2}{b^2}\psi\delta\mu = 0 \tag{4.10a}$$

and

$$K' + K\left(\frac{2}{r} + \frac{\psi'}{\psi}\right) - 2E\frac{a}{b}R + E\frac{a}{b}\psi(\delta\mu - \delta\lambda) = 0. \tag{4.10b}$$

Similarly, the perturbed Einstein equations are

$$2\frac{\delta\mu'}{r} + a^2\frac{E^2}{b^2}\psi^2\delta\mu + \delta\lambda\left[\psi'^2 - \frac{2}{r}\left(2\nu' + \frac{1}{r}\right)\right]$$
$$-\frac{\omega^2}{E}\frac{a}{b}\frac{2K}{r\psi} - a^2R\psi\left(\frac{E^2}{b^2} - 1\right) - \psi'R' = 0\,, \tag{4.10c}$$

$$2\delta\lambda/r - R\psi' - E(a/b)K\psi = 0. \tag{4.10d}$$

As one may now readily verify, the perturbed KGE equations (4.10) are not functions of ω but rather of ω^2, and have real coefficients. Thus if ω is an eigenvalue, then $-\omega$ is also an eigenvalue. As was shown in the author's dissertation,[12] ω^2 must be real for the zero-node state. Thus a necessary condition for the KG geon to be stable is that all eigenvalues of ω^2 be positive.[15]

[15] Tullio Regge and John A. Wheeler, Phys. Rev. **108**, 1063 (1957).

V. THERMODYNAMICS OF KG GEON

Considering the success that has been achieved by applying thermodynamical principles to the problem of the stability of gravitating systems, it is worthwhile to pause and investigate the possibility of applying the same principles to the problem of the stability of the KG geon, which is also a gravitating system. To do so, we first discuss the equilibrium states, and then investigate the perturbed equations for radial perturbations from a thermodynamical viewpoint. We finally show that, unlike the majority of gravitating systems, the assumption that adiabatic radial perturbations exist is not valid for the KG geon. Thus all radial perturbations of the KG geon must be nonadiabatic.

A. State Variables for Equilibrium States

Since the KG geon has anisotropic stresses, it is necessary to define another state variable in addition to the energy density ρ, the pressure p, and the number density n. We shall call this new variable q and define all the variables as follows:

$$\rho \equiv T^4{}_4\,, \tag{5.1a}$$

$$p \equiv -T^1{}_1\,, \tag{5.1b}$$

$$q \equiv T^2{}_2 - T^1{}_1 = T^3{}_3 - T^1{}_1\,, \tag{5.1c}$$

$$n \equiv (J^\nu J_\nu)^{1/2}\,\mathrm{sign}(J^4)\,, \tag{5.1d}$$

and then from (3.4), (3.5), (3.9), and (3.8), the state variables at equilibrium are given by

$$q = a^{-2}\psi'^2\,, \tag{5.2a}$$

$$\rho = \tfrac{1}{2}(E^2/b^2+1)\psi^2 + \tfrac{1}{2}q\,, \tag{5.2b}$$

$$p = \tfrac{1}{2}(E^2/b^2-1)\psi^2 + \tfrac{1}{2}q\,, \tag{5.2c}$$

$$n = (E/b)\psi^2. \tag{5.2d}$$

ψ satisfies the KG equation (3.9), which is

$$\psi'' + (2/r + \nu' - \lambda')\psi' + a^2(E^2/b^2 - 1)\psi = 0. \tag{5.3}$$

The Einstein equations are equivalent to

$$p' + \nu'(\rho + p) + (2/r)q = 0\,, \tag{5.4a}$$

$$2\nu'/r = a^2(p + m^*/r^3)\,, \tag{5.4b}$$

where

$$a^2 = (1 - m^*/r)^{-1} \tag{5.4c}$$

and

$$m^*(r) = \int_0^r r^2\rho dr. \tag{5.4d}$$

Equations (5.1) and (5.4) are general relationships between the metric components and the state variables of a spherically symmetric, gravitating system in equilibrium. Equations (5.2) and (5.3) are the "equations of state" for the KG geon. There is one algebraic relation contained in (5.2):

$$q = q(\rho, p, n) = \rho + p - n^2/(\rho - p). \tag{5.5}$$

Equation (5.5) is an equation of state for q, which gives q as a function of the other state variables. As we shall show later, (5.5) is also valid when the geon is infinitesimally perturbed, which is also another requirement for an equation of state.

However, (5.5) is the only *algebraic* equation of state. All others contained in (5.2) must be *differential* equations of state, similar to (5.3). Thus the thermodynamical variables of the KG geon will exhibit a nonlocal behavior upon perturbation, since, in addition to these differential equations, boundary conditions must be specified.

Although the equations of state for the KG geon are nonlocal, the equilibrium states and the equilibrium values of the state variables are nevertheless very reasonable. And, they do seem analogous to certain systems that obey *local* equations of state.

As seen from (5.2), ρ, n, and q are positive-definite functions. p as defined by (5.1b) is the radial pressure, and it is also positive for all eigenstates, as can be seen from (5.2c) and (5.4a). At $r=0$, $E>b(0)$, so that the central pressure $p_c=p(0)$ is greater than zero. From (5.4a), $p'\leq 0$ for all values of r, and since p vanishes like ψ^2 as $r\to\infty$, and thus approaches zero, then p must be non-negative everywhere.

Also, ρ' is negative so that ρ, as well as p, is a monotonically decreasing function. From (5.2b) and (5.2c),

$$\rho'=p'+2\psi'\psi\,, \tag{5.6}$$

and since the product $\psi'\psi$ is negative (see Fig. 1), $\rho'\leq 0$.

From these relations it is easy to see that the state variables of the KG geon are identical to those of a system where the equations of state are local, and are given by a γ law, provided, of course, that γ is not constant and is defined in such a manner that it is given by

$$\gamma=p'\rho/\rho' p\,. \tag{5.7}$$

Since (5.5) is dependent on n, another local equation of state would be required to give n. But since

$$n'=(E/b)(2\psi'\psi-\nu'\psi^2)\leq 0\,, \tag{5.8}$$

a similar γ law for n could be given.

Therefore, with these considerations in mind, it is not surprising that the N and m curves given in Fig. 2 closely resemble those of gravitating fluids which obey a γ-law equation of state. The only real difference in the equilibrium states appears to be the anisotropic stresses, which we now comment on.

About the center of the geon, q is very small since ψ' is zero at the center. In this region, the geon will closely resemble a perfect fluid with isotropic stresses. As one goes out from the center, the stresses become more and more anisotropic. Although the radial pressure p remains positive, it vanishes faster than the angular pressure $(-T^2{}_2=-T^3{}_3)$, which eventually goes negative. This behavior can be clearly exhibited by using the asymptotic solutions to calculate p and q.

At $r\to\infty$, the asymptotic solution of the KGE equations is given by (3.16). From these equations and (5.2), we find that to the lowest order

$$p\simeq\rho\kappa/r\,, \tag{5.9a}$$

$$q\simeq\kappa^2\rho+r^{-1}\rho(2\kappa-\kappa^3-E^2m)\,, \tag{5.9b}$$

so that

$$-T^2{}_2=-T^3{}_3=p-q\simeq-\kappa^2\rho[1-(E^2/\kappa^2r)(m-\kappa)]\,. \tag{5.9c}$$

As can be clearly seen from (5.9), as $r\to\infty$ the ratio of p/q vanishes and the angular stresses will correspond to a negative pressure, or rather elastic restoring forces. Consequently, a very crude model for representing the internal stresses of the KG geon would be an inflated balloon. The internal pressure in the balloon is positive and isotropic while the angular pressure on the balloon's skin is negative and the radial pressure is positive.

B. Adiabatic, Radial Perturbations

For time-dependent, radial motion, Eqs. (5.1) and (5.4b)–(5.4d) are still valid if one allows all quantities to contain a time dependence. To first order in the perturbations, an additional term must be added to (5.4a), giving

$$p'+\nu'(\rho+p)+(2/r)q-b^{-2}\partial_4T_{14}=0\,. \tag{5.10a}$$

The 14 component of the Einstein equations must also be included in these equations:

$$2\partial_4\lambda=rT_{14}\,. \tag{5.10b}$$

In the standard manner, we define the 4-velocity of the differential elements of the geon by

$$J^\nu\equiv nU^\nu\,, \tag{5.11}$$

where n is still given by (5.1d). From (5.11) and (5.1d), $U^4>0$.

The calculation of the perturbed quantities can now be carried out from the definitions, Eqs. (5.1) and (5.11). Using (4.1), (4.3), and (4.9), we have

$$\delta q=(2/a^2)\psi'R'-(2/a^2)\psi'^2\delta\lambda\,, \tag{5.12a}$$

$$\delta\rho=\psi R(E^2/b^2+1)-(E^2/b^2)\psi^2\delta\mu+\tfrac{1}{2}\delta q\,, \tag{5.12b}$$

$$\delta p=\delta\rho-2R\psi\,, \tag{5.12c}$$

$$\delta n=2(E/b)R\psi-(E/b)\psi^2\delta\mu\,, \tag{5.12d}$$

$$T_{14}=i\omega[\psi'R+E(a/b)K\psi]\,, \tag{5.12e}$$

$$U^1=-(i\omega/aE)K/\psi\,. \tag{5.12f}$$

From (5.12), one may now readily verify that (5.5) is valid to first order in the perturbations.

Equations (5.12) give the Eulerian variations of the respective parameters, and since the standard thermodynamical equations are only valid for Lagrangian variations (which we will denote by Δ), whereby one follows the differential elements of the system, it is necessary to determine the relation between the

Lagrangian variation of the Schwarzschild radial coordinate ξ and the Eulerian variation of the KG field. This is given by (5.12f) and

$$U^1=\frac{d\xi}{ds}=\frac{dt}{ds}\frac{d\xi}{dt}=\frac{i\omega}{b}\xi\,, \tag{5.13}$$

so that

$$K=-a(E/b)\psi\xi\,. \tag{5.14}$$

For adiabatic motion with anisotropic stresses, the thermodynamical identity is no longer given by

$$\Delta\rho=(\rho+p)\Delta n/n\,. \tag{5.15}$$

To determine the correct generalization of (5.15), one may use the general relativistic elastic theory as developed by Hernandez,[16] which for our special case gives for adiabatic motion[12]

$$\Delta\rho=(\rho+p)\Delta n/n+(2/r)q\xi\,. \tag{5.16a}$$

Also requiring the perturbation to conserve particle number gives the following relation for Δn^2:

$$\Delta n/n+\Delta\lambda+(2/r)\xi+\xi'=0\,. \tag{5.16b}$$

The last term in (5.16a) vanishes if the stress-energy tensor is isotropic, and then (5.16a) reduces to (5.15).

Now from these equations we shall show that no adiabatic perturbations exist. This will be done by showing that (5.16a) is inconsistent with the perturbed KGE equations.

First, the tensor component T_{14} may be expressed as

$$T_{14}=(\rho+p)U_1U_4+B\,, \tag{5.17}$$

and we shall proceed to show that if (5.16a) is true, and if T_{14} is finite at $r=0$, then B must be zero. From (5.10b), (5.13), and (5.17),

$$2\delta\lambda=-ra^2(\rho+p)\xi+(r/i\omega)B\,. \tag{5.18}$$

From the definition of λ, (5.4c) and (5.4d), $\delta\lambda$ satisfies the equation

$$\frac{d}{dr}\left(\frac{2}{a^2}r\delta\lambda\right)=r^2\delta\rho\,. \tag{5.19}$$

Now, from (5.16), (5.18), (5.19), and the equilibrium equations (4.6) and (4.7), one can obtain

$$\frac{1}{r^2ab}\frac{d}{dr}\left(\frac{r^2B}{i\omega ab}\right)=0\,, \tag{5.20}$$

and thus

$$B(r)=(i\omega ab/r^2)c\,, \tag{5.21}$$

where c is some constant. Since B as given by (5.21) diverges at $r=0$, then if T_{14} is to be finite at $r=0$, c must be zero. Therefore, the Einstein equations, conservation of particle number, and adiabatic motion *require* T_{14} to be of the special form given by (5.22):

$$T_{14}=(\rho+p)U_1U_4 \tag{5.22a}$$

$$=-i\omega a^2(\rho+p)\xi\,. \tag{5.22b}$$

Now consider (5.12e) and (5.22b). Using (5.2) and (5.14), (5.22b) requires that

$$R+\psi'\xi=0\,. \tag{5.23}$$

Equations (5.14) and (5.23) give the perturbed KG field in terms of ξ. Thus if (5.14) and (5.23) are consistent with the perturbed KG equations (4.10a) and (4.10b), then a certain class of the perturbations would be adiabatic. However, this is not the case. Although (5.14) is consistent with the perturbed KGE equations, (5.23) is an additional, *independent* equation which causes the solutions to be overdetermined so that only trivial solutions are allowed.[12] Thus the assumption that adiabatic perturbations exist leads to a contradiction, which then implies that all radial perturbations of the KG geon must be nonadiabatic.

C. Do Nonlocal Equations of State Induce Stability?

Earlier, in Sec. V A, we indicated that although the equations of state for the KG geon are nonlocal, the equilibrium states are nevertheless analogous to those which may be obtained from a certain local, γ-law equation of state. The question that we now want to answer is whether or not this latter system would be stable. In other words, if the KG geon *did* satisfy a γ-law equation of state, and adiabatic perturbations *were* allowed, would the KG geon be stable for radial perturbations? This question, in the Newtonian limit, was answered in the author's dissertation,[12] with the following conclusions.

When ρ and p are related by a local, γ-law equation of state, then, upon carrying out a stability analysis similar to that followed by Chandrasekhar,[17] one finds that the KG geon would be unstable in the Newtonian limit. However, when one considers the complete set of the perturbed KGE equations in the Newtonian limit, one can obtain a variational principle for ω^2, when the perturbations are spherically symmetric. And from this variational principle it was possible to show[12] that *all* eigenvalues of ω^2 *must* be positive for radial perturbations. Consequently, a nonlocal equation of state can cause a normally unstable system to become stable.

VI. SPHERICALLY SYMMETRIC PERTURBATIONS

We now discuss perhaps the most interesting aspect of this work, the case of radial perturbations and the possibility of gravitational collapse of the KG geon. As has been mentioned before, even in the general case, ω^2 must be real[12] for radial perturbations, and thus there

[16] Walter C. Hernandez, Jr., Ph.D. thesis, University of Maryland, 1966 (unpublished).

[17] S. Chandrasekhar, Phys. Rev. Letters **12**, 114 (1964).

TABLE II. The lowest eigenvalue of ω as obtained by numerical integration.

β	ω	$\omega^2/(1-E)^2$
0.3	0.100	0.677
0.4	0.115	0.578
0.5	0.119	0.452
0.6	0.117	0.350
0.8	0.106	0.224
1.0	0.095	0.169

are two general methods that may be used to determine whether or not the lowest eigenvalue of ω^2 is negative. If it is negative, then the KG geon would be unstable.

The first method would be to numerically integrate the perturbed KGE equations, and the second one would be to use a variational principle with trial functions. Both of these methods were used, but since the first method would give an unconditional answer, it was used first.

In the case of a perfect fluid, when the central pressure rises above the value corresponding to the first peak in the mass curve,[2] the first instability occurs. From Fig. 2, this point corresponds to $\beta \simeq 0.4$, and it was anticipated that for $\beta > 0.4$, the KG geon would be unstable. However, such is not the case and the KG geon is *stable* for radial perturbations.

The first results showed that, for β between 0.3 and 1.0, no eigenvalues existed for $\omega^2 < 0$ and that the lowest eigenvalue must be positive. Thus in this region the KG geon is stable, whereas all systems that obey a local equation of state are unstable.

Next, estimates were obtained for the lowest eigenvalue; however, some of these values are not too accurate. This is because the nature of the boundary conditions at $r=\infty$ is different for $\omega^2>0$ and for $\omega^2 \leq 0$. When $\omega^2<0$, or is very slightly positive, the boundary conditions can be set very accurately by means of an asymptotic series. But when ω^2 increases from zero and approaches $(1-E)^2$, then the asymptotic series rapidly loses accuracy. No estimates of the errors were obtained and results are summarized in Table II. As one can see from column 3, the values of ω should become more accurate as β increases. This is also indicated to be true by the results of the variational calculations, which we shall consider next.

When $\beta>1$, exact numerical integration of the equations becomes much more difficult, chiefly due to the rapidly increasing number of increments required. And, in order to obtain estimates in this region, a variational method was used, which we now describe and derive from the perturbed KGE equations.

The required equations are given by (4.10), and upon using (4.10d) to eliminate $\delta\lambda$, and using (4.10c) to eliminate $\delta\mu'$ from (4.10a), we have the following three equations remaining:

$$R''+R'\left(\frac{2}{r}+\nu'-\lambda'\right)+a^2R\left[\frac{E^2+\omega^2}{b^2}-1+r\psi'\psi\left(\frac{E^2}{b^2}-2\right)+\psi'^2(1-\tfrac{1}{2}r^2\psi^2)\right]+\frac{aE}{b}Ka^2\left[r\psi^2\left(\frac{E^2}{b^2}-1\right)+\psi'\psi(1-\tfrac{1}{2}r^2\psi^2)\right]-2a^2\frac{E^2}{b^2}\psi\delta\mu=0\,, \quad (6.1a)$$

$$K'+K\left(\frac{2}{r}+\frac{\psi'}{\psi}-\tfrac{1}{2}ra^2\frac{E^2}{b^2}\psi^2\right)-2\frac{aE}{b}R(1+\tfrac{1}{4}r\psi'\psi)+\frac{aE}{b}\psi\delta\mu=0\,, \quad (6.1b)$$

$$\frac{2}{r}\delta\mu'+a^2\frac{E^2}{b^2}\psi^2\delta\mu-R'\psi'-a^2R\left[\psi\left(\frac{E^2}{b^2}-1\right)(1+\tfrac{1}{2}r\psi'\psi)+\frac{\psi'}{r}\right]-\omega^2\frac{2aK}{rEb\psi}-\frac{aE}{b}a^2\psi K\left[\tfrac{1}{2}r\psi^2\left(\frac{E^2}{b^2}-1\right)+\frac{1}{r}\right]=0\,. \quad (6.1c)$$

Now take (6.1b) as *defining* $\delta\mu$. Then (6.1a) and (6.1c) reduce to the fourth-order system:

$$R''+R'(2/r+\nu'-\lambda')+R(\omega^2a^2/b^2-V_1)+2\frac{aE}{b}\left(K'+\frac{2}{r}K\right)+KV_2=0\,, \quad (6.2a)$$

$$K''+K'(2/r+\nu'-\lambda')+K(\omega^2a^2/b^2-V_3)-2(aE/b)R'+RV_2=0\,, \quad (6.2b)$$

where

$$V_1=a^2(3E^2/b^2+1)+2a^2r\psi'\psi-\psi'^2(1+r\nu'-r\lambda')\,, \quad (6.3a)$$

$$V_2=2(aE/b)[\psi'/\psi-\tfrac{1}{2}ra^2\psi^2+\tfrac{1}{2}\psi'\psi(1+r\nu'-r\lambda')]\,, \quad (6.3b)$$

$$V_3=2\left(\frac{\psi'}{\psi}+\frac{1}{r}\right)^2+\frac{2}{r}(\lambda'-\nu')+a^2\left(\frac{E^2}{b^2}-1\right)-a^2(E^2/b^2)\psi^2(1+r\nu'-r\lambda')\,. \quad (6.3c)$$

Equations (6.2) are derivable from the variational principle given by

$$\omega^2=\int_0^\infty r^2\frac{b}{a}dr\left[R'^2+K'^2+4\frac{aE}{b}R'K+R^2V_1-2RKV_2+K^2V_3\right]\bigg/\int_0^\infty r^2\frac{a}{b}dr\,(R^2+K^2)\,. \quad (6.4)$$

It is Eq. (6.4) that we shall use to obtain values for the lowest eigenvalues by means of trial functions. As to the choice of the trial functions, we take

$$I=r\psi, \tag{6.5a}$$

$$R=C_1\psi+C_2r\psi', \tag{6.5b}$$

where C_1 and C_2 are constants to be varied, and the justification of this choice is as follows. As shown in Sec. V, K is related to the Lagrangian displacement ξ by Eq. (5.14). Now since the simple trial function $\xi=r$ is sufficiently accurate to give negative eigenvalues for gravitating fluids, we shall use the same trial function, except that we shall drop the metric terms, and thus (5.14) becomes (6.5a).

The form for R in (6.5b) is obtained as follows. In the Newtonian limit, (6.1b) will approach

$$K'+K(2/r+\psi'/\psi)-2R=0, \tag{6.6}$$

and thus becomes an equation which can be used to define R. From (6.6) and (6.5a),

$$R=\tfrac{3}{2}\psi+r\psi'. \tag{6.7}$$

Since $\psi>0$ and $\psi'\leq 0$, (6.7) shows that R has a node and this is because, in the Newtonian limit, R must be orthogonal to ψ so that the first-order change in the mass energy will be zero. Now generalize (6.7) to get (6.5b), and let the two constants be varied to find a minimum in ω^2.

The results of these calculations are given in Table III, and, as one can see, ω is real for all the states. As mentioned before, the results in Table II are not too accurate except for possibly $\beta=0.8$ or 1.0, and the results of the variational calculations bear this out. If Table II were exact, then ω in column 2 of Table III would always be larger than the ω value in column 5, since variational calculations always give an upper bound. Since this is not so for the lower values of β, as expected, these values contain a respectable error. Nevertheless, the result that ω^2 must be positive for the states given in Table II is still true.

Table III. The lowest eigenvalue of ω as obtained by variational calculations.

β	ω	c_1	c_2	ω (from Table II)
0.01	0.0022	1.4884	0.9944	...
0.1	0.0229	1.3875	0.9456	...
0.2	0.0462	1.2826	0.8939	...
0.3	0.0675	1.1853	0.8454	0.1000
0.4	0.0855	1.0954	0.8000	0.1152
0.5	0.0996	1.0124	0.7577	0.1190
0.6	0.1094	0.9358	0.7182	0.1171
0.8	0.1166	0.8000	0.6465	0.1064
1.0	0.1115	0.6878	0.5846	0.0954
1.2	0.1123	0.6063	0.5368	...
1.5	0.1556	0.5839	0.5208	...
2.0	0.1601	0.7329	0.6100	...
3.0	0.1579	0.7068	0.5948	...
4.0	0.1575	0.7079	0.5955	...
6.0	0.1575	0.7018	0.5905	...

VII. CONCLUSION

To summarize the results of this paper: First we have shown that eigenstates of the Klein-Gordon Einstein equations exist, and that these eigenstates are well localized in that the Klein-Gordon field vanishes exponentially.

In Sec. V, the thermodynamical properties of Klein-Gordon geons were thoroughly investigated. The equilibrium states of these geons seem analogous to certain other gravitating systems, but since the equation of state is a differential equation (i.e., the Klein-Gordon equation) and is not an algebraic equation, then upon being perturbed the Klein-Gordon geon behaves significantly different from normal gravitating systems. In fact, adiabatic perturbations are not allowed and, at least for small radial perturbations, the geon cannot undergo spherically symmetric gravitational collapse. Even though all eigenstates with zero nodes are stable for radial perturbations, this does not imply that they are stable for nonspherically symmetric perturbations. Especially for the solutions with an energy excess, one would expect an instability would occur when $l=2$. Because then, the geon could radiate away its excess energy by means of gravitational radiation and "decay" into a state with the same N value, but with a lower energy (see Fig. 2). Consequently, it is to be expected that only the states below the first peak in the mass curve (at $\beta\simeq 0.4$) are stable for small perturbations, since for a fixed N they are the states with the lowest energy.

As mentioned in Sec. I, cylindrical systems are counterexamples to the conjecture that gravitational collapse is inevitable. However, in a sense they are not realistic counterexamples, since their stability may be related to their infinite length and infinite mass.

Consider a cylindrical gravitating system of *finite* length. One would normally expect this system to be unstable, since if the matter were rearranged into a spherically symmetric system, the potential energy would decrease and become more negative. At least this is so in the Newtonian limit. Thus (assuming that the kinetic energy will not significantly affect this argument), it should be possible to deform any cylindrical system, regardless of its length, provided that the length is finite, into a spherically symmetric system without having to do work. Now if the equation of state is local and if the system is initially sufficiently massive, then gravitational collapse can occur. Thus it appears highly likely that the stability properties of an infinite cylindrical system will be considerably different from those of a finite cylindrical system.

If this were the case, then if we would only consider *finite* systems that obey local equations of state, the conjecture of the inevitability of gravitational collapse would appear to be true.

The results of this paper indicate the conditions under which the conjecture is false. By a specific example, we have shown that the Klein-Gordon geon is resistant to gravitational collapse, and this resistance to collapse appears to be due to the nonlocal properties of the equation of state. Thus if the usual concept of an equation of state breaks down at the high densities required for collapse to occur, then the conjecture that gravitational collapse is inevitable may not be true.

Note added in proof. In a recent paper, Feinblum and McKinley[18] have also discussed the eigenstates of the Klein-Gordon geon, but their results differ from this author's results because of two requirements that we do not agree with. First, they required the Schwarzschild mass energy to be equal to the mass energy of the eigenvalue E. Although this may seem to be reasonable, we can give a counter example. Consider the Hartree-Fock equations for a two-electron atom. We identify the expectation value of the Hamiltonian as the total energy of the system. If we now use the Hartree-Fock equations to evaluate the energy, we find that the total energy is always less than the sum of the eigenvalues of the Hartree-Fock equations, because the eigenvalues contain the electron-electron interaction energy counted *twice*. For a two-electron atom with both electrons in the same S state, these equations are very analogous to the Newtonian limit of the Klein-Gordon, Einstein equations, which differ only in the sign of the self-interaction and in the point charge at the origin. Thus, in analogy, we would expect the total mass energy of the KG geon to be *more* than the mass energy of the eigenvalue, and the results of Sec. III confirm that this is always so, for the zero-node states.

But, the above statement turns out to be very much dependent on the normalization used, and Feinblum and McKinley[18] did use a normalization different from ours. Whereas we set the integral of the fourth component of the number density equal to unity, they set it equal to E, which is always less than unity (see Table I). And, the combination of these two conditions then allows eigenstates to exist. Using their normalization, Eqs. (3.18) and (3.19) would have a factor of E included on the right-hand side, which changed Eq. (3.21) to $m_g=(1-B)ME$. But, if $m_g=ME$, then $B=0$, and thus, their geons have a binding energy of exactly zero. Also, it is rather surprising that these conditions uniquely determine the value of the bare mass M. From Eq. (3.20), upon setting $m_g=EM$, we can solve for M^2 and obtain

$$M^2=\hbar cm/2GE.$$

From Table I, $B=0$ requires that $\beta\simeq 0.76$, and also that the values of m and E are 1.11 and 0.78, respectively. This gives that $M=1.84\times10^{-5}$ g, and thus, only for this value of M, will a solution, regular at the origin, exist. However, since they used a value of 1.28×10^{-12} g for the bare mass, we would suggest that the irregularity of their solution at the origin[18] is caused by using the wrong value of the bare mass.

Finally, since the submission of this paper, it has been pointed out to the author by R. Ruffini that equations similar to the KGE equations also occur when one quantizes the KGE system with the Hartree-Fock approximation.[19–21] In fact, for the two-particle problem, the equations for the ground state are identical to the KGE equations used here.[19]

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