

GEONS AND (4+1) GRAVITY

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Gravitational-electromagnetic entities “geons” are singularity-free solutions of the Einstein-Maxwell equations. These structures in cylindrical symmetry are considered here through the noncompactified Kaluza-Klein theory which describes geometrically the gravitation field and its sources.

Key words: geons, 5D gravity, Kaluza-Klein gravity.

1. INTRODUCTION

“Geons” is a term coined by Wheeler[1][2] to specify objects of sourceless electromagnetic fields contained by the curvature of space-time associated with their own energy density. Wheeler has argued that the solutions of the Einstein-Maxwell equations describing such structures are singularity free and this was confirmed by numerical investigations for the cases of spherical[1] and cylindrical [3] symmetry. He also argued that spherical geons must be unstable tending to transform into the more stable toroidal geons. For a wide mass range, geons are pure classical bodies in that they can be described solely by general relativity and electromagnetism.

Recently, Wesson and coworkers introduced a theory of (4+1) gravity that unifies the gravitational field and its sources[4][5]. It is a version of Kaluza-Klein gravity theory where the source of gravitation is generated geometrically by the existence and non compactification of the fifth dimension. In this theory, the 5D Einstein equations for apparent vacuum (lower case Latin letters run from 0 to 4 and Greek letters from 0 to 3)

$$\widehat{R}_{ab} = 0 \tag{1}$$

recover general relativity where the source of gravitation is induced geometrically if all the fifth-dimensional extra terms are equated to the four-dimensional stress tensor $T_{\alpha\beta}$.

The aim of this paper is to reconsider geons from the point of view of Wesson's (4+1) gravity. As pointed out by Melvin[6], Wheeler's geons have dynamic wave structures and only certain averages over these structures represent rigorous solutions of the time-independent Einstein-Maxwell equations; they are thus a sort of cloud of photons with no Poynting vector. In contrast to this, the numerical solution obtained by Ernst[3], which represents the gravitational field created by the electromagnetic field of a monochromatic standing wave confined to an infinitely long circular cylinder, has a Poynting vector along the axis of the cylinder and so the photon cloud image does not apply. The third case we consider is the exact Melvin solution[2]

$$ds^2 = (1 + kr^2)^2 (dt^2 - dr^2 - dz^2) - \frac{r^2}{(1 + kr^2)^2} d\phi^2, \tag{2}$$

representing a cylindrically symmetric magnetic flux held together by its own gravitational attraction (the constant k is related to the magnetic field strength). Here as for the first case there is no Poynting vector. Throughout this paper geometrized units $c = G = 1$ are used.

2. CYLINDRICAL SYMMETRIC SOLUTIONS

The cloud of photons matter is described by a perfect fluid energy-momentum tensor with a radiation-like equation of state. So we first consider solutions with radiation-like induced matter as this may be relevant to Wheeler's geons. In cylindrical symmetry the metric is

$$ds^2 = A^2(r) dt^2 - B^2(r) (dr^2 + dz^2) - C^2(r) d\phi^2 - D^2(r) d\psi^2; \tag{3}$$

A , B , C , and D depend only on r , as it can be shown that this leads only to radiation-like induced matter[7]. For such metrics which are independent of the fifth dimension the noncompactified Kaluza-Klein theory is indistinguishable from the usual theory. In fact only the scalar wave equation is contained in the field equations for this case [4]. Spherically symmetric solutions of this type are extended distributions of a cloud of photons with no event horizon. They are thus naked singularities[4]. This last feature prevents us from identifying them to Wheeler's spherical geons which are singularity free. In fact among all

the available solutions of Wesson's gravity theory, only the cosmological ones are singularity free[8]. The general solution of the field equations (1) for this metric are found to be

$$\begin{aligned} A(r) &= (ar - b)^\beta, \quad B(r) = (ar - b)^\nu, \\ C(r) &= e^\delta (ar - b)^{1-\alpha-\beta}, \quad D(r) = (ar - b)^\alpha, \end{aligned} \quad (4)$$

with the consistency relation

$$\nu = \beta^2 - \beta + \alpha\beta + \alpha^2 - \alpha. \quad (5)$$

Here a and b are positive constants; b is dimensionless and a has the dimension of an inverse of length if r and ψ are lengths.

The quantities α , β , and ν are dimensionless free parameters. The constant δ refers to the deficit angle and can be set to zero without loss of generality.

The induced stress four-tensor for the metric (3) is

$$\begin{aligned} 8\pi T_0^0 &= \frac{1}{B^2} \left(\frac{D''}{D} + \frac{D'C'}{DC} \right), \\ 8\pi T_1^1 &= \frac{1}{B^2} \left(\frac{D'B'}{DB} + \frac{A'D'}{AD} + \frac{D'C'}{DC} \right), \\ 8\pi T_2^2 &= \frac{1}{B^2} \left(\frac{D''}{D} + \frac{A'D'}{AD} \right), \\ 8\pi T_3^3 &= \frac{1}{B^2} \left(\frac{D''}{D} - \frac{D'B'}{DB} + \frac{A'D'}{AD} + \frac{D'C'}{DC} \right). \end{aligned} \quad (6)$$

With the density and pressure given by

$$\rho = T_0^0, \quad p = -(T_1^1 + T_2^2 + T_3^3)/3, \quad (7)$$

we obtain a radiation-like equation of state

$$\rho(r) = -\alpha\beta \frac{a^2}{8\pi} (ar - b)^{-2\nu-2}, \quad p = \rho/3. \quad (8)$$

The gravitational mass (Tolman-Whittaker mass) is defined by[9]

$$M_g(r) = \int (T_0^0 - T_1^1 - T_2^2 - T_3^3) \sqrt{-g} dV, \quad (9)$$

and thus we have a mass

$$M_g(r) = 2\beta a(ar - b)^{-\alpha} \quad (10)$$

per unit length in the direction of the axis of symmetry z . In order to satisfy the energy constraints, *i.e.*, positive density and positive gravitational mass we must have $\alpha < 0$ and $\beta > 0$.

The center of the structure is at $r = b/a$ for the surface area of cylindrical 2-shells varies as $(r - b/a)$. Here in contrast to the spherical solution[5] (the 5D soliton) the center may be at $r = 0$ (b can be set to zero). Because in this case b is not related to the mass or density of the object whereas for the spherical soliton such a constant is related to mass. We note also that solution (4) is singular at the center. Indeed the Kretschmann scalar in 5D geometry

$$\hat{I} = \hat{R}^{abcd}\hat{R}_{abcd} = \frac{f(\alpha, \beta)}{(ar - b)^{4+4\nu}} \quad (11)$$

is divergent at $r = b/a$. More precisely, this is not true for $\nu < -1$; but one can see that this is not permitted by the consistency relation (5). Indeed, for this case α and β are not real. The 4D Kretschmann scalar is qualitatively the same as (11) thus the four-dimensional matter is singular at $r = b/a$ and this also can be seen from expression (8) of the density which is singular for $\nu > -1$ at $r = b/a$. It is interesting to remark that the condition of a central object with density decreasing with distance (as the case of the geons) imply also $\nu > -1$. The solution (4) thus represents cylindrical cloud of photons with a naked linear singularity.

For $\alpha = 0$ and $b = 0$, the metric (4) reduces on the hypersurface $\psi = \text{const.}$ to the general relativity vacuum solution in cylindrical symmetry which represents a line mass[10];

$$ds^2 = (ar)^{2\beta} dt^2 - (ar)^{2\beta^2-2\beta} (dr^2 + dz^2) - (ar)^{-2\beta} r^2 d\phi^2. \quad (12)$$

The Levi-Cevita solution is obtained for $\beta = 2$.

Linear cylindrical geons are good models for the central region of toroidal geons[3]. So, by the coordinate change $z = \rho \sin \theta$, $r = R + \rho \cos \theta$, the solution (4) describes then a toroidal structure with the major radius R :

$$ds^2 = (aR - b + a\rho \cos \theta)^{2\beta} dt^2 - (aR - b + a\rho \cos \theta)^{2\nu} (d\rho^2 + \rho^2 d\theta^2) - (aR - b + a\rho \cos \theta)^{2(1-\alpha-\beta)} d\phi^2 - (aR - b + a\rho \cos \theta)^{2\alpha} d\psi^2. \quad (13)$$

It is interesting to note that the five-dimensional part of metric (4) for $\alpha < 0$, $\beta > 0$ (imposed by the energy constraints) asymptotically shrinks to zero.

To find other possible solutions we must generalize the metric (3). In the usual Kaluza-Klein theory this is done by considering a metric with off-diagonal components between 4D space-time and the extra coordinate: in this case one may obtain the Melvin geon. However in the noncompactified approach the Kaluza-Klein theory is 5D-covariant so we can use the coordinate degrees of freedom to set the off-diagonal components $g_{\alpha 4} = 0$ but with ψ -dependent metric coefficients[4][5]. All electromagnetic sources can thus be obtained. A simple solution with radiation-like matter for the ψ -dependent metric form is

$$ds^2 = A^2(r) G^2(\psi) dt^2 - A^{-2}(r) G^{-2}(\psi) dr^2 - E^2(\psi) r^2 d\phi^2 - E^2(\psi) dz^2 - D^2(r) F^2(\psi) d\psi^2, \quad (14)$$

where

$$\begin{aligned} A(r) &= \sqrt{a/r}, \quad D(r) = r/a, \quad F(\psi) = \text{arbitrary}, \\ G(\psi) &= 1/E(\psi), \\ E(\psi) &= \exp \left\{ \int \frac{F(\psi)}{2 \int F(\psi) d\psi - b} d\psi \right\}; \end{aligned} \quad (15)$$

a and b are constants and a is positive with the dimension of length. The density and pressure are

$$\rho(r) = \frac{1}{16\pi} \frac{a}{r^3} \exp \left\{ -2 \int \frac{F(\psi)}{2 \int F(\psi) d\psi - b} d\psi \right\}, \quad p = \rho/3, \quad (16)$$

and the gravitational mass is

$$M_g(r) = -\frac{a}{4r}. \quad (17)$$

In addition to being singular at $r = 0$, this last solution (14-15) is different from the first one by the divergence of its gravitational mass. In this latter case the strong energy constraint is satisfied (ρ positive) but the weak energy constraint is not (M_g negative). As the Melvin solution possesses a radiation-like equation of state it is instructive to compare it with this solution. We note first that Melvin solution satisfies another condition: The radial stress density (pressure) is equal and opposite in sign to the longitudinal stress density [6] or

$$T_1^1 = -T_2^2 = T_3^3 = -T_0^0. \quad (18)$$

One can readily verify that our solution (14-15) does not satisfy this condition. Therefore this solution cannot represent Wheeler's or Melvin's geons.

3. CONCLUSION

It remains then to seek solutions for the more general form of the cylindrical metric

$$ds^2 = A^2(r, \psi) dt^2 - B^2(r, \psi) (dr^2 + dz^2) - r^2 C^2(r, \psi) d\phi^2 + \varepsilon D^2(r, \psi) d\psi^2, \quad (19)$$

with $\varepsilon = \pm 1$ (spacelike or timelike fifth dimension). Work is in progress to solve the field equations numerically for concentrated objects and to compare the solution to Melvin's. The results will be reported in a forthcoming paper. The cylindrical geon of Ref[3] cannot be obtained with the metric (19) unless $D = D(t, r, z, \psi)$ for if this is so there may be a Poynting vector along the z -direction[5]. In the case one is interested to find only the Melvin geon, it is possible to use Weyl's metric (with two gravitational potentials in 4D) since the condition on the components of the stress tensor required by this form will be satisfied[2][6]. Whether one can obtain solutions which give back Melvin solution in 4D is an important point because one can set new experiments to test the predictions of 5D Wesson's gravity and general relativity coupled to electromagnetism. Indeed, Ivanov[10] had established the solution of the gravitational field of a long solenoid which reduces to Melvin solution in the central region of the solenoid. If such experiment can be built and generate a sufficiently high magnetic field strength ($\gtrsim 10^{10}$ G) one can test the solutions of both theories.

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