Nonexistence of marginally trapped surfaces and geons in $2+1$ gravity

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Abstract

We consider the existence of marginally outer trapped surfaces (MOTSs) in $2+1$ gravity and use these results to obtain nonexistence of geons in $2+1$ gravity. In particular, our results show that any $2+1$ initial data set, which obeys the dominant energy condition with cosmological constant $\Lambda \geq 0$ and which satisfies a mild asymptotic condition, must have trivial topology. Moreover, any data set obeying these conditions cannot contain a MOTS. The asymptotic condition involves a cutoff at a finite boundary at which a null mean convexity condition is assumed to hold; this null mean convexity condition is satisfied by all the standard asymptotic boundary conditions. The results presented here strengthen various aspects of previous related results in the literature. These results not only have implications for classical $2+1$ gravity but also apply to quantum $2+1$ gravity when formulated using Witten's solution space quantization.

1. INTRODUCTION

Solitons, an interesting feature of many nonlinear field theories, are stable solutions that exhibit the characteristics of particles, including properties such as mass, charge and spin. When present, they interact with other particles and fields in the nonlinear theory with important physical consequences. In gravity, the existence of such solutions, termed geons, was first proposed by Wheeler in both classical and quantum contexts [1]. In the original framework, geons are asymptotically flat solutions of Einstein-Maxwell theory. Initial investigations into their existence and properties were carried out in a series of papers by Wheeler and collaborators [2–6]. It was discovered that geons with trivial topology were classically unstable on short timescales. In contrast, topological geons do not disperse classically as their nontrivial spatial topology is preserved by evolution under the Einstein equations. Their nontrivial topology also can produce electric charge without the presence of charged matter sources; however, simple types of topological geons, for example those with the topology of a handle, also produce magnetic charge, in contradiction to observed properties of matter coupled to electromagnetism.

An explanation resolving this contradiction and other novel results led to renewed interest in topological geons as quantum particles in $3+1$-dimensional quantum gravity. Sorkin demonstrated that the nonorientable handle produced electric charges without also producing magnetic monopoles [7]. Additionally, an interesting formal argument in $3+1$-dimensional quantum gravity demonstrated that certain topological geons produce spin $1/2$ quantum states even though no fermionic matter sources are included [8–10]. A detailed analysis of the formal existence of spin $1/2$ states from quantum geons yielded interesting ties to the topology of 3-manifolds, as described in the series of papers [11–14].[1] Furthermore, physically reasonable initial data sets for the Einstein equations can be constructed on all smooth 3-manifolds [15]; consequently, classical topological geons exist in $3+1$-dimensional gravity. Thus, by

[1] These results also yielded counter-examples to some conjectures in 3-dimensional topology.

the correspondence principle, so should their quantum counterparts in a theory of $3 + 1$-dimensional quantum gravity.

Though intriguing, these formal arguments regarding the properties of topological geons cannot be more rigorously developed in a quantum context as no complete theory of $3 + 1$-dimensional quantum gravity is known. However, the potential for such studies exists in one lower dimension; as shown by Witten using a solution space quantization, $2+1$-dimensional quantum gravity is a well defined theory [16]. Though initial work concentrated on its formulation for spatially closed 2-manifolds [16–18], more recent investigations have been in the context of $2+1$-dimensional anti-de Sitter spacetimes [19] and related $2 + 1$-dimensional theories with asymptotic regions such as topologically massive gravity [20, 21] and chiral gravity [22]. Consequently, $2 + 1$-dimensional quantum gravity may provide a natural testbed for rigorously exploring the quantum properties of topological geons.

A natural first step toward the study of quantum geons in $2+1$-dimensional gravity is the identification of classical $2 + 1$-dimensional geons. This paper will rigorously address this issue; are there classical topological geons in $2 + 1$ gravity? This question was recently considered for asymptotically flat spacetimes obeying the dominant energy condition in [23]. They proved the nonexistence of asymptotically flat geons in $2+1$-dimensional vacuum spacetimes and under the more general assumption that spacetime is analytic.[2] It follows that there are no quantum geons in its corresponding solution space quantization. The proof of nonexistence of geons given in [23] is based on a spacetime approach that makes use of topological censorship techniques [24–26], combined with a refinement of the marginally trapped surface results in $2+1$ gravity considered in [27].

The aim of the present work is to strengthen various aspects of the nonexistence result obtained in [23], which, in the process, involves improvements of results of [27]. Here we take an initial data set approach, and hence our results are localized

[2] Analyticity is used to handle the case of equality in the dominant energy condition. As pointed out in [23], in $2 + 1$ dimensions, analyticity necessarily holds for vacuum spacetimes.

in time. Moreover, we are able to remove the analyticity assumption in [23]; smooth (or sufficiently differentiable, C^2, say) initial data sets suffice. Also, in [23] implicit assumptions were made about the existence of outermost marginally outer trapped surfaces (outermost MOTSs). Here we make careful use of recently established existence results for outermost MOTSs [28–31].

The main result of the paper (Theorem 4.1) is presented in Section 4. In it, we prove that bounded domains, satisfying a mild and physically natural boundary convexity condition, in 2+1-dimensional initial data sets obeying the dominant energy condition, with cosmological constant $\Lambda \geq 0$, are necessarily topological disks and do not contain MOTSs. (In this work, as will be seen, the cosmological constant is not considered as a source.) We give two different proofs of this. The first proof makes use of the theory of MOTSs, and the second proof makes use of Jang's equation with a Dirichlet boundary condition (as in [32]), combined with the Gauss-Bonnet theorem. The advantage of using the Dirichlet boundary condition is that no asymptotic fall-off conditions are needed; the boundary convexity condition mentioned above suffices.

In Section 2, we present some background material on MOTSs and obtain a strengthening of the results on trapped surfaces in $2 + 1$ gravity given in [27]. This allows the weakening of the regularity condition used in [23]; see especially, Theorem 2.3, which extends the main rigidity result obtained in [33]. Background material and relevant results on Jang's equation are presented in Section 3. We emphasize the connection between MOTSs and Jang's equation to obtain the so-called Schoen-Yau stability inequality, which plays a key role in the proof of Theorem 4.1 via Jang's equation.

While our results rule out the existence of MOTSs and nontrivial topology in $2 + 1$-dimensional asymptotically flat initial data sets obeying the dominant energy condition with $\Lambda \geq 0$, they do not do so for $\Lambda < 0$. Indeed, there are well-known examples of $2 + 1$-dimensional asymptotically AdS spacetimes which have MOTSs and nontrivial topology, such as the BTZ black holes and related spacetimes [34–37]. Hence the study of the quantum properties of $2 + 1$-geons in asymptotically AdS spacetimes remains an intriguing possibility.

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2. MARGINALLY TRAPPED SURFACES

Let Σ be a co-dimension two spacelike submanifold of a spacetime M. Under suitable orientation assumptions, there exists two families of future directed null geodesics issuing orthogonally from Σ. If one of the families has vanishing expansion along Σ then Σ is called a marginally outer trapped surface. The notion of a marginally outer trapped surface (MOTS) was introduced early on in the development of the theory of black holes [38]. Under suitable circumstances, the occurrence of a MOTS signals the presence of a black hole [38, 39]. For this and other reasons MOTSs have played a fundamental role in quasi-local descriptions of black holes; see e.g., [40]. MOTSs arose in a more purely mathematical context in the work of Schoen and Yau [41] concerning the existence of solutions to Jang's equation (see Section 3), in connection with their proof of positivity of mass.

In the following subsections we give precise definitions and present some results about MOTSs relevant to the present work.

2.1. MOTSs in initial data sets

In this paper we are primarily interested in initial data sets, and MOTSs therein.

Let (M^{n+1}, g) denote a spacetime, by which we mean a smooth (Hausdorff, paracompact) manifold M of dimension $n+1$, $n \geq 2$, equipped with a metric g of Lorentz signature $(- + \cdots +)$, such that, with respect to g, M is time oriented. An initial data set in (M^{n+1}, g) is a triple (V^n, h, K), where V is a spacelike hypersurface in M, and

h and K are the induced metric and second fundamental form, respectively, of V. To set sign conventions, for vectors $X, Y \in T_pV$, K is defined as, $K(X,Y) = \langle \nabla_X u, Y\rangle$, where ∇ is the Levi-Civita connection of M and u is the future directed timelike unit vector field to V. Note that a triple (V^n, h, K), where V is a smooth manifold, h is a Riemannian metric on V, and K is a covariant symmetric 2-tensor on V, is always the initial data set of some spacetime (e.g., let M' be a sufficiently small neighborhood of $\{0\} \times V$ in $\mathbb{R} \times V$, equipped with the metric, $g' = -dt^2 + h_t$, where $h_t = h + tK$). However, we will only be interested in *physically relevant* initial data sets, i.e., initial data sets associated with spacetimes that satisfy the Einstein equations (see Section 2.2).

Let (V^n, h, K) be an initial data set, and let Σ^{n-1} be a closed (compact without boundary) two-sided hypersurface in V^n. Then Σ admits a smooth unit normal field ν in V, unique up to sign. By convention, refer to such a choice as outward pointing. Then $l_+ = u + \nu$ (resp. $l_- = u - \nu$) is a future directed outward (resp., future directed inward) pointing null normal vector field along Σ, unique up to positive scaling.

The second fundamental form of Σ can be decomposed into two scalar valued *null second forms* χ_+ and χ_-, associated to l_+ and l_-, respectively. For each $p \in \Sigma$, $\chi_\pm$ is the bilinear form defined by,

$$\chi_\pm : T_p\Sigma \times T_p\Sigma \to \mathbb{R}, \qquad \chi_\pm(X,Y) = g(\nabla_X l_\pm, Y)\,. \tag{2.1}$$

The null expansion scalars $\theta_\pm$ of Σ are obtained by tracing $\chi_\pm$ with respect to the induced metric γ on Σ,

$$\theta_\pm = \mathrm{tr}_\gamma \chi_\pm = \gamma^{AB}\chi_{\pm AB} = \mathrm{div}\,_\Sigma l_\pm\,. \tag{2.2}$$

where γ is the induced metric on Σ. It is well known that the sign of $\theta_\pm$ is invariant under positive scaling of the null vector field $l_\pm$. Physically, θ_+ (resp., θ_-) measures the divergence of the outgoing (resp., ingoing) light rays emanating from Σ. In terms of the initial data (V^n, h, K),

$$\theta_\pm = \mathrm{tr}_\gamma K \pm H\,, \tag{2.3}$$

where H is the mean curvature of Σ within V (given by the divergence of ν along Σ).

We say that Σ is an *outer trapped surface* (resp., *weakly outer trapped surface*) if $\theta_+ < 0$ (resp., $\theta_+ \leq 0$). If θ_+ vanishes, we say that Σ is a *marginally outer trapped surface*, or MOTS for short. Geometrically, MOTSs may be viewed as spacetime analogues of minimal surfaces in Riemannian manifolds. In fact, in the time-symmetric case ($K = 0$), a MOTS Σ is just a minimal surface in V. In recent years MOTSs have been shown to share a number of properties in common with minimal surfaces. In particular MOTSs admit a notion of stability analogous to that of minimal surfaces [42, 43]. Here, stability is associated with variations of the null expansion under deformations of a MOTS.

2.2. Variation of the null expansion

Let (V^n, h, K) be an initial data set in a spacetime (M^{n+1}, g) that obeys the Einstein equation with cosmological term,

$$\mathrm{Ric} - \frac{1}{2}Rg + \Lambda g = \mathcal{T}\,, \tag{2.4}$$

where $\mathcal{T}$ is the energy-momentum tensor. The Gauss-Codazzi equations imply the Einstein constraint equations,

$$\frac{1}{2}\left(S_V + (\mathrm{tr}\,K)^2 - |K|^2\right) = \rho + \Lambda \tag{2.5}$$

$$\mathrm{div}K - d(\mathrm{tr}\,K) = J\,, \tag{2.6}$$

where $\rho = \mathcal{T}(u,u)$, $J = \mathcal{T}(u,\cdot)$, and S_V is the scalar curvature of V. For a given choice of Λ, ρ and J are completely determined by the initial data.

The energy-momentum tensor $\mathcal{T}$ is said to obey the dominant energy condition (DEC) provided, $\mathcal{T}(X,Y) = T_{ij}X^iY^j \geq 0$ for all future directed causal vectors X and Y. One verifies that $\mathcal{T}$ obeys the DEC if and only for all initial data sets (V^n, h, K) in (M^{n+1}, g), $\rho \geq |J|$.

We now want to consider variations in the null expansion due to deformations of a MOTS. Hawking [38, 45] introduced such variational techniques to obtain results about the topology of black holes in 3+1 dimensions. These results were more recently generalized to higher dimensions [33, 44]. Ida [27] adapted Hawking's argument to $2+1$ dimensions to obtain restrictions on the existence of certain types of MOTSs.

Let (V^n, h, K), $n \geq 2$, be an initial data set in a spacetime obeying the Einstein equations. Let Σ be a connected MOTS in V with outward unit normal ν. We consider variations $t \to \Sigma_t$ of $\Sigma = \Sigma_0$, $-\epsilon < t < \epsilon$, with variation vector field $\mathcal{V} = \frac{\partial}{\partial t}\big|_{t=0} = \phi\nu$, $\phi \in C^\infty(\Sigma)$. Let $\theta(t)$ denote the null expansion of Σ_t with respect to $l_t = u + \nu_t$, where u is the future directed timelike unit normal to V and ν_t is the outer unit normal to Σ_t in V. A computation shows, [43]

$$\left.\frac{\partial \theta}{\partial t}\right|_{t=0} = L(\phi) \tag{2.7}$$

where $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ is the operator,

$$L(\phi) = -\triangle\phi + \langle X, \nabla\phi\rangle + \left(\frac{1}{2}S - P + \operatorname{div} X - |X|^2\right)\phi\,, \tag{2.8}$$

where,

$$S = \begin{cases} 0\,, & \text{if } n = 2 \\ \text{the scalar curvature of } \Sigma\,, & \text{if } n \geq 3\,, \end{cases} \tag{2.9}$$

$$P = \rho + J(\nu) + \Lambda + \frac{1}{2}|\chi|^2 \tag{2.10}$$

($\chi =$ the outward null second fundamental form of Σ), and where X is the vector field on Σ metrically dual to the one-form, $K(\nu, \cdot)$, and $\langle \cdot, \cdot \rangle = \gamma$ is the induced metric on Σ.

In the time-symmetric case ($K = 0$), θ becomes the mean curvature H, the vector field X vanishes and L reduces to the classical stability operator of minimal surface theory. In analogy with the minimal surface case, we refer to L in (2.8) as the stability operator associated with variations in the null expansion θ. Although in general L is not self-adjoint, its principal eigenvalue (eigenvalue with smallest real part) $\lambda_1(L)$ is real. Moreover there exists an associated eigenfunction $\phi \in C^\infty(\Sigma)$ which is strictly positive.

As an application of the variational formula (2.7-2.8), we consider the following result, which summarizes several results in the literature [27, 44–46].

Theorem 2.1. *Let (V^n, h, K), $n \geq 2$, be an initial data set in a spacetime satisfying the Einstein equations, with $\Lambda \geq 0$. Let Σ be a MOTS in V such that either (1) $\Lambda = 0$, and $\rho > |J|$ along Σ, or (2) $\Lambda > 0$, and $\rho \geq |J|$ along Σ. Suppose, further, that one of the following conditions holds.*

(i) $n = 2$.

(ii) $n \geq 3$ *and* $\int_\Sigma S d\mu \leq 0$.

(iii) $n \geq 3$ *and* Σ *is not of positive Yamabe type, i.e.,* Σ *does not admit a metric of positive scalar curvature.*

Then Σ can be deformed outward to a strictly outer trapped surface.

Ida's [27] main observation is the case $n = 2$. Note that in this case Σ is one-dimensional and hence is topologically a circle (or disjoint union of circles).

Proof. We present here a fairly uniform proof of Theorem 2.1, which is relevant to the proof of Theorem 2.3 below.

By applying the argument below to each component of Σ, we may assume without loss of generality that Σ is connected. Note that, by the energy conditions, the scalar quantity P in (2.8) is strictly positive.

Consider the "symmetrized" operator $L_0 : C^\infty(\Sigma) \to C^\infty(\Sigma)$,

$$L_0(\phi) = -\triangle\phi + (\tfrac{1}{2}S - P)\phi\,, \tag{2.11}$$

obtained from (2.8) by formally setting $X = 0$. The main argument in [44] establishes the following (see also [43], [33]).

Proposition 2.2. $\lambda_1(L) \leq \lambda_1(L_0)$.

For self-adjoint operators of the form (2.11), the Rayleigh formula [47] and an integration by parts, gives the following standard characterization of the principle eigenvalue,

$$\lambda_1(L_0) = \inf_{\phi \not\equiv 0} \frac{\int_\Sigma |\nabla \phi|^2 + (\frac{1}{2}S - P)\phi^2 \, d\mu}{\int_\Sigma \phi^2 \, d\mu} \,. \tag{2.12}$$

In the cases (i) and (ii), we have $\int_\Sigma S d\mu \le 0$. Hence, by setting $\phi = 1$ in the expression on the right hand side of (2.12), and using the fact that $P > 0$, we see that $\lambda_1(L_0) < 0$. Thus, by Proposition 2.2, $\lambda_1(L) < 0$.

Now let ϕ be an eigenfunction associated to $\lambda_1(L)$, $L(\phi) = \lambda_1(L)\phi$; ϕ can be chosen to be strictly positive. Using this ϕ to define our variation $t \to \Sigma_t$, we have from (2.7),

$$\left.\frac{\partial \theta}{\partial t}\right|_{t=0} = \lambda_1(L)\phi < 0 \,. \tag{2.13}$$

Together with the fact that $\theta = 0$ on Σ, this implies that for $t > 0$ sufficiently small, Σ_t is outer trapped, as desired.

Now consider case (iii). First suppose $n = 3$. Then Σ is 2-dimensional, and by the Gauss-Bonnet theorem, the assumption that Σ does not carry a metric of positive curvature implies $\int_\Sigma S d\mu \le 0$. Thus, the argument is the same as in cases (i) and (ii). For $n \ge 4$, consider the conformal Laplacian, $L_{cf} : C^\infty(\Sigma^{n-1}) \to C^\infty(\Sigma^{n-1})$,

$$L_{cf}(\phi) = -4\frac{n-2}{n-3} \triangle \phi + S\phi \,. \tag{2.14}$$

If Σ does not carry a metric of positive scalar curvature then we must have, $\lambda_1(L_{cf}) \le 0$. The Rayleigh formula applied to L_{cf} gives,

$$\lambda_1(L_{cf}) = \inf_{\phi \not\equiv 0} \frac{\int_\Sigma \frac{4(n-2)}{n-3} |\nabla \phi|^2 + S\phi^2 \, d\mu}{\int_\Sigma \phi^2 \, d\mu} \,. \tag{2.15}$$

Comparing (2.12) and (2.15), and using the positivity of P, one easily obtains, $\lambda_1(L_0) < \frac{1}{2}\lambda_1(L_{cf})$. Hence, $\lambda_1(L_0) < 0$, and so by Proposition 2.2, we again arrive at, $\lambda_1(L) < 0$. We may then proceed as before. $\square$

One can see, by a simple modification of the proof, that in the case $\Lambda = 0$, it is sufficient to require $\rho \ge |J|$ on Σ, with strict inequality somewhere.

With somewhat more effort, one can obtain the following refinement of Theorem 2.1 which does not require any strictness in the energy conditions.

Theorem 2.3. *Let (V^n, h, K), $n \geq 2$, be an initial data set in a spacetime satisfying the Einstein equations (2.4) with $\Lambda \geq 0$, such that $\mathcal{T}$ satisfies the DEC. Suppose Σ is a MOTS in V such that in some neighborhood $U \subset V$ of Σ there are no (strictly) outer trapped surfaces outside of, and homologous, to Σ. Suppose, further, that one of the following conditions holds.*

(i) $n = 2$.

(ii) $n = 3$ *and* $\int_\Sigma S d\mu \leq 0$.

(iii) $n \geq 3$ *and Σ is not of positive Yamabe type, i.e., Σ does not admit a metric of positive scalar curvature.*

Then there exists an outer half-neighborhood U^+ of Σ foliated by MOTSs, i.e., $U^+ \approx [0, \epsilon) \times \Sigma$, such that each slice $\Sigma_t = \{t\} \times \Sigma$, $t \in [0, \epsilon)$ is a MOTS.

Remarks on the proof. Case (iii) of Theorem 2.3 is proved in [33]. The proof in this case consists of two steps. In the first step, one obtains an outer foliation $t \to \Sigma_t$, $0 \leq t \leq \epsilon$, of surfaces Σ_t of *constant* outer null expansion, $\theta(t) = c_t$. The second step involves showing that the constants $c_t = 0$. This latter step requires a reduction to the case that V has nonpositive mean curvature, $\tau \leq 0$ near Σ. For this it is necessary to know that the DEC holds in a spacetime neighborhood of Σ. The proof makes use of the formula for the t-derivative, $\frac{\partial\theta}{\partial t}$, not just at $t = 0$ where $\theta = 0$, but all along the foliation $t \to \Sigma_t$, where, a priori, $\theta(t)$ need not be zero. Thus, additional terms appear in the expression for $\frac{\partial\theta}{\partial t}$ beyond those in (2.7), including a term involving the mean curvature of V, which need to be accounted for. The proof of case (iii) given in [33] can be easily modified to give a proof of Theorem 2.3 in the cases (i) and (ii), by using arguments like those used in the proof of the cases (i) and (ii) in Theorem 2.1 above. For Theorem 2.3, it is necessary to restrict the dimension in case (ii) to $n = 3$ in order to control, via Gauss-Bonnet, the total scalar curvature of each Σ_t. $\square$

Let us say that a MOTS Σ in an initial data set is *locally outermost* if, with respect to some neighborhood of $U \subset V$ of Σ, there are no weakly outer trapped surfaces outside of, and homologous, to Σ in U. Theorem 2.3(i) shows that for initial data sets in $2+1$-dimensional spacetimes satisfying the Einstein equations (2.4) with $\Lambda \geq 0$, such that $\mathcal{T}$ satisfies the DEC, there can be no locally outermost MOTSs. This strengthens Ida's results and those of [23] by removing any strictness in the energy inequalities, restriction to the vacuum case or assumption of analyticity. Theorem 2.3(i) rules out, in particular, the existence of $2+1$-dimensional stationary black hole spacetimes obeying the stated energy conditions.

In Section 4, we obtain a more comprehensive result which rules out MOTSs altogether, locally outermost or otherwise.

2.3. Existence of MOTSs

Substantial progress has been made in recent years concerning the existence of MOTSs. Following an approach of Schoen [48], Andersson and Metzger [28] established in low dimensions the existence of MOTSs under natural barrier conditions. Combining this existence result with the compactness result established in [28], they were able to establish the existence of outermost MOTSs in 3-dimensional initial data sets [29]. Such an outermost MOTS is realized as the boundary of the so-called *trapped region*, suitably defined. Using what he refers to as the C-almost minimizing property of $MOTSs$, Eichmair [30, 31, 49] is able to extend the results of Andersson and Metzger to dimensions n, $2 \leq n \leq 7$; see especially [31, Theorem 5.1], which, except for the last sentence, holds for apparent horizons as well as generalized apparent horizons, and holds in dimension two, as well.

These results may be formulated as follows.

Theorem 2.4. *Let (V^n, h, K) be an initial data set, $2 \leq n \leq 7$, and let W^n be a connected compact n-manifold-with-boundary in V^n. Suppose that the boundary ∂W can be expressed as a disjoint union, $\partial W = \Sigma_{inn} \cup \Sigma_{out}$, such that $\theta^+ < 0$ along Σ_{inn} with respect to the null normal whose projection points into W, and $\theta^+ > 0$ along Σ_{out} with respect to the null normal whose projection points out of W. Then there exists* *a smooth compact outermost MOTS Σ in the interior of W homologous to Σ_{out}.*

Some remarks are in order.

1. If, as the notation suggests, we think of Σ_{inn} as an *inner* boundary and Σ_{out} as an *outer* boundary, then we are assuming that Σ_{inn} is outer trapped and Σ_{out} is outer untrapped.

2. By Σ being *homologous* to Σ_{out}, we mean explicitly that there exists an open set $U \subset W$ such that $\partial U = \Sigma \cup \Sigma_{out}$. Then θ_+ is defined with respect to the null normal whose projection points into U.

3. By Σ being *outermost* we mean that if Σ' is a weakly outer trapped ($\theta^+ \leq 0$) surface in $\overline{U}$ homologous to Σ_{out} then $\Sigma' = \Sigma$. In other words, Σ must enclose all weakly outer trapped surfaces in W homologous to Σ_{out}.

4. It is important to note for applications that Σ_{inn} and Σ_{out} need not be connected. Also the MOTS Σ will not in general be connected (even if Σ_{inn} and Σ_{out} are).

5. Finally, Andersson and Metzger [29] have shown, by a technique of modifying the initial data near the inner boundary to get a strict barrier, that it is sufficient in Theorem 2.4 to require that Σ_{inn} be only weakly outer trapped, $\theta_+ \leq 0$. Then the outermost MOTS Σ may have components that agree with components of Σ_{inn}.

Note the tension between Theorem 2.3(i) and Theorem 2.4, the former implying that there are no locally outermost MOTSs under appropriate energy conditions, and the latter providing conditions for the existence of outermost MOTSs. We will exploit this tension to give a proof of the main result in Section 4.

The proof of the basic existence result for MOTSs alluded to at the beginning of this subsection is based on Jang's equation [50], which we discuss in the next section. (Variational methods used to establish the existence of minimal surfaces in Riemannian manifolds under sutaible barrier conditions are not in general available to establish the existence of MOTSs.) Schoen and Yau [41] established existence and regularity for Jang's equation with respect to asymptotically flat initial sets, as part of their approach to proving the positive mass theorem for general, nonmaximal, initial data sets. In the process they discovered an obstruction to global existence: Solutions to Jang's equation tend to blow-up in the presence of MOTSs in the initial data (V^n, h, K). Turning the situation around, this behavior has been exploited in [29, 31] to establish the existence of MOTSs by *inducing* blow-up of Jang's.

3. JANG'S EQUATION AND THE SCHOEN-YAU STABILITY INEQUALITY

Let (V^n, h, K) be an initial data set. Then Jang's equation is the equation,

$$\gamma^{ij}\left(\frac{D_i D_j f}{\sqrt{1+|Df|^2}} - K_{ij}\right) = 0\,, \tag{3.1}$$

where f is a function on V, D is the Levi-Civita connection of h, and $\gamma^{ij} = g^{ij} - \frac{f^i f^j}{1+|Df|^2}$. Introducing the Riemannian product manifold, $\bar{V} = V \times \mathbb{R}$, $\bar{h} = h + dz^2$, we notice that the γ^{ij}'s are the contravariant components of the induced metric γ_f on $\Sigma_f = \text{graph}\, f$ in $\bar{V}$, and, moreover, that,

$$H(f) := -\frac{\gamma^{ij} D_i D_j f}{\sqrt{1+|Df|^2}}$$

is the mean curvature of Σ_f, computed with respect to the *upward pointing*[3] unit normal ν. Thus, Jang's equation becomes,

$$H(f) + \operatorname{tr}_{\gamma_f} \bar{K} = 0\,, \tag{3.2}$$

where $\bar{K}$ is the pullback, via projection along the z-factor, of K to $\bar{V}$. Comparing with Equation (2.3), we see that, geometrically, Jang's equation is the requirement

[3] We note that in [41] the mean curvature of Σ_f is considered with respect to the downward pointing normal. Our choice results in some minor sign differences.

that the graph Σ_f has vanishing null expansion, $\theta_+ = 0$, i.e., is a MOTS, in the initial data set $(\bar{V}^n, \bar{h}, \bar{K})$.

Given a solution f to Jang's equation, we can use Equations (2.7-2.8) to obtain a formula for the scalar curvature S_f of Σ_f. Consider the variation $t \to \Sigma(t)$ of Σ_f obtained by shifting Σ_f up and down the z-axis, i.e., $\Sigma(t) =$ the graph of $f + t$. This may be viewed as a normal variation, with variation vector field,

$$\mathcal{V} = \phi\nu\,, \quad \phi = \bar{h}(\nu, \partial_z) \tag{3.3}$$

where ν is the upward pointing unit normal along Σ_f.

Let $\theta(t)$ denote the null expansion of $\Sigma(t)$. Because Jang's equation is translation invariant, in the sense that if f is a solution then $f + t$ is also a solution, we have that $\theta(t) = 0$ for all t. Hence, $\frac{\partial\theta}{\partial t}\big|_{t=0} = 0$, and Equations (2.7-2.8) give along Σ_f,

$$-\triangle\phi + \langle\bar{X}, \nabla\phi\rangle + \left(\frac{1}{2}S_f - P + \operatorname{div}\bar{X} - |\bar{X}|^2\right)\phi = 0\,, \tag{3.4}$$

where $\bar{X}$ is the vector field on Σ_f metrically dual to the one-form, $\bar{K}(\nu, \cdot)$, and

$$P = \bar{\rho} + \bar{J}(\nu) + \Lambda + \frac{1}{2}|\chi|^2\,, \tag{3.5}$$

where $\bar{\rho}$ and $\bar{J}$ are the pullback of ρ and J, respectively, via projection along the z-factor.

By setting $\phi = e^u$ in (3.4) and completing the square, we obtain,

$$\frac{1}{2}S_f + \operatorname{div}(\bar{X} - \nabla u) - |\bar{X} - \nabla u|^2 = \bar{\rho} + \bar{J}(\nu) + \Lambda + \frac{1}{2}|\chi|^2 \geq 0\,, \tag{3.6}$$

where the inequality holds provided $\Lambda \geq 0$ and the DEC, $\rho \geq |J|$, holds with respect to the original initial data set (V^n, h, K). This inequality is equivalent to the "on shell" *Schoen-Yau stability inequality*[4] obtained in [41]; cf., (2.29) on p. 240. Hence, assuming $\Lambda \geq 0$ and the DEC holds, we arrive at,

$$S_f \geq -2\operatorname{div}(\bar{X} - \nabla u)\,, \tag{3.7}$$

[4] Here *stability* relates to the fact that with respect to the variation being considered, the null expansion is nondecreasing in the "outward" direction.

where $u = \ln \bar{h}(\nu, \partial_z)$ and $\bar{X}$ is the vector field on Σ_f metrically dual to the one-form $\bar{K}(\nu, \cdot)$.

In [41] Schoen and Yau studied extensively the existence and regularity of solutions f to Jang's equations over complete asymptotically flat 3-dimensional initial data sets (V^3, h, K), with suitable decay on the asymptotically Euclidean ends. In [32], Yau described the modifications necessary to obtain solutions to Jang's equation, with Dirichlet boundary data $f = 0$, on compact manifolds W, with *null convex* boundaries ∂W (as defined in the next section). These results establish the existence of regular solutions to Jang's equation in regions $W' \subset W$ external to a finite number of MOTSs homologous to ∂W, at which the solution asymptotes in a C^2 fashion to the vertical cylinders over these MOTSs; cf., [41, Proposition 4 and Corollary 2] for precise statements.

Since many of the key estimates in [41] hold in dimensions $n \leq 5$, we will assume in the next section that these results remain valid in dimension $n = 2$ without proof. Moreover, we are using Dirichlet boundary data which means the asymptotic fall off conditions used in [41] are not needed, which simplifies the estimates.

4. MAIN RESULT

Let W^n, $n \geq 2$, be a connected compact manifold-with-boundary in an initial data set (V^n, h, K). We say that the boundary ∂W^n is *null mean convex* provided it has positive outward null expansion, $\theta^+ > 0$, and negative inward null expansion, $\theta^- < 0$. Note that round spheres in Euclidean slices of Minkowski space, and, more generally, large "radial" spheres in asymptotically flat initial data sets are null mean convex.

The aim of this section is to prove the following result about *2-dimensional* initial data sets.

Theorem 4.1. *Let* (V^2, h, K)*, be a* 2*-dimensional initial data set in a spacetime satisfying the Einstein equations* (2.4) *with* $\Lambda \geq 0$*, such that* $\mathcal{T}$ *satisfies the DEC. If* W^2 *is a connected compact* 2*-manifold with null mean convex boundary* ∂W^2 *in* V^2*,* *then W^2 is diffeomorphic to a disk, and there are no MOTSs in W^2.*

We actually present two proofs of this result, the first makes use of Theorem 2.3(i), and the second makes use of Jang's equation.

First proof. We first show that W^2 is diffeomorphic to a disk. This is a consequence of the following claim.

Claim. Let (V^2, h, K), be a 2-dimensional initial data set in a spacetime satisfying the Einstein equations (2.4) with $\Lambda \geq 0$, such that $\mathcal{T}$ satisfies the DEC. If W^2 is a connected compact 2-manifold with null mean convex boundary ∂W^2 in V^2, then ∂W^2 is connected.

Proof of the claim. Suppose ∂W^2 is not connected. Then designate one component as Σ_{inn} and the union of others as Σ_{out}. Note that, by the null mean convexity of ∂W^2, Σ_{inn} and Σ_{out} obey the null expansion conditions of Theorem 2.4. Thus there exists an outermost MOTS Σ, as defined in Section 2.2, homologous to Σ_{out}. But, by applying Theorem 2.3 to any component of Σ, we see immediately that there exists a MOTS Σ' homologous to Σ_{out} which is not enclosed by Σ. Thus, ∂W must be connected.

To continue with the proof that W^2 is diffeomorphic to a disk, we know from the claim that ∂W is connected, and hence is a circle. Suppose that W is not orientable. Then we can pass to the orientable 2-sheeted covering $(\tilde{W}, \pi)$ of W [51]. Since ∂W is orientable, $\partial \tilde{W} = \pi^{-1}(\partial W)$ will consist of two components. On the other hand, the conditions of Theorem 2.4 lift to $\tilde{W}$; in particular $\partial \tilde{W}$ will be null mean convex. Hence, by the claim, $\partial \tilde{W}$ must be connected, which a contradiction.

Thus, W^2 is orientable, with a single circular boundary. Then, by the classification of surfaces, W^2 is diffeomorphic to a disk with $g \geq 0$ handles. We want to show that $g = 0$. Suppose $g \geq 1$, i.e., suppose there is at least one handle. Then there is an imbedded circle Σ in the interior of W that does not separate W. In particular there are loops in W with nonzero intersection number with respect to Σ. Let G be the subgroup of the fundamental group $\pi_1(W)$ of W consisting of elements of $\pi_1(W)$

having *even* intersection number with respect to Σ. Let $(\tilde{W}, \pi)$ be the covering of W associated with the subgroup G [51]. Since G has index two in $\pi_1(W)$, $(\tilde{W}, \pi)$ is a two-sheeted covering of W.[5] Then, as ∂W has zero intersection number with respect to Σ, $\partial\tilde{W} = \pi^{-1}(\partial W)$ consists of two components, which again leads to a violation of the the claim. Thus, $g = 0$ and W^2 is diffeomorphic to a disk.

To complete the proof, we need to show that there are no MOTSs in W. Suppose, to the contrary, that Σ is a MOTS in W. By focusing on a single component, we may assume Σ is connected. Since W is a disk, Σ must be homologous to the boundary. By considering, if necessary, the time-dual of our initial data set, we may assume that Σ satisfies $\theta_+ = 0$ with respect to the normal pointing outwards towards the boundary. Let $W' \subset W$ be the region bounded by Σ and ∂W. We can apply Theorem 2.4 to W' to conclude that there is an outermost MOTS in W'. But this again is contradicted by Theorem 2.3. Thus there can be no MOTSs in W. $\square$

Second proof. As noted above, this proof makes use of Jang's equation. As before, we first show that W^2 is diffeomorphic to a disk. By passing to a double cover if necessary we may assume without loss of generality that W is orientable.

For simplicity, let's assume first that there exists a globally regular solution $f : W \to \mathbb{R}$ to Jang's equation, with $f = 0$ on ∂W. As in Section 3, we consider $\Sigma_f =$ graph f in the metric γ_f induced from the product metric $\langle \cdot, \cdot \rangle = h + dz^2$. We introduce an orthonormal frame e_1, e_2, e_3 along Σ_f near $\partial\Sigma_f = \partial W$. Take $e_3 = \nu$, and let e_1 and e_2 be tangent to Σ_f, such that e_1 is tangent to $\partial\Sigma_f$ and e_2 is normal to $\partial\Sigma_f$ and outward pointing.

By the Gauss-Bonnet formula applied to (Σ_f, γ_f), we have,

$$\begin{aligned} \iint_{\Sigma_f} \mathcal{K} dA + \int_{\partial\Sigma_f} \kappa ds &= 2\pi\chi(\Sigma_f) = 2\pi\chi(W) \\ &= 2\pi\left(2 - 2g - k\right). \end{aligned} \tag{4.1}$$

[5] $\tilde{W}$ has a simple description in terms of cut-and-paste operations: By making a cut along Σ, we obtain a compact 2-manifold W' whose boundary consists of ∂W and two copies of Σ. By suitably gluing together two copies of W' along each of their two copies of Σ, one obtains $\tilde{W}$.

where g is the number of handles and k is the number of boundary components of W.

To show that $g = 0$ and $k = 1$, and hence that W is a disk, it is sufficient to show that the left hand side of (4.1) is strictly positive. From (3.7), the Gaussian curvature $\mathcal{K}$ satisfies, $\mathcal{K} \geq -\mathrm{div}(\bar{X} - \nabla u)$, where $u = \ln\langle e_3, \partial_z\rangle$ and $\bar{X}$ is the vector field on Σ_f metrically dual to the one-form $\bar{K}(\nu, \cdot)$. The geodesic curvature κ is given by $\kappa = -\langle \nabla_{e_1} e_1, e_2\rangle = \bar{H}_{\partial W}$, the mean curvature of ∂W in (Σ_f, γ_f). Then, applying the divergence theorem,

$$\begin{aligned}\iint_{\Sigma_f} \mathcal{K} dA + \int_{\partial \Sigma_f} \kappa ds &\geq \int_{\partial W} \bar{H}_{\partial W} - \langle \bar{X}, e_2\rangle + \langle \nabla u, e_2\rangle \, ds \\ &= \int_{\partial W} \bar{H}_{\partial W} - \bar{K}(e_3, e_2) + e_2(u) \, ds \,. \end{aligned} \tag{4.2}$$

By analyzing each term in the integrand in a manner similar to what is done in [32, p. 9f], we show that the integrand is strictly positive.

Let w be the unit normal field to ∂W tangent to V. Then note, since ∂_z is parallel,

$$\bar{H}_{\partial W} = -\langle \nabla_{e_1} e_1, e_2\rangle = -\langle e_2, w\rangle \langle \nabla_{e_1} e_1, w\rangle = \langle e_2, w\rangle H_{\partial W} \,, \tag{4.3}$$

where $H_{\partial W}$ is the mean curvature of ∂W in (V, h). Also, since $\bar{K}(\partial_z, \cdot) = 0$,

$$\bar{K}(e_3, e_2) = \langle e_3, w\rangle \bar{K}(w, e_2) = \frac{\langle e_3, w\rangle}{\langle e_2, w\rangle} \bar{K}(e_2, e_2) \,. \tag{4.4}$$

For the term $e_2(u)$, we have,

$$\begin{aligned} e_2(u) &= \frac{1}{\langle e_3, \partial_z\rangle} \, e_2\langle e_3, \partial_z\rangle = \frac{1}{\langle e_3, \partial_z\rangle} \langle \nabla_{e_2} e_3, \partial_z\rangle \\ &= \frac{\langle e_2, \partial_z\rangle}{\langle e_3, \partial_z\rangle} \langle \nabla_{e_2} e_3, e_2\rangle = -\frac{\langle e_2, \partial_z\rangle}{\langle e_3, \partial_z\rangle} \langle \nabla_{e_2} e_2, e_3\rangle \\ &= \frac{\langle e_3, w\rangle}{\langle e_2, w\rangle} \langle \nabla_{e_2} e_2, e_3\rangle \,. \end{aligned} \tag{4.5}$$

Since,

$$\begin{aligned} -\mathrm{tr}\, \bar{K} = H(f) &= -\langle \nabla_{e_1} e_1, e_3\rangle - \langle \nabla_{e_2} e_2, e_3\rangle = \\ &= \langle e_3, w\rangle H_{\partial W} - \langle \nabla_{e_2} e_2, e_3\rangle \,, \end{aligned} \tag{4.6}$$

Equation (4.5) can be written as

$$e_2(u) = \frac{\langle e_3, w\rangle}{\langle e_2, w\rangle}\left(\langle e_3, w\rangle H_{\partial W} + \operatorname{tr}\bar{K}\right) . \tag{4.7}$$

Combining (4.3), (4.4), and (4.7), we obtain,

$$\begin{aligned}\bar{H}_{\partial W} - \bar{K}(e_3, e_2) + e_2(u) &= \left(\langle e_2, w\rangle + \frac{\langle e_3, w\rangle^2}{\langle e_2, w\rangle}\right) H_{\partial W} + \frac{\langle e_3, w\rangle}{\langle e_2, w\rangle}\left(\operatorname{tr}\bar{K} - \bar{K}(e_2, e_2)\right) \\ &= \langle e_2, w\rangle^{-1}\left(H_{\partial W} - \langle e_3, w\rangle \operatorname{tr}_{\partial W} K\right) \\ &\geq \langle e_2, w\rangle^{-1}\left(H_{\partial W} - |\operatorname{tr}_{\partial W} K|\right) . \end{aligned} \tag{4.8}$$

But observe that the quantity $H_{\partial W} - |\operatorname{tr}_{\partial W} K|$ is positive if and only if ∂W is null mean convex, and moreover that, $H_{\partial W} - |\operatorname{tr}_{\partial W} K| \geq \min(\theta_+, -\theta_-)$.

Hence, using (4.8) in (4.2) we obtain,

$$\iint_{\Sigma_f} \mathcal{K} dA + \int_{\partial\Sigma_f} \kappa ds \geq \int_{\partial W} H_{\partial W} - |\operatorname{tr}_{\partial W} K| \, ds > 0 \, , \tag{4.9}$$

from which we conclude that W^2 is diffeomophic to a disk in the case that there is a global solution to Jang's equation.

We must now consider the possibility that the solution to Jang's equation with $f = 0$ on ∂W has cylindrical blow-ups in the interior of W. Thus, in accordance with Proposition 1 and Corollary 2 in [41], assume there exist (i) an open set $W' \subset W$ such that $\partial W' = \partial W \cup \sigma$, where σ is a union of ℓ MOTSs homologous to ∂W, and (ii) a regular solution $f : W' \cup \partial W \to \mathbb{R}$ to Jang's equation, with $f = 0$ on ∂W, such that the graph Σ_f asymptotes in a C^2 fashion to the vertical cylinders over the components of σ. Let $\Sigma_{f,a}$ denote the portion of Σ_f lying between $z = -a$ and $z = a$. For a sufficiently large, $\Sigma_{f,a}$ is a smooth compact manifold with boundary $\partial\Sigma_{f,a} = \partial W \cup \sigma_a$, where each component of σ_a corresponds to a cylindrical end of Σ_f.

By the Gauss-Bonnet theorem, and for a large enough,

$$\begin{aligned}\iint_{\Sigma_{f,a}} \mathcal{K} dA + \int_{\partial\Sigma_{f,a}} \kappa ds &= 2\pi\chi(\Sigma_{f,a}) = 2\pi\chi(W') \\ &= 2\pi\left(2 - 2g - k - \ell\right) . \end{aligned} \tag{4.10}$$

As in our earlier analysis we obtain,

$$\begin{aligned}\iint_{\Sigma_{f,a}} \mathcal{K} dA + \int_{\partial\Sigma_{f,a}} \kappa ds &\geq \int_{\partial\Sigma_{f,a}} \bar{H}_{\partial\Sigma_{f,a}} - \bar{K}(e_3, e_2) + e_2(u)\, ds \\ &\geq \int_{\partial W} H_{\partial W} - |\mathrm{tr}_{\partial W} K|\, ds \\ &\qquad + \int_{\sigma_a} \bar{H}_{\sigma_a} - \bar{K}(e_3, e_2) + e_2(u)\, ds\,. \qquad (4.11)\end{aligned}$$

Note that ∂W is the asymptotic boundary where the convexity condition holds and σ_a is the boundary produce on the interior of W due to the blow up.

Focusing on the second integral on the right hand side of (4.11), one can show that each term in the integrand goes to zero as $a \to \infty$. Consider the term $e_2(u)$. From (4.5), we have,

$$e_2(u) = -\frac{\langle e_2, \partial_z \rangle}{\langle e_3, \partial_z \rangle} B(e_2, e_2)\,, \qquad (4.12)$$

where B is the second fundamental form of Σ_f; in terms of coordinates, B is given by, $B_{ij} = -(1 + |Df|^2)^{-\frac{1}{2}} D_i D_j f$. Expressing the right hand side of (4.12) in terms of f leads to,

$$e_2(u) = \frac{1}{|Df|(1 + |Df|^2)^{\frac{3}{2}}} \mathrm{Hess} f(Df, Df)\,.$$

Using $\mathrm{Hess} f(Df, Df) = \langle D_{Df} Df, Df \rangle = \frac{1}{2} Df(|Df|^2) = |Df| Df(|Df|)$ in the above gives,

$$e_2(u) = \frac{Df(|Df|)}{(1 + |Df|^2)^{\frac{3}{2}}}\,. \qquad (4.13)$$

One can now take a Riemannian collared neighborhood of each component of σ analogous to what is done in [52, Lemma 2.2]. Namely, one can use the level curves of f near each component $\sigma(i)$ of σ to introduce orthogonal coordinates (r, θ), with $r = 0$ at $\sigma(i)$, such that, with respect to these coordinates $f = \pm\frac{1}{r}$ with $r \to 0$ as $a \to \infty$. It then follows from (4.13) that $e_2(u) = O(r)$. This implies that $e_2(u) \to 0$ as $a \to \infty$, as claimed. Simpler arguments show that the other two terms in the integrand in the second integral on the right hand side of (4.11) tend to zero as $a \to \infty$.

Since, as has already been shown, the first integral on the right hand side of (4.11) is positive, it follows that, for a sufficiently large, the entire right hand side of (4.11) will be positive. Since $g \geq 0$ and $k \geq 1$, Equation (4.10) then implies that $\ell = 0$, which means that there can be no cylindrical ends. That is, for 2-dimensional initial data sets obeying the DEC, solutions to Jang's equation cannot blow up[6], and we are back to the first case.

We have now shown in general that W is a disk. Again, to complete the proof of Theorem 4.1, we need to show that there are no MOTSs in W. Arguing by contradiction as in the first proof, we obtain an outermost MOTS Σ' in W'. Now, instead of appealing to Theorem 2.3, one can make use of a result of Metzger [53] to give an argument based on Jang's equation. By Theorem 3.1 in [53], which we assume holds for 2-dimensional initial data sets, since Σ' is outermost there exists a solution f to Jang's equation, with $f = 0$ on ∂W, such that $f \to +\infty$ on approach to Σ'. (There may, in addition, be other blow-ups.) But our arguments based on Gauss-Bonnet have shown that there can be no blow-ups. $\square$

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