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Space-Time Defect Solutions of the Einstein Field Equations

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Ideas from the theory of defects in crystalline matter are combined with results from the direct gauge theory for the Poincaré group to obtain exact solutions of the Einstein field equations. Many of the solutions are sufficiently simple that the equations for geodesic motion can be solved in closed form. Some of these solutions exhibit unexpected behaviors and properties, such as geodesic motions with hyperlight speed and local time reversals relative to observers in the asymptotic Minkowski space-time at large distances from the defect core regions. However, these same geodesic motions are regular in the frames of reference attached to observers that move along the geodesics, and hence no established physical laws are broken by such solutions.

1. INTRODUCTION

One of the original reasons for constructing a gauge theory for the Poincaré group was the belief that the four-parameter translation subgroup would lead to a description of gravity that agreed with the Einstein formulation and was compatible with the other fundamental gauge theories. Although it is now known that gravitational effects arise from the compensating 1-forms for local action of Lorentz boosts, local action of the translation subgroup is still essential. This is because the Einstein field equations are obtained, either directly or as approximations, from the field equations that are the Euler–Lagrange equations that result from variation of an appropriate Lagrangian function with respect to the components of the compensating fields for the local action of the translation group. The question thus naturally arises as to what specific effects or physical proper-

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ties are associated with the presence of compensating fields for the local action of the translation group. A partial answer to this question is obtained by use of corresponding ideas and methods of analysis developed in the classical theory of defects (dislocations and disclinations) in crystalline matter. The results are exact descriptions of regions of space-time where the Einstein field equations are satisfied and where unusual patterns of geodesic motion are obtained.

2. THE UNDERLYING GAUGE GEOMETRY

The results collected together in this section have been established elsewhere (Edelen, 1986, 1989, and the references therein). The reader is referred to these papers for proofs of the assertions made below.

Let M_4 be the standard Minkowski space-time with global coordinates $\{x, y, z, t\}$. and let L_4 be the space-time that is generated from the local action of the Poincaré group on M_4. The space-time L_4 is a four-dimensional, Riemann–Cartan manifold, in general, with both curvature and torsion, that shares the coordinate functions $\{x, y, z, t\}$ with the underlying M_4. The structure of the manifold L_4 is as follows.

Local action of the four-parameter translation group $T(4)$ gives rise to the four compensating 1-forms

$$\phi^i = \phi^i_j(x^k)\, dx^j, \qquad 1 \le i \le 4 \tag{1}$$

while local action of the six-parameter, proper, orthochronous Lorentz group $L(6)$ gives rise to the six compensating 1-forms

$$W^\alpha = W^\alpha_i(x^k)\, dx^i, \qquad 1 \le \alpha \le 6 \tag{2}$$

The semidirect product structure of the ten-parameter Poincaré group $P(10)$ as a subgroup of $GL(5, \mathbb{R})$ yields the minimal replacement $dx^i \to B^i$, with

$$B^i = B^i_j(x^k)\, dx^j = (\delta^i_j + W^\alpha_j l^i_{k\alpha} x^k + \phi^i_j)\, dx^j \tag{3}$$

Here, $l^i_{j\alpha}$, $1 \le \alpha \le 6$, $1 \le i, j \le 4$, are the components of a matrix basis for the matrix Lie algebra of the Lorentz group. The action of this minimal replacement is what lifts M_4 up to L_4.

The *distortion* 1-forms $\{B^i | 1 \le i \le 4\}$ constitute a basis for the vector space Λ^1 of forms on L_4 provided

$$B^1 \wedge V^2 \wedge B^3 \wedge B^4 = B\mu \ne 0 \tag{4}$$

where

$$B = \det(B^i_j), \qquad \mu = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \tag{5}$$

The dual basis $\{b_i | 1 \le i \le 4\}$ is defined by

$$b_i \rfloor B^j = \delta^i_j, \qquad b_i = b^j_i(x^k)\partial_j \tag{6}$$

The collections $\{B^i | 1 \le i \le 4\}$ and $\{b_i | 1 \le i \le 4\}$ form the fundamental *frame* and *coframe* fields of L_4, respectively. The new space-time L_4 that is obtained from M_4 by minimal replacement has the line element

$$dS^2 = g_{ij}\, dx^i \otimes dx^j \tag{7}$$

with

$$g_{ij} = B^r_i h_{rs} B^s_j, \qquad g = \deg(g_{ij}) = -B^2 \tag{8}$$

while the line element for M_4 is

$$ds^2 = h_{ij}\, dx^i \otimes dx^j = (dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{9}$$

in a system of units for which the speed of light is unity.

The new space-time L_4 has both curvature and torsion. The Cartan torsion 2-forms $\{\Sigma^i | 1 \le i \le 4\}$ are defined by $\Sigma^i = dB^i + W^\alpha l^i_{j\alpha} \wedge B^j$. The holonomic torsion 2-forms $S^k = \frac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji})\, dx^i \wedge dx^j$ are then determined in terms of the Σ^i by $S^k = b^k_r \Sigma^r$. Explicit calculation of the exterior derivatives and products gives the evaluations

$$\Sigma^i = \theta^\alpha l^i_{j\alpha} x^j + d\phi^i + W^\alpha l^i_{j\alpha} \wedge \phi^j, \qquad 1 \le i \le 4 \tag{10}$$

where

$$\theta^\alpha = dW^\alpha + \frac{1}{2} C_\rho{}^\alpha{}_\nu W^\rho \wedge W^\nu = \frac{1}{2} \theta^\alpha_{rs}\, dx^r \wedge dx^s, \qquad 1 \le \alpha \le 6 \tag{11}$$

are the six curvature 2-forms for the Lorentz sector of the Poincaré group. The corresponding components of the curvature tensor and the Ricci tensor of L_4 have the determination

$$R^i_{rsj} = \theta^\alpha_{rs} L^i_{j\alpha}, \qquad R_{ij} = R^k_{kij} = \theta^\alpha_{ki} L^k_{j\alpha} \tag{12}$$

Here,

$$L^i_{j\alpha} = b^i_r l^r_{s\alpha} B^s_j, \qquad 1 \le \alpha \le 6, \quad 1 \le i, j \le 4 \tag{13}$$

define a basis for the Lie algebra of the Lorentz group lifted to the new space L_4; that is,

$$L^k_{i\alpha} g_{kj} + L^k_{j\alpha} g_{ki} = 0 \tag{14}$$

Gauge theories of defects in condensed matter (Kadić and Edelen, 1983; Edelen and Lagoudas, 1988) refer to the quantities such as Σ^i as the *dislocation* density and current 2-forms, while θ^α are referred to as the *disclination* density and current 2-forms. We shall take over these names even though defect dynamics deals only with the gauge theory that arises from the local action of the group that is the semidirect product of $T(3)$ with $SO(3)$. This would seem reasonable in the light of the fact that the smaller gauge group used in defect dynamics is a six-parameter subgroup of the Poincaré group.

The differential equations for the determination of geodesics in the new space-time L_4 take a particularly simple form when resolved on the fundamental frame basis given by (6). We therefore set

$$V^i = b^i_k v^k, \qquad v^i = B^i_k V^k \tag{15}$$

and

$$\omega^\alpha = V^j W^\alpha_j \tag{16}$$

Let $x^i = u^i(\tau)$ be the parametric equations for a geodesic in L_4 with path parameter τ. The four functions $\{u^i(\tau) | 1 \le i \le 4\}$ are to be determined by solving the system of differential equations

$$\mathring{u}^i = V^i = b^i_k v^k, \qquad \mathring{v}^i + \omega^\alpha l^i_{j\alpha} v^j = 0, \qquad 1 \le i \le 4 \tag{17}$$

If we let the first three l's be a basis for the Lie algebra of the subgroup $SO(3, \mathbb{R})$ and the remaining three l's generate Lorentz boosts in the (x, t), (y, t), and (z, t) planes, then the second half of the system (17) becomes

$$\mathring{v}^x + \omega^1 v^y + \omega^2 v^z + \omega^4 v^t = 0 \tag{18}$$

$$\mathring{v}^y - \omega^1 v^x + \omega^3 v^z + \omega^5 v^t = 0 \tag{19}$$

$$\mathring{v}^z - \omega^2 v^x - \omega^3 v^y + \omega^6 v^t = 0 \tag{20}$$

$$\mathring{v}^t + \omega^4 v^x + \omega^5 v^y + \omega^6 v^z = 0 \tag{21}$$

We note in particular that the system (18)–(21) admits the quadratic first integral

$$v^i h_{ij} v^j = V^i g_{ij} V^j = \text{const} \tag{22}$$

A geodesic is thus timelike, null, or spacelike along its entire orbit, depending upon whether the constant in (22) is positive, zero, or negative, respectively.

It was pointed out in Edelen (1989) that (18)–(22) show that gravitational effects arise through the quantities $\{\omega^4, \omega^5, \omega^6\}$. Thus, (16) shows that gravitational effects arise through the compensating 1-forms for the Lorentz boosts; namely, W^4, W^5, and W^6. This conclusion can be further corroborated by noting that (11) and (12) tell us that all of the components of the curvature tensor of L_4 vanish when all of the W's vanish throughout L_4, and hence the Einstein paradigm relating curvature and gravity cannot be valid without local action of the Lorentz group!

3. ELEMENTARY DISLOCATION SOLUTIONS

The simplest possible situation is that in which all of the six W's vanish throughout L_4. If this is the case, then the relations (11) and (12) show that

$$\theta^\alpha = 0, \qquad R^i_{rsj} = 0 \tag{23}$$

and hence L_4 is a curvature-free, four-dimensional space-time. Conversely, if all six of the curvature 2-forms θ^α vanish throughout L_4, then there exists a choice of gauge for which all six of the 1-forms W^α will vanish throughout L_4. There are still the compensating 1-forms ϕ^i for the translation subgroup, however, and hence L_4 can still have torsion. In fact, under the stipulation $W^\alpha = 0$, $1 \le \alpha \le 6$, (10) shows that we have the dislocation density and current (Cartan torsion) 2-forms

$$\Sigma^i = d\phi^i \tag{24}$$

and the distortion 1-forms have the evaluation

$$B^i = dx^i + \phi^i \tag{25}$$

A standard practice in the classical theory of dislocations in crystalline solids (Kovacs and Zsoldos, 1973; Lardner, 1974; Mura, 1982; Nabarro, 1979, Zorawski, 1967; Seeger, 1955) is to specify the dislocation density 2-forms and then to calculate the response of the crystalline solid. It thus seems useful to turn our problem around and specify the dislocation density and current 2-forms Σ^i on L_4. To this end, we consider the particularly simple situation where

$$\Sigma^i = A^i(x, y)\, dx \wedge dy \tag{26}$$

and the functions $\{A^i(x, y)\}$ are piecewise smooth and bounded throughout L_4; that is, where the dislocation density and current 2-forms are *space supported.* Space-time-supported dislocation density and current 2-forms of the form $A^i(z, t)\, dz \wedge dt$ will be considered later. The restriction that the coefficient functions $\{A^i | 1 \le i \le 4\}$ depend only on the two variables x and y follows from the fact that (24) implies that we must satisfy the consistency conditions $d\Sigma^i = 0$. An integration of the system (24) gives

$$\phi^i = a^i(x, y)\{x\, dy - y\, dx\} \tag{27}$$

where the coefficient functions $\{a^i(x, y) | 1 \le i \le 4\}$ have the evaluation

$$a^i(x, y) = \int_0^1 A^i(\lambda x, \lambda y)\lambda\, d\lambda \tag{28}$$

If we use polar coordinates (r, ϑ) in the $x-y$ plane and set

$$\tilde{A}^i(r, \vartheta) = A^i(r \cos \vartheta, r \sin \vartheta),$$

then (28) gives

$$a^i(x, y) = r^{-2} \int_0^r \tilde{A}^i(l, \vartheta) l \, dl \tag{29}$$

If the dislocation densities all vanish outside of an infinite tube of radius r_0 parallel to and including the z axis, then we have the evaluation (29) for $r \le r_0$, while

$$a^i(x, y) = r^{-2} \int_0^{r_0} \tilde{A}^i(l, \vartheta) l \, dl \tag{30}$$

for $r > r_0$. Defect dynamics refers to the region $r \le r_0$ as the *dislocation core,* while the region $r > r_0$ is the field region. Noting that $x\,dy - y\,dx = r^2\,d\vartheta$, use of (27) shows that an integration of the 1-form ϕ^i or B^i around a circle $\{\gamma | r = r_1 > r_0\}$ in the field region gives

$$\int_\gamma B^i = \int_\gamma \phi^i = \int_0^{2\pi} \int_0^{r_0} \tilde{A}^i(l, \vartheta) l \, dl \, d\vartheta$$

Thus, although there is no dislocation density or current in the field region, the presence of a nontrivial dislocation core can be detected in exactly the same way that the presence of an electric current in a conductor can be detected by observations of the field external to the conductor. Use of (27) and (30) shows that the dislocation density and current (Cartan torsion) 2-forms Σ^i all vanish for $r \geq r_0$. In fact, if the a^i are functions of x and y that are homogeneous of degree -2, then we will have $\Sigma^i = 0$. We have already assumed, however, that L_4 is curvature-free. Accordingly, the region $r > r_0$ is both curvature-free and torsion-free. *The region $r > r_0$ of L_4 is therefore a region in which the vacuum Einstein field equations are satisfied*! Solutions to the Einstein field equations that are obtained in this way will be referred to as *space-supported elementary dislocation solutions.* These solutions are not trivial because they are defined on the exterior, $r > r_0$, of a torus of infinite extent, and hence on a region with nontrivial topology. Further, the core region $r < r_0$ of L_4 has the nontrivial dislocation density and current (Cartan torsion) 2-forms $\{A^i(x, y)\, dx \wedge dy | 1 \le i \le 4\}$, which are the consequence of compensation for a nontrivial action of the translation subgroup of the gauge group $P(10)$. Particularly simple examples of elementary dislocation solutions will be analyzed in the following two sections.

The essential point to recognize here is the relative ease with which exact solutions of the Einstein field equations can be obtained. In fact, the procedure admits a useful generalization. Let $\alpha(x^i)$ and $\beta(x^i)$ be two functionally independent functions defined on L_4 (i.e., $d\alpha \wedge d\beta \neq 0$ throughout L_4). The Σ^i are taken to be simple 2-forms with the representation

$$\Sigma^i = A^i(\alpha, \beta)\, d\alpha \wedge d\beta \tag{31}$$

where the functions $A^i(\alpha, \beta)$ are piecewise smooth functions. This form of the Σ^i is clearly sufficient in order to satisfy the compatibility conditions $d\Sigma^i = 0$. Since $\Sigma^i = dB^i$, an integration yields

$$B^i = B^i_j\, dx^j = dx^i + a^i(\alpha, \beta)\{\alpha\, d\beta - \beta\, d\alpha\} \tag{32}$$

where

$$a^i(\alpha, \beta) = \int_0^1 A^i(\lambda\alpha, \lambda\beta)\lambda\, d\lambda \tag{33}$$

In fact, (33) gives particular solutions of the partial differential equations

$$A^i(\alpha, \beta) = \alpha \frac{\partial a^i}{\partial \alpha} + \beta \frac{\partial a^i}{\partial \beta} + 2a^i \tag{34}$$

which result from the demand that

$$d\{a^i(\alpha, \beta)\{\alpha\, d\beta - \beta\, d\alpha\}\} = A^i(\alpha, \beta)\, d\alpha \wedge d\beta \tag{35}$$

It is thus clear that any region of L_4 where all of the a^i are homogeneous functions of $\{\alpha, \beta\}$ of degree -2 is a region where all of the A^i and hence the Σ^i vanish. Such regions therefore have both vanishing curvature and vanishing torsion, and hence the line element $dS^2 = B^r_i h_{rs} B^S_j\, dx^j \otimes dx^j$ is a solution of the Einstein field equations.

4. SPACE-SUPPORTED ELEMENTARY SPACE DISLOCATION SOLUTIONS

We first consider the case where $A^1 = A^2 = A^4 = 0$, while $A^3 = A^3(r)$ is a piecewise smooth, bounded function of the single argument $r = (x^2 + y^2)^{1/2}$ that vanishes for $r > r_0 > 0$. Since only A^3 is nontrivial, the dislocation structure of L_4 is spatial in nature and corresponds to what is referred to as a screw dislocation in the classical literature on defects in crystal structures. Accordingly, we have

$$a^1 = a^2 = a^4 = 0, \qquad a^3 = f(r) = Kr^{-2} \tag{36}$$

for $r > r_0$, where K is a constant that is given by

$$K = \int_0^{r_0} A^3(l) l\, dl$$

The fundamental frame and coframe fields thus have the evaluations

$$b_1 = \partial_1 + yf\partial_z, \qquad b_2 = \partial_y - xf\partial_z, \qquad b_3 \doteq \partial_z, \qquad b_4 = \partial_t \tag{37}$$

$$B^1 = dx, \qquad B^2 = dy, \qquad B^3 = -yf\,dx + xf\,dy + dz, \qquad B^4 = dt \tag{38}$$

Use of (7) shows that the line element for the region $r > r_0$ of L_4 is given by

$$\begin{aligned} dS^2 = &-(1+f^2y^2)\,dx^2 + 2f^2xy\,dx\,dy + 2fy\,dx\,dz \\ &-(1+f^2x^2)\,dy^2 - 2fx\,dy\,dz - dz^2 + dt^2 \end{aligned} \tag{39}$$

This line element is an exact solution of the Einstein field equations in the region $r > r_0$ because this region is both curvature-free and torsion-free. The solution is nontrivial, however, because of the nontrivial topology of the region $r > r_0$; there are loops in this region that cannot be shrunk to a point while remaining in the region. Inspection of (39) shows that this solution is a static solution. In addition, it is asymptotically Minkowskian as r tends to infinity because $f = K/r^2$. It is therefore convenient to view this L_4 as embedded in M_4 for interpretation of the geometry of L_4.

The easiest way of understanding the nature of this solution of the Einstein field equations is to examine the geodesics in the region $r > r_0$. Since all of the W^α vanish, all of the ω^α vanish and the geodesic equations reduce to

$$\mathring{v}^i = 0, \qquad \mathring{u}^i = b^i_j v^j \tag{40}$$

in the region $r > r_0$. Clearly, the first half of this system has the solutions $v^i = k^i$, where the k's are constants. For the purposes of this discussion, let

$$\{v^i\} = \{0, k, 0, 1\} \tag{41}$$

For $0 \le k < 1$, the geodesic will be timelike, while for $k = 1$ it will be a null geodesic (remember that we have taken units so that $c = 1$). The spacelike geodesics that obtain for $k > 1$ will be ignored since they do not correspond to motions of test particles in the Einstein theory.

The evaluations of the frame fields given by (37) and the first of the system of relations (15) give the velocity vector field

$$\{V^i\} = \left\{0, k, \frac{-kxK}{x^2+y^2}, 1\right\} \tag{42}$$

Likewise, (37) and the second set of the system (40) give the following equations for the determination of the geodesic orbits:

$$\begin{aligned} \frac{dx}{d\tau} &= 0; \qquad x(0) = X > r_0 \\ \frac{dy}{d\tau} &= k, \qquad y(0) = 0 \\ \frac{dz}{d\tau} &= -kxK\{x^2 + y^2\}^{-1}, \qquad z(0) = 0 \\ \frac{dt}{d\tau} &= 1, \qquad t(0) = 0 \end{aligned} \tag{43}$$

A direct integration of this initial value problem yields the orbital equations

$$x(\tau) = X, \qquad y(\tau) = k\tau, \qquad z(\tau) = K \arctan\left(\frac{k\tau}{X}\right), \qquad t(\tau) = \tau \tag{44}$$

Accordingly, (42) and (44) yield

$$\{V^i\} = \left\{0, k, \frac{-kXK}{X^2 + k^2\tau^2}, 1\right\} \tag{45}$$

The simplest interpretation of these results is from the vantage point of the asymptotically attached Minkowski space-time. Although the vector field with components V^i gives $V^i g_{ij} V^j = v^i h_{ij} v^j = 1 - k^2 > 0$ for this timelike geodesic with respect to the metric structure of L_4, an observer in the asymptotic Minkowski space at infinity would obtain $V^i h_{ij} V^j = 1 - k^2 - k^2X^2K^2/(X^2 + k^2\tau^2)^2$. In fact, the asymptotic Minkowskian observer would say that the point traversing this geodesic has an effective spatial velocity $\mathscr{V}$ that has the evaluation

$$\mathscr{V}^2 = k^2\left(1 + \frac{X^2K^2}{(X^2 + k^2\tau^2)^2}\right) \tag{46}$$

Now, $X > r_0$ is the value of the closest approach of this geodesic to the dislocation core, as measured by an asymptotic Minkowskian observer, while K is the "total amplitude" of the dislocation core, which is as yet unassigned. It is thus clear that we can always choose K sufficiently large that $\mathscr{V} > 1$ ($\mathscr{V}$ is greater than the speed of light) for some open interval around $\tau = 0$ (around the point of closest approach to the dislocation core). Thus, although an observer moving along this timelike geodesic would record a constant spatial velocity $k < 1$ (less than the light velocity) with respect to the metric structure of the L_4 through which he is passing, a fixed external Minkowskian observer at very large distance from the dislocation core would assign a spatial velocity *greater* than the light velocity for the moving observer in a neighborhood of nearest approach to the dislocation core. In fact, since K can be assigned any finite value, it is clear that there are L_4's with spatial dislocations for which an external Minkowskian observer would record motion along a geodesic with spatial velocity that is any large, but finite multiple of the light speed on closest approach $(X > r_0)$ to the dislocation core.

Another interesting aspect of the above solution of the geodesic equations is the resulting z motion. Observe first that the motion in the (x, y, t) space-time is what would be obtained for the motion of a particle in Minkowski space with sublight spatial velocity $k < 1$ and closest approach X to the origin (z axis); namely

$$x(\tau) = X, \qquad y(\tau) = k\tau, \qquad t(\tau) = \tau$$

The presence of the dislocation core is manifest in the resulting motion along the z axis (i.e., parallel to the dislocation core),

$$z(\tau) = -K \arctan\left(\frac{k\tau}{X}\right)$$

Thus, during a time interval $[0, T]$, the change in the z location is given by

$$\Delta z = K \arctan\left(\frac{kT}{X}\right)$$

which is asymptotic to $\Delta z = K\pi/2$ for large values of T. Appropriate choices of the field strength K thus result in any large but finite value of Δz during the time interval $[0, T]$. Note also that definite motion is required in the (x, y) plane because there is no motion along the z axis if $k = 0$.

5. SPACE-SUPPORTED ELEMENTARY TIME DISLOCATION SOLUTIONS

This time we take $A^1 = A^2 = A^3 = 0$ and $A^4 = A^4(r)$, with $A^4(r)$ a piecewise smooth function of r that vanishes for $r > r_0$. Since only A^4 is nonzero, this situation corresponds to a spatially supported time dislocation. This type of dislocation structure has no analog in the classical literature of defects in crystal structures. With these values of the A^i we necessarily have

$$a^1 = a^2 = a^3 = 0, \qquad a^4 = f(r) = Kr^{-2} \tag{47}$$

with

$$K = \int_0^{r_0} A^3(l) l \, dl \tag{48}$$

The resulting frame and coframe fields are thus given by

$$b_1 = \partial_x + yf\partial_t, \qquad b_2 = \partial_y - xf\partial_t, \qquad b_3 = \partial_z, \qquad b_4 = \partial_t \tag{49}$$

$$B^1 = dx, \qquad B^2 = dy, \qquad B^3 = dz, \qquad B^4 = dt - yf\,dx + xf\,dy \tag{50}$$

These in turn determine the metric structure of the resulting L_4 in the region $r > r_0$ by (39):

$$\begin{aligned} dS^2 = {} & (-1 + f^2y^2)\,dx^2 - 2f^2xy\,dx\,dy - 2fy\,dx\,dt \\ & + (-1 + f^2x^2)\,dy^2 + 2fx\,dy\,dt - dz^2 + dt^2 \end{aligned} \tag{51}$$

This line element represents a static solution of the Einstein field equations in the region $r > r_0$ that becomes asymptotically Minkowskian for sufficiently large r. We may thus view L_4 from the vantage point of the Minkowski space-time that can be asymptotically attached as r approaches infinity.

Again, we investigate the space-time L_4 by solving the geodesic equations. In direct analogy with the previous section, we take $\{v^i\} = \{0, k, 0, 1\}$, with $k^2 < 1$, and hence

$$\{V^i\} = \{0, k, 0, 1 - kxf\} \tag{52}$$

If we use the same initial data as before, namely $\{x^i(0)\} = \{X, 0, 0, 0\}$, the orbital equations for this geodesic are

$$x(\tau) = X, \qquad y(\tau) = k\tau, \qquad z(\tau) = 0, \qquad t(\tau) = \tau - K\arctan\left(\frac{k\tau}{X}\right) \tag{53}$$

Consider two observers, one riding along the geodesic and one at rest at a very large value of r. The observer riding along the geodesic will determine his speed by $V^i g_{ij} V^j = v^i h_{ij} v^j = 1 - k^2 > 0$, and hence he is a proper test particle observer of the space-time L_4. The fixed observer at large r would use his Minkowski metric structure to assign the speed $\mathscr{V}$ by

$$\mathscr{V}^2 = V^i h_{ij} V^j = \left(1 - \frac{kXK}{X^2 + k^2\tau^2}\right)^2 - k^2$$

Clearly, this can be positive, zero, or negative at $\tau = 0$ (at closest approach to the dislocation core) by appropriate choices of the dislocation field strength coefficient K.

Of greater interest here is the "time orbit"

$$t(\tau) = \tau - K\arctan\left(\frac{k\tau}{X}\right) \tag{54}$$

In view of the first three orbital equations given by (53), it is reasonable to interpret τ as the canonical time variable in the asymptotically attached

Minkowski space-time. Since $t(\tau)$ was obtained by solving the geodesic equations, $t(\tau)$ can be interpreted as the time variable for the observer moving along the geodesic. As such, (54) shows that we can always choose the dislocation field strength K so that the observer moving along the geodesic will have *time reversal* in a neighborhood of the closest approach of the geodesic to the dislocation core relative to the canonical time variable of an observer in the asymptotic Minkowski space-time at large distances from the dislocation core.

6. SPACE-TIME-SUPPORTED ELEMENTARY DISLOCATION SOLUTIONS

We again assume that all six of the W^α vanish throughout L_4, and hence we have $\theta^\alpha = 0$, $R^i_{rsj} = 0$ throughout L_4. The situations to be considered here are those for which

$$\Sigma^i = P^i(z, t)\, dz \wedge dt \tag{55}$$

that is, where the dislocation density and current 2-forms are simple and space-time-supported. They give rise to distortion 1-forms that have the evaluations

$$B^i = dx^i + p^i(z, t)\{z\, dt - t\, dt\} \tag{56}$$

because the compensating 1-forms for the translation subgroup are given by

$$\phi^i = p^i(z, t)\{z\, dt - t\, dz\} \tag{57}$$

Here, the p^i are evaluated in terms of the P^i by

$$p^i(z, t) = \int_0^1 P^i(\lambda z, \lambda t)\lambda\, d\lambda \tag{58}$$

Since all of the components of the curvature tensor vanish throughout L_4, it follows from (55) that any region of L_4 where all four of the Σ^i vanish is a region where the Einstein field equations are satisfied. The line element for such a region is given by

$$dS^2 = B^r_i h_{rs} B^s_j\, dx^i \otimes dx^j \tag{59}$$

Solutions obtained in this manner will be referred to as *space-time-supported elementary dislocation solutions* of the Einstein field equations.

A space-time-supported elementary space dislocation solution is defined by

$$P^1 = P(z^2 + t^2), \qquad P^2 = P^3 = P^4 = 0 \tag{60}$$

where $P(z^2 + t^2)$ is a piecewise smooth, bounded function of its argument that vanishes for $z^2 + t^2 = T^2 > T_0^2$. The region $T \le T_0$ is the core region, while the region $T > T_0$ is the external field region. With the evaluations (60), (58) gives

$$p^1 = \mathscr{P} T^{-2} = p, \qquad p^2 = p^3 = p^4 = 0 \tag{61}$$

with

$$\mathscr{P} = \int_0^{T_0} P(l) l \, dl \tag{62}$$

in the external field region $T > T_0$. The corresponding frame and coframe fields are thus given by

$$b_1 = \partial_x, \qquad b_2 = \partial_y, \qquad b_3 = pt\partial_x + \partial_z, \qquad b_4 = -pz\partial_x + \partial_t \tag{63}$$

$$B^1 = dx - pt\,dz + pz\,dt, \qquad B^2 = dy, \qquad B^3 = dz, \qquad B^4 = dt \tag{64}$$

and the line element has the evaluation

$$\begin{aligned} dS^2 = & -dx^2 + 2pt\,dx\,dz - 2pz\,dx\,dt - dy^2 \\ & -(1 + p^2t^2)\,dz^2 + 2p^2tz\,dz\,dt + (1 - p^2z^2)\,dt^2 \end{aligned} \tag{65}$$

This line element is clearly not static, but still reduces to the corresponding Minkowski line element sufficiently far away from the dislocation core (i.e., for $T^2 = z^2 + t^2$ very large). It might seem that the corresponding metric tensor g_{ij} could become singular in view of the fact that we have $g_{44} = 1 - p^2z^2$. This is not the case, however, for a direct calculation gives $\det(g_{ij}) = -1$.

The equations for the geodesics reduce in this case to

$$\mathring{v}^i = 0, \qquad \mathring{u}^i = b^i_j v^j \tag{66}$$

The solution of the first part of this system, with the initial data $\{v^i\} = \{0, 0, k, 1\}$, $k < 1$, is $\{v^i(\tau)\} = \{0, 0, k, 1\}$. The second part of the system (66) thus gives

$$\begin{aligned} \frac{dx}{d\tau} &= kpt - pz, \qquad x(0) = 0 \\ \frac{dy}{d\tau} &= 0, \qquad y(0) = 0 \\ \frac{dz}{d\tau} &= k, \qquad z(0) = Z > T_0 \\ \frac{dt}{d\tau} &= 1, \qquad t(0) = 0 \end{aligned} \tag{67}$$

where the restriction $Z > T_0$ is required in order to ensure that the geodesic does not penetrate the core region $T \leq T_0$. Use of (61) and direct integration gives

$$x(\tau) = -\mathscr{P} \arctan\left(\frac{\tau}{Z + k\tau}\right) \tag{68}$$

$$y(\tau) = 0, \qquad z(\tau) = Z + k\tau, \qquad t(\tau) = \tau \tag{69}$$

A combination of (67) and (69) thus gives

$$\frac{dx(\tau)}{d\tau} = \frac{-\mathscr{P}Z}{\tau^2 + (Z + k\tau)^2} \tag{70}$$

and hence the maximal x-velocity component is $V^x_{\max} = -\mathscr{P}/Z$. Since the dislocation strength $\mathscr{P}$ is still at our disposal, it is clear that we can choose $\mathscr{P}$ so that $V^x_{\max}$ can exceed any finite multiple of the speed of light $(=1)$. Interpretations similar to those made in previous sections can also be made here since (69) represents a linear motion in the (y, z) plane in the asymptotic Minkowski space-time.

A space-time-supported time dislocation solution is obtained by choosing

$$P^1 = P^2 = P^3 = 0, \qquad P^4 = P(z^2 + t^2)\, dz \wedge dt \tag{71}$$

where $P(z^2 + t^2)$ is a piecewise smooth function that vanishes for $T^2 = z^2 + t^2 > T_0^2$. As before, we set $p = \mathscr{P}T^{-2}$, where $\mathscr{P}$ is given by (62). The only nonzero ϕ-field is therefore $\phi^4 = p\{z\, dt - t\, dz\}$. An exactly similar analysis gives the frame and coframe field evaluations

$$b_1 = \partial_x, \qquad b_2 = \partial_y, \qquad b_3 = \partial_z + \frac{pt}{1 + pz}\,\partial_t, \qquad b_t = \frac{\partial_t}{1 + pz} \tag{72}$$

$$B^1 = dx, \qquad B^2 = dy, \qquad B^3 = dz, \qquad B^4 = -pt\, dz + (1 + pz)\, dt \tag{73}$$

The line element in the exterior region $T > T_0$ is thus given by

$$dS^2 = -dx^2 - dy^2 + (p^2t^2 - 1)\, dz^2 - 2pt(1 + pz)\, dz\, dt + (1 + pz)^2\, dt^2 \tag{74}$$

Again, since $\det(g_{ij}) = -(1 + pz)^2 < 0$, this line element is regular throughout the region $z^2 + t^2 > T_0^2$. It is also not static, but reduces to the Minkowski line element at very large values of $T^2 = z^2 + t^2$. An analysis of the geodesics produces results similar to those obtained in Section 5.

7. ELEMENTARY DISCLINATION SOLUTIONS

We now turn to the somewhat more involved situations in which all four of the dislocation compensating 1-forms ϕ^i vanish throughout L_4.

Under these conditions, the relevant geometric quantities have the evaluations

$$\Sigma^i = \theta^\alpha l^i_{j\alpha} x^j, \qquad \theta^\alpha = dW^\alpha + \frac{1}{2} C_\beta{}^\alpha{}_\gamma W^\beta \wedge W^\gamma \tag{75}$$

The resulting space-time L_4 is said to have a *space-supported elementary disclination* if only one of the six W^α is nonzero, and the corresponding nonzero curvature, say θ^q, has the form

$$\theta^q = \Theta(x, y)\, dx \wedge dy \tag{76}$$

where $\Theta(x, y)$ is a piecewise smooth, bounded function. Since (75) now gives $\theta^q = dW^q$, an integration yields

$$W^q = w(x, y)\{x\, dy - y\, dx\} \tag{77}$$

with

$$w(x, y) = \int_0^1 \Theta(\lambda x, \lambda y)\lambda\, d\lambda \tag{78}$$

Accordingly, the Cartan torsion 2-forms have the evaluations

$$\Sigma^i = \Theta(x, y)\, dx \wedge dy\, l^i_{jq} x^j \tag{79}$$

We now make the critical assumption that $\Theta(x, y)$ vanishes for $r^2 = x^2 + y^2 > r_0^2$. The region $r \leq r_0$ of L_4 may be identified with the core of the disclination, while the exterior region $r \geq r_0$ is the field region. For simplicity, we take $\Theta(x, y) = \bar{\Theta}(x^2 + y^2)$, so that

$$w = \mathscr{W} r^{-2}$$

for $r > r_0$. Now, simply observe that the exterior region $r > r_0$ has both vanishing curvature and vanishing torsion, and hence the line element for this exterior region is a solution of the Einstein field equations, a *space-supported elementary Lorentz disclination solution.* In particular, we have

$$B^i = dx^i + w\{x\, dy - y\, dx\} l^i_{jq} x^j \tag{80}$$

and

$$dS^2 = B^r_i h_{rs} B^s_j\, dx^i \otimes dx^j$$

All that remains is to select which of the six possible values for q is to be used for further study. The previously chosen basis matrices for the Lorentz group show that $q = \{1, 2, \text{ or } 3\}$ would correspond to local action of the subgroup $SO(3, \mathbb{R})$, while $q = \{4, 5, \text{ or } 6\}$ correspond to Lorentz boosts in the (x, t), (y, t), and (z, t) planes, and this latter set of three alternatives is known to be associated with gravitational phenomena. Let us therefore take $q = 6$. The evaluations (80) now give

$$B^1 = dx, \qquad B^2 = dy, \qquad B^3 = dz + tw\{x\,dy - y\,dx\} \tag{81}$$
$$B^4 = dt + zw\{x\,dy - y\,dx\}$$

The corresponding frame fields are

$$b_1 = \partial_x + twy\partial_z + zwy\partial_t, \qquad b_2 = \partial_y - twx\partial_z - zwx\partial_t \tag{82}$$
$$b_3 = \partial_z, \qquad b_4 = \partial_t$$

and the line element for the resulting solution of the Einstein field equations is given by

$$\begin{aligned} dS^2 = {} & -(1 + t^2w^2y^2 - z^2w^2y^2)\,dx^2 + 2xyw^2(t^2 - z^2)\,dx\,dy \\ & + 2twy\,dx\,dz - 2zwy\,dx\,dt - (1 + t^2w^2x^2 - z^2w^2x^2)\,dy^2 \\ & - 2twx\,dy\,dz + 2zwx\,dy\,dt - dz^2 + dt^2 \end{aligned} \tag{83}$$

This line element is not static, but it is nonsingular throughout L_4 because (83) gives $\det(g_{ij}) = -1$. It is also asymptotically Minkowskian for large $r^2 = x^2 + y^2$ because $w = \mathscr{W}r^{-2}$. As before, we can use this asymptotic Minkowskian space-time as a "container," so to speak, for the analysis and understanding of the solution represented by (83).

In order to obtain the governing differential equations for geodesics in this L_4, we first have to compute $\omega^6 = v^j b_j^i W_j^6$. Use of the above evaluations gives

$$\omega^6 = w\{xv^y - yv^x\} \tag{84}$$

The geodesic equations now split into two parts. The first part deals with motion in the (x, y) plane. It has the form

$$\mathring{v}^x = 0, \qquad v^x(0) = 0, \qquad \mathring{v}^y = 0, \qquad v^y(0) = k \tag{85}$$

with $k^2 < 1$, and

$$\mathring{x} = v^x, \qquad x(0) = X > r_0, \qquad \mathring{y} = v^y, \qquad y(0) = 0 \tag{86}$$

These equations have the solution

$$v^x(\tau) = 0, \qquad v^y(\tau) = k, \qquad x(\tau) = X, \qquad y(\tau) = k\tau \tag{87}$$

Accordingly, the motion in the (x, y) plane is *rectilinear motion* parallel to the y axis that has the minimal separation $X - r_0 > 0$ from the disclination core, where this separation is measured in the asymptotic Minkowskian space-time. Thus, although there are gravitational forces present because $\omega^6 \neq 0$, these forces are axially polarized, so that there is no gravitational influence in the (x, y) plane. The space-supported Lorentz disclination solution represented by (83) is thus of a fundamentally different nature than previously published solutions of the Einstein field equations.

The simplicity of the first half of the geodesic equations is deceptive, for the remaining equations for the motion in the (z, t) plane are more complicated. When the results given by (86) and (87) are used, the second half of the geodesic equations take the form

$$\mathring{v}^z + \hat{w}Xkv^t = 0, \qquad v^z(0) = 0, \tag{88}$$

$$\mathring{v}^t + \hat{w}Xkv^z = 0, \qquad v^t(0) = 1 \tag{89}$$

$$\mathring{z} = -\hat{w}Xkt + v^z, \qquad z(0) = 0 \tag{90}$$

$$\mathring{t} = -\hat{w}Xkz + v^t, \qquad t(0) = 0 \tag{91}$$

with

$$\hat{w} = \frac{\mathscr{W}}{X^2 + k^2\tau^2} \tag{92}$$

Although the system (88), (89) can be solved first, and the results then put into the remaining equations (90), (91), the τ dependence of $\hat{w}$ that is given by (92) makes these equations difficult to solve in closed form. However, if we introduce a new independent variable by the transformation

$$u(\tau) = \frac{1}{kX}\arctan\left(\frac{k\tau}{X}\right) \tag{93}$$

the system (88), (89) can be solved to give

$$v^z = \sinh[\mathscr{W}kXu(\tau)] = \sinh\left[\mathscr{W}\arctan\left(\frac{k\tau}{X}\right)\right] \tag{94}$$

$$v^t = \cosh[\mathscr{W}kXu(\tau)] = \cosh\left[\mathscr{W}\arctan\left(\frac{k\tau}{X}\right)\right] \tag{95}$$

These relations show that the Lorentz disclination field strength $\mathscr{W}$ determines the ranges of $v^z(\tau)$ and $v^t(\tau)$ during the geodesic motion. An apparent speed $\hat{\mathscr{V}}$ along the z-axis can be defined by $\hat{\mathscr{V}} = v^z/v^t$. Use of (94), (95) gives

$$\hat{\mathscr{V}} = \tanh\left[\mathscr{W}\arctan\left(\frac{k\tau}{X}\right)\right] \tag{96}$$

and hence the motion is always sublight (i.e., $\hat{\mathscr{V}}^2 < 1$), as is known to be the case with geodesic motion in a gravitational field. The "particle" moving along the geodesic thus decelerates along the z axis for negative τ, comes to rest at $\tau = 0$, and then accelerates along the z axis for positive τ, all the while undergoing rectilinear motion in the (x, y) plane. The reader should carefully note that some motion in the (x, y) plane is required in view of the degeneracy that results in (94), (95) when $k = 0$.

When the solutions (94), (95) are substituted into (90), (91), a system of first-order differential equations with variable coefficients is obtained for the determination of $z(\tau)$ and $t(\tau)$. This system is sufficiently complicated that it would appear to require numerical integration even after the new independent variable $u(\tau)$ is introduced.

A significant simplification was achieved by the choice $q = 6$ because happenings in the (x, y) plane and the (z, t) plane decouple to a useful extent. If we were to choose $q = 4$, then

$$B^i = dx^i + w\{x\,dy - y\,dx\}l^i_{j4}x^j$$

and hence we have

$$B^1 = (1 - twy)\,dx + twx\,dy, \qquad B^2 = dy, \qquad B^3 = dz \tag{97}$$

$$B^4 = dt - xwy\,dx + x^2w\,dy$$

The corresponding line element is

$$\begin{aligned} dS^2 = {} & (w^2x^2y^2 - (1 - twy)^2)\,dx^2 + 2(twx(-1 + wty) - w^2x^3y\}\,dx\,dy \\ & - 2wxy\,dx\,dt - \{1 + t^2w^2x^2 - w^2x^4\}\,dy^2 \\ & + 2wx^2\,dy\,dt - dz^2 + dt^2 \end{aligned} \tag{98}$$

This line element is not static and is not necessarily regular because

$$\det(g_{ij}) = -(1 - wty)^2 \tag{99}$$

and $w = \mathscr{W}r^{-2}$. The equations for the geodesics for this line element are considerably more complicated than those for (83) and will not be reported here.

The solutions become even more complicated when the nonzero curvature 2-form is space-time-supported:

$$\theta^q = P(x^2 + t^2)\,dx \wedge dt \tag{100}$$

It is assumed that P is a piecewise smooth function of its arguments that vanishes for $x^2 + t^2 = T^2 > T_0^2$. The corresponding W-field is given by

$$W^q = p(x, t)\{x\,dt - t\,dx\} \tag{101}$$

with

$$p = \frac{\mathscr{P}}{x^2 + t^2} \tag{102}$$

With $q = 6$, the coframe fields are given by

$$B^1 = dx, \qquad B^2 = dy, \qquad B^3 = -t^2 p\, dx + tpx\, dy + dz$$
$$B^4 = xpt\, dx + (1 + x^2 p)\, dt \tag{103}$$

The corresponding line element is

$$\begin{aligned} dS^2 = {} & (1 + p^2 t^4 - p^2 t^2 x^2)\, dx^2 + 2p^2 t^3 x\, dx\, dy + 2pt^2\, dx\, dz \\ & + 2ptx(1 + px^2)\, dx\, dt - (1 - p^2 t^2 x^2)\, dy^2 \\ & - 2ptx\, dy\, dz - dz^2 + (1 + px^2)^2\, dt^2 \end{aligned} \tag{104}$$

Again, this line element is not static. It is, however, regular throughout the region $x^2 + t^2 > T_0^2$ because

$$\det(g_{ij}) = -(1 + px^2)^2 \tag{105}$$

and $p = \mathscr{P}/(x^2 + t^2)$. Further, the core region for this solution is the locus of points in L_4 for which $x^2 + t^2 \leq T_0^2$, and hence there is no core region for t in the intervals $(-\infty, T_0)$ and (T_0, ∞). The line element (104) thus describes the external field of a violent event that occurs during the time interval $[-T_0, T_0]$ in the entire (x, y) plane.

In order to complete this picture, we give an example of a disclination solution generated by the compensating field for the local action of an element of $SO(3, \mathbb{R})$. Here we take

$$W^1 = f\{x\, dy - y\, dx\} \tag{106}$$

in the region $r^2 = x^2 + y^2 > r_0^2$ with

$$f = Kr^{-2} \tag{107}$$

The resulting coframe fields are

$$B^1 = (1 - fy^2)\, dx + fxy\, dy, \qquad B^2 = fxy\, dx + (1 - fx^2)\, dy$$
$$B^3 = dz, \qquad B^4 = dt \tag{108}$$

These coframe fields yield the line element

$$\begin{aligned} dS^2 = {} & -\{f^2 x^2 y^2 + (fy^2 - 1)^2\}\, dx^2 + 2fxy(fx^2 + fy^2 - 2)\, dx\, dy \\ & - \{f^2 x^2 y^2 + (fx^2 - 1)^2\}\, dy^2 - dz^2 + dt^2 \end{aligned} \tag{109}$$

This is obviously a static line element that is regular in the region $r > r_0$ for $K \neq 1$ because $\det(g_{ij}) = -(fx^2 + fy^2 - 1)^2 = -(K - 1)^2$. It is also asymptotically Minkowskian for sufficiently large r. The geodesic equations are, unfortunately, very complicated and lead to little further understanding.

8. DISCUSSION

A combination of direct gauge theory for the Poincaré group with ideas from the classical theory of defects in crystalline materials has led to a method for obtaining collections of new solutions to the Einstein field equations. These solutions have either dislocation or disclination core regions where things happen so that in the regions exterior to the cores there are frame and coframe fields that lead to line elements that satisfy the Einstein field equations. In addition, these line elements are asymptotically Minkowskian sufficiently far away from the core regions. We do not make specific statements about what physically happens in the core regions other than to note that such regions always carry nontrivial torsion fields and may or may not carry nontrivial curvature fields. This situation is a direct analog of current practices in the classical theory of defects, where dislocation and disclination cores are posited and the external fields that are created by the presence of the cores are analyzed. Examples have been given where geodesic motion in such fields results in hyperlight velocities and time reversals relative to observers in the Minkowski space-time at infinity. It is therefore of particular importance that investigations of the properties and possible methods of generation of such core regions be pursued.

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