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Static Electromagnetic Geon

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A static, spherically symmetric, and asymptotically flat solution of coupled Einstein–Born–Infeld equations is presented. When the internal mass of the system is zero the resulting space-time is regular and describes static electromagnetic geon.

Early in the fifties John Wheeler introduced the notion of an electromagnetic geon[(1)]: In the Einstein–Maxwell theory an electromagnetic geon is formed of electromagnetic radiation held together by mutual gravitational attraction. The gravitational field and the electromagnetic field of a geon is regular everywhere.

Here we would like to show that there is a spherically symmetric static solution of the Einstein field equation coupled with the nonlinear Born–Infeld electrodynamics, which is regular everywhere.

The space-time is described by a spherically symmetric line element

$$ds^2 = e^{\nu}\, dt^2 - e^{\lambda}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1}$$

where ν and λ are functions of r only.

The electromagnetic field is determined by an antisymmetric tensor $F_{\mu\nu}$, which is related in the usual way to the covariant potential A_{μ}, $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. The electric and magnetic fields are defined by the relations

$$E_a = F_{\mu\nu} u^{\mu} v^{\nu}_{a}, \qquad B^a = -F^{\mu\nu} u_{\mu} v^{a}_{\nu} \tag{2}$$

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where $F^{\mu\nu}$ is the dual tensor to $F^{\mu\nu}$, u^μ is the four velocity of an observer, and v_a^μ are three orthonormal spacelike vectors. The two invariants of the electromagnetic field tensor $S = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $P = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ are used to construct the Lagrangian. The Born–Infeld electrodynamics is derived from the Lagrangian of the form[2]

$$L = b^2[1 - \sqrt{1 - 2b^{-2}S - b^{-4}P^2}] \tag{3}$$

where b is a constant which has the physical interpretation of a critical field strength.

The auxiliary fields D and H are defined with the help of an auxiliary tensor $H^{\mu\nu} = \partial L/\partial F_{\mu\nu}$

$$D^a = H^{\mu\nu}u_\mu v_\nu^a, \qquad H_a = H_{\mu\nu}u^\mu v_a^\nu \tag{4}$$

The field equations of the Born–Infeld electrodynamics assume the form

$$\nabla_\mu F^{\mu\nu} = 0, \qquad \nabla_\mu H^{\mu\nu} = 4\Pi j^\nu \tag{5}$$

The field equations could be easily generalized to include contribution from a magnetic monopole.

The spherically symmetric solution of the Born–Infeld equations describing a field produced by a point charge q is

$$\begin{aligned} D_r &= \frac{q}{r^2}\, e^{-(1/2)(\nu+\lambda)} \\ E_r &= \frac{q}{\sqrt{r^4 + q^2/b^2}}\, e^{(1/2)(\nu+\lambda)} \end{aligned} \tag{6}$$

The energy momentum tensor of the Born–Infeld electrodynamics is

$$\begin{aligned} 4\Pi T_\nu^\mu = {} & \frac{1}{\sqrt{1 - 2b^{-2}S - b^{-4}P^2}}\, F^{\mu\lambda}F_{\lambda\nu} \\ & + g_\nu^\mu b^2 \left[\frac{1 - 2b^{-2}S}{\sqrt{1 - 2b^{-2}S - b^{-4}P^2}} - 1\right] \end{aligned} \tag{7}$$

For the field of a point charge we have

$$\begin{aligned} 4\Pi T_0^0 = 4\Pi T_1^1 = {} & \frac{E_r^2\, e^{-(\nu+\lambda)}}{\sqrt{1 - E_r^2 b^{-2}\, e^{-(\nu+\lambda)}}} \\ & + b^2[\sqrt{1 - E_r^2 b^{-2}\, e^{-(\nu+\lambda)}} - 1] \\ 4\Pi T_2^2 = 4\Pi T_3^3 = {} & b^2[\sqrt{1 - E_r^2 b^{-2}\, e^{-(\nu+\lambda)}} - 1] \end{aligned} \tag{8}$$

The Einstein field equations for the stationary spherically symmetric space-time assume the form

$$-8\Pi T_1^1 = e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2} \tag{9}$$

$$-8\Pi T_0^0 = e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)-\frac{1}{r^2} \tag{10}$$

$$-8\Pi T_2^2 = -8\Pi T_3^3 = \frac{1}{2}e^{-\lambda}\left(\nu''+\frac{\nu'2}{2}+\frac{\nu'-\lambda'}{r}-\frac{\nu'\lambda'}{2}\right) \tag{11}$$

Since the energy-momentum tensor satisfies $T_0^0 = T_1^1$, Eqs. (9) and (10) imply $\nu'+\lambda'=0$, and without losing generality we can assume that $\nu+\lambda=0$.

Equation (10) can be now written in the form

$$-2b^2\left(\frac{\sqrt{r^4+a^4}}{r^2}-1\right)=\frac{1}{r^2}(r\,e^{-\lambda})'-\frac{1}{r^2} \tag{12}$$

where $a^2 = q/b$. The general solution of this equation is

$$e^{-\lambda}=e^{\nu}=1-\frac{2m}{r}-\frac{2q^2}{3}\frac{1}{\sqrt{r^4+a^4}+r^2}-\frac{4q^2}{3}\frac{1}{r}\int\frac{dr}{\sqrt{r^4+a^4}} \tag{13}$$

here m is an integration constant, which can be interpreted as an intrinsic mass. One can easily generalize this solution to incorporate a magnetic monopole. The gravitational field produced by a magnetic monopole g and electric charge q is described by

$$e^{-\lambda}=e^{\nu}=1-\frac{2m}{r}-\frac{2(q^2+g^2)}{3\sqrt{r^4+a^4}+r^2} \\ -\frac{4(q^2+g^2)}{3}\frac{1}{r}\int\frac{dr}{\sqrt{r^4+a^4}} \tag{14}$$

where $a^4 = (q^2+g^2/b^2)$, and m is an intrinsic mass.

This solution can be explicitly given in terms of elliptic functions

$$e^{-\lambda}=e^{\nu}=1-\frac{2m}{r}-\frac{2(q^2+g^2)}{3\sqrt{r^4+a^4}+r^2} \\ -\frac{2(q^2+g^2)}{3}\frac{1}{r}F\left(\frac{1}{2},\arccos\frac{a^2-r^2}{a^2+r^2}\right) \tag{15}$$

where $F(k, \phi)$ is the elliptic function of the first kind.

For large r we have the following asymptotic expansion:

$$e^{\nu}=1-\frac{2m}{r}-\frac{4(q^2+g^2)\,K(1/2)}{3ar}+\frac{(q^2+g^2)}{r^2}+\cdots \tag{16}$$

A distant observer will associate with this solution a total mass $M=m+[2(q^2+g^2)\,K(1/2)]/3a$, total charge q, and total magnetic charge g.

The spherically symmetric, static, static solution of the coupled Einstein–Born–Infeld equtions possesses interesting properties. When the intrinsic mass m is zero the line element is regular everywhere. It is not very difficult to check that the Riemann tensor is also regular everywhere and hence the space-time is singularity free. It is therefore an example of a static electromagnetic geon.

The purely electromagnetic mass of the geon is $m_{\text{el}}=2q^{3/2}b^{1/2}K(1/2)/3$. Let us compute the electromagnetic mass of a geon produced by an elementary charge assuming that the critical field strength $b=e/r_e^2$, where r_e is the classical electromagnetic radius. In this case $m_{\text{el}}=(2K(1/2)/3)\,m_e=1.236m_e$, where m_e is the electron mass. It is more reasonable, however, to use for the critical field strength the value arising from quantum considerations. Assuming that the potential difference across the Compton wavelength of an electron is equal to $2m_ec^2$ for the critical field strength, we obtain $b=2m_e^2c^3/eh$. This value of the critical field strength leads to $m_{\text{el}}=0.15m_e$. This means that in a realistic situation the electromagnetic mass of an electron is smaller than its total. The static geon cannot therefore surve as a classical model of an electron.

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