PROBING THE GRAVITATIONAL GEON

F. I. Cooperstock,

V. Faraoni

and

G. P. Perry

Department of Physics and Astronomy, University of Victoria
P.O. Box 3055, Victoria, B.C. V8W 3P6 (Canada)

Abstract

The Brill–Hartle gravitational geon construct as a spherical shell of small amplitude, high frequency gravitational waves is reviewed and critically analyzed. The Regge–Wheeler formalism is used to represent the most general gravitational wave perturbation of the spherical background as a superposition of tensor spherical harmonics and an attempt is made to build a non–singular solution to meet the requirements of a gravitational geon. High–frequency waves are seen to be a necessary condition for the geon and the field equations are decomposed accordingly. It is shown that this leads to the impossibility of forming a spherical gravitational geon. The spherical shell in the proposed Brill–Hartle geon does not meet the regularity conditions required for a non–singular source and hence cannot be regarded as an adequate geon construct. Since it is the high frequency attribute which is the essential cause of the geon non–viability, it is argued that a geon with less symmetry is an unlikely prospect. The broader implications of the result are discussed with particular reference to the problem of gravitational energy.

To appear in *Int. J. Mod. Phys. D*

1 Introduction

Almost forty years ago, the geon concept was introduced [1]: zero rest mass field concentrations held together for long periods of time by their gravitational attraction. Such constructs were motivated by studies of the motion of bodies in general relativity. More recent interest arises from the study of the entropy of radiation [2] and from the analogy between electromagnetic geons and quark stars [3]. Electromagnetic, neutrino and mixed type geons were studied [1], [4]–[9] and it was suggested that it should be possible to construct a geon from gravitational waves [10]. Brill and Hartle [11] (henceforth referred to as BH) attempted the construction of a gravitational geon model in detail. Later papers ([12, 13] – see also [14]) assumed the correctness of the BH model. In their approach, BH considered a strongly curved static or quasi–static "background geometry" $\gamma_{\mu\nu}$ on top of which a small ripple $h_{\mu\nu}$ resided, satisfying a linear wave equation. The wave frequency was assumed to be so high as to create a sufficiently large effective energy density which served as the source of the background $\gamma_{\mu\nu}$, taken to be spherically symmetric on a time average. For their analysis, they took the Regge–Wheeler [10] (henceforth referred to as RW) decomposition of $h_{\mu\nu}$ in a spherical background in terms of waves characterized by the usual quantum numbers l, m related to the angular momentum operators, and by the frequency ω. They claimed to have found a solution with a flat–space spherical interior, a Schwarzschild exterior and a thin shell separation meant to be created by high frequency gravitational waves. With the mass M identified from the exterior metric, there would follow an unambiguous realization of the gravitational geon as described above.

To be complete, however, two conditions must be satisfied. Firstly, the gravitational geon must be a non–singular solution of the Einstein equations in vacuum. Any singularities present would indicate the presence of non–gravitational sources $T_{\mu\nu}$ compactified into points, curves or surfaces, negating the desired non–singular purely field structure. Secondly, the consistency of the solution must be demonstrated, namely that the background $\gamma_{\mu\nu}$ is consistent with the time–averaged effective density constructed from $h_{\mu\nu}$ as source in the region of non–vanishing $h_{\mu\nu}$. Regarding the first condition, it is straightforward to show that the junction conditions for regularity are not satisfied by the BH solution and hence as it stands, cannot be taken as singularity–free. With the first condition violated, there is no basis for proceeding with a consideration of the second.

One might reasonably argue that while the given structure is inadequate as it stands, an expansion of the shell region into one of finite extent would reveal a well–posed geon solution with both regularity and consistency. Our analysis is sufficiently general to include this and other geometries in which the gravitational field decays sufficiently rapidly at spatial infinity, and to consider also the possibility of geons "leaking" radiation to the exterior. Both even and odd high frequency modes in the RW formalism were analyzed in conjunction with a static and a time–dependent spherically symmetric background metric $\gamma_{\mu\nu}$. It was found that the Einstein equations forced the elimination of the waves in all cases and hence a spherical gravitational geon cannot exist. While a more general case was not yet analyzed, it would be unexpected that such a geon could be found when the most primitive case is excluded. Moreover, the key factor which leads to the non–existence of the spherical geon is not the spatial symmetry but rather the high frequency. This fortifies the expectation that the result is general.

A concise description of this work was published in [15]. The present paper provides details of the calculations and an expanded study of the gravitational geon problem. In Sec. 2, we review the basic mathematical formalism for the construction of gravitational geons. This is used in Sec. 3 to analyze the proposed BH solution. In Sec. 4, we attempt the construction of a non–singular solution for a general gravitational geon with spherical symmetry and demonstrate that the Einstein equations do not permit the realization of the geon. We conclude with a discussion of the results and their potential ramifications in Sec. 5.

2 Gravitational geons

We consider the spacetime metric given by [1]

$$g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu} \ , \tag{2.1}$$

where we assume that $g_{\mu\nu}$ is asymptotically flat, that $\gamma_{\mu\nu}$ is a static, spherically symmetric, asymptotically flat metric and $h_{\mu\nu}$ are small perturbations ($|h_{\mu\nu}| << 1$) representing gravitational waves. In a system of Schwarzschild–like coordinates $\{x^\alpha\} = \{t, r, \theta, \varphi\}$, the background metric is given by

$$(\gamma_{\mu\nu}) = \mathrm{diag}\left(-\mathrm{e}^{\nu}, \mathrm{e}^{\lambda}, r^2, r^2 \sin^2\theta\right) \ , \tag{2.2}$$

where

$$\lambda = \lambda(r) \, , \qquad \nu = \nu(r) \tag{2.3}$$

and

$$h_{\mu\nu} = h_{\mu\nu}(t, r, \theta, \varphi) \, . \tag{2.4}$$

Following BH, we represent the most general gravitational wave perturbation $h_{\mu\nu}$ of the spherical background as a superposition of tensor spherical harmonics:

$$h_{\mu\nu} = \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \int_0^{+\infty} d\omega \, h_{\mu\nu}^{(lm\omega)}(r, \theta, \varphi) \, \mathrm{e}^{i\omega t} \ + \ \mathrm{c.c.} \tag{2.5}$$

[1]The metric signature is $- + ++$. We use units in which $G = c = 1$. Greek indices run from 0 to 3 and Latin indices run from 1 to 3 (apart from Appendix B, where they assume the values 0, 2 and 3). A comma and a semicolon denote, respectively, ordinary and covariant differentiation with respect to the background metric. The Ricci tensor is given by $R_{\mu\nu} = \Gamma^{\sigma}_{\mu\sigma,\nu} - \Gamma^{\sigma}_{\mu\nu,\sigma} + \Gamma^{\sigma}_{\rho\nu}\Gamma^{\rho}_{\mu\sigma} - \Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma}$.

This is justified by the fact that the dynamics of the gravitational waves in the present context are governed by the *linearized* Einstein equations around the background $\gamma_{\mu\nu}$ and therefore a superposition principle holds. Due to linearity, we can restrict ourselves to a study of the evolution of the single tensor spherical modes. For ease of comparison with the BH paper, we will use the RW set of tensor spherical harmonics ([10], [16]–[18]; see [19] for a review and for relations with other sets of tensor spherical harmonics). An "even mode" (also called "polar mode" by other authors [20]) in the RW formalism is factorized as the product of functions dependent only on time, radius, and angles respectively. The angular part is determined by the numbers l and m related to the usual scalar spherical harmonics. The even modes have the form

$$h_{\mu\nu}^{(\text{even})}(t,r,\theta,\varphi) = \begin{pmatrix} -e^{\nu}H_0(r) & H_1(r) & 0 & 0 \\ H_1(r) & e^{\lambda}H_2(r) & 0 & 0 \\ 0 & 0 & r^2K(r) & 0 \\ 0 & 0 & 0 & r^2K(r)\sin^2\theta \end{pmatrix} Y^{lm}\cos(\omega t)\,, \quad (2.6)$$

where $Y^{lm}(\theta,\varphi)$ are the usual spherical harmonics [2]. These modes have parity $(-)^l$. The "odd modes" (in the RW terminology – also called "axial modes") are given by

$$h_{\mu\nu}^{(\text{odd})}(t,r,\theta,\varphi) = \begin{pmatrix} 0 & 0 & -h_0(r)\,(\sin\theta)^{-1}\,\partial Y^{lm}/\partial\varphi & h_0(r)\sin\theta\,\partial Y^{lm}/\partial\theta \\ 0 & 0 & -h_1(r)\,(\sin\theta)^{-1}\,\partial Y^{lm}/\partial\varphi & h_1(r)\sin\theta\,\partial Y^{lm}/\partial\theta \\ \text{Sym} & \text{Sym} & 0 & 0 \\ \text{Sym} & \text{Sym} & 0 & 0 \end{pmatrix} \cos(\omega t) \quad (2.7)$$

and have parity $(-)^{l+1}$. We will consider the case of odd and even modes separately.

A **gravitational geon** is defined as a bounded configuration of gravitational waves whose gravity is sufficiently strong to keep them confined. It is required that no matter or fields other than the gravitational field be present. Although one may consider the possibility of strong gravitational waves, and the definition of gravitational geon allows for this possibility, in this paper we will restrict ourselves to the case in which the amplitude of gravitational waves is

[2] Strictly speaking, the radial functions in Eqs. (2.6) and (2.7) depend on ω, l and m and should be labelled accordingly. However, this would result in a cumbersome notation that is preferably avoided.

small. This permits us to apply the linearized Einstein theory to the propagation of each single wave in the background created by the average action of all the waves composing the geon. Furthermore, it is required that the configuration represented by the metric $\gamma_{\mu\nu}$ be stable over a time scale much larger than the typical period of its gravitational wave constituents, and that the gravitational field becomes asymptotically flat at spatial infinity. Gravitational geons were introduced on the basis of the analogy with electromagnetic and neutrino geons in the RW paper and were studied in greater detail by BH. Wheeler's method of building an electromagnetic geon was to replace the details of the electromagnetic field by the time average of the components of the electromagnetic stress energy tensor. Upon averaging over many modes of oscillation of the electromagnetic field, one obtains a stress-energy tensor, and as a consequence, a gravitational field and metric which are spherically symmetric. Any given mode of oscillation is taken to propagate in the spherically symmetric gravitational field created by the rest of the radiation. The attempt to build a geon resembles the construction, in other fields of physics, of a system with many (almost) identical components, each of which introduces a negligible perturbation in the dynamics of the whole system and has an evolution governed by the averaged action of all the other components. An example of such a system in Newtonian theory is a galaxy described by the potential created by the mass distribution of many stars (here we neglect dark matter, and the fact that a potential–density pair usually describes only a single component of a galaxy, and is adequate only for certain types of galaxies [22]). Each star gives a very small contribution to this potential and its orbit is determined by the global galactic potential.

Consistent with this idea, it is required that

$$\gamma_{\mu\nu} = \langle g_{\mu\nu} \rangle \ . \tag{2.8}$$

We also have

$$\langle h_{\mu\nu} \rangle = \left\langle \frac{\partial h_{\mu\nu}}{\partial x^\alpha} \right\rangle = \left\langle \frac{\partial^2 h_{\mu\nu}}{\partial x^\alpha \partial x^\beta} \right\rangle = 0 \ , \tag{2.9}$$

where $\langle \quad \rangle$ denotes an average over a time that is much longer than the typical gravitational wave wavelength λ ("Brill–Hartle average"). A mathematically rigourous treatment of this concept is contained in the paper by MacCallum and Taub [23]. This idea has proved very valuable and the averaging process has been used by many authors after BH, and is well defined only if it is assumed that the typical wavelength λ [3] is much smaller than the space and time

[3] The term "typical gravitational wavelength" λ may be source of confusion to some readers. Since we are decomposing the general wave form into an infinite set of Regge–Wheeler modes, one may think that λ represents the wavelength of each mode, and that Eq. (2.10) is only valid if the geon was composed of one and only one mode. However, when one is analyzing a general wave form, it is justifiable to assign a *single* parameter describing the scale of variation of the wave form. In the present context, λ is the scale over which the wave form varies. Equation (2.10) is easily derived from Eq. (2.23) if one keeps in mind that $h_{\mu\nu,\alpha} \sim \epsilon/\lambda$ etc. (see [24]) and that λ represents the scale of variation of $h_{\mu\nu}$.

scale of variation L of the background metric $\gamma_{\mu\nu}$ (high frequency approximation) [21]:

$$\epsilon \equiv \frac{\lambda}{L} << 1 \, . \tag{2.10}$$

This assumption provides us with a smallness parameter ϵ to be used as an expansion parameter. Following [21], we measure times and lengths in units of L so that $\lambda = \epsilon$. We have also

$$h_{\mu\nu} = \mathrm{O}(\epsilon) \, , \tag{2.11}$$

$$\omega = \frac{2\pi}{\lambda} = \mathrm{O}\left(\frac{1}{\epsilon}\right) \, , \tag{2.12}$$

$$\mathrm{O}\left(\frac{\partial h_{\mu\nu}}{\partial x^\alpha}\right) = \mathrm{O}(1) \, , \tag{2.13}$$

$$\mathrm{O}\left(\frac{\partial^2 h_{\mu\nu}}{\partial x^\alpha \partial x^\beta}\right) = \mathrm{O}\left(\frac{1}{\epsilon}\right) \, . \tag{2.14}$$

In our notation, $\mathrm{O}(1) \equiv \mathrm{O}(\epsilon^0)$. Equation (2.11) is derived in [21, 24, 25]. It is to be noted that, in the most general case of high frequency gravitational waves on a curved spacetime, two smallness parameters are involved: the dimensionless amplitude of the waves and the ratio λ/L. These two parameters coincide in the specific case under consideration, in which the only source of the background curvature are the gravitational waves. One can conceive of situations in which more than one parameter arises from the high frequency approximation, and these cases have been considered in the literature (see e.g. [26]). However, in these situations, gravitational waves are not the only source of curvature. When gravitational waves are the only source of curvature, as in the gravitational geon, these multiple parameters reduce to the single parameter ϵ. Equation (2.13) implies that the quantum numbers l and m are of order $\mathrm{O}(1/\epsilon)$.

The Ricci tensor can be expanded in the form [11, 21]

$$R_{\alpha\beta}(g) = R^{(0)}_{\alpha\beta}(\gamma) + R^{(1)}_{\alpha\beta}(\gamma, h) + R^{(2)}_{\alpha\beta}(\gamma, h) + \cdots \, , \tag{2.15}$$

where ([11, 21] and references therein)

$$R^{(1)}_{\alpha\beta} = \frac{1}{2} \gamma^{\rho\tau} \left(h_{\rho\tau;\alpha\beta} + h_{\alpha\beta;\rho\tau} - h_{\tau\alpha;\beta\rho} - h_{\tau\beta;\alpha\rho} \right) \, , \tag{2.16}$$

$$\begin{aligned} R^{(2)}_{\alpha\beta} = & -\frac{1}{2} \left[\frac{h^{\rho\tau}{}_{;\beta}}{2} h_{\rho\tau;\alpha} + h^{\rho\tau} \left(h_{\tau\rho;\alpha\beta} + h_{\alpha\beta;\rho\tau} - h_{\tau\alpha;\beta\rho} - h_{\tau\beta;\alpha\rho} \right) \right. \\ & \left. + h_\beta{}^{\tau;\rho} \left(h_{\tau\alpha;\rho} - h_{\rho\alpha;\tau} \right) - \left(h^{\rho\tau}{}_{;\rho} - \frac{h^{;\tau}}{2} \right) \left(h_{\tau\alpha;\beta} + h_{\tau\beta;\alpha} - h_{\alpha\beta;\tau} \right) \right] \, , \end{aligned} \tag{2.17}$$

and $h \equiv h^{\alpha}{}_{\alpha}$. The term $R^{(0)}_{\alpha\beta}(\gamma)$ is the Ricci tensor of the background metric $\gamma_{\mu\nu}$, whereas $R^{(1)}_{\alpha\beta}$ and $R^{(2)}_{\alpha\beta}$ are, respectively, the parts of the Ricci tensor linear and quadratic in $h_{\mu\nu}$ and their derivatives. In the absence of high frequency waves (or on a flat background), $h_{\mu\nu}$ and their derivatives are all of order O(ϵ). In this case the superscripts on the expansion terms of Eq. (2.15) also indicate its order in powers of ϵ. However, in the high frequency approximation it is clear that $R^{(1)}_{\mu\nu}$ contains terms of order O($1/\epsilon$) and O(1) as well as O(ϵ) [21]. Similarly, $R^{(2)}_{\mu\nu}$ is comprised of terms of order O(1), O(ϵ), etc. Solving the vacuum field equations

$$R_{\mu\nu}(g) = 0 \tag{2.18}$$

consistently to any order of approximation requires that we set each order in the expansion parameter ϵ equal to zero. We express Eqs. (2.16) and (2.17) as

$$R^{(1)}_{\mu\nu}(\gamma, h) = R^{(1)}_{\mu\nu}\left[\epsilon^{-1}\right] + R^{(1)}_{\mu\nu}\left[\epsilon^{0}\right] + \cdots\,, \tag{2.19}$$

$$R^{(2)}_{\mu\nu}(\gamma, h) = R^{(2)}_{\mu\nu}\left[\epsilon^{0}\right] + R^{(2)}_{\mu\nu}\left[\epsilon\right] + \cdots\,, \tag{2.20}$$

where $R^{(k)}_{\mu\nu}[\epsilon^n]$ denotes the term of order O(ϵ^n) in $R^{(k)}_{\mu\nu}$. The first order approximation is thus

$$R^{(1)}_{\mu\nu}\left[\epsilon^{-1}\right] = 0\,. \tag{2.21}$$

The second order approximation requires that terms of order O(1) be set equal to zero. The field equations to this order are

$$R^{(0)}_{\mu\nu}(\gamma) + R^{(1)}_{\mu\nu}\left[\epsilon^{0}\right] + R^{(2)}_{\mu\nu}\left[\epsilon^{0}\right] = 0\,. \tag{2.22}$$

Performing the Brill–Hartle average on Eq. (2.22), one obtains

$$R^{(0)}_{\mu\nu}(\gamma) = -\left\langle R^{(2)}_{\mu\nu}\left[\epsilon^{0}\right]\right\rangle\,. \tag{2.23}$$

Note that from Eq. (2.9)

$$\left\langle R^{(1)}_{\mu\nu}\left[\epsilon^{-1}\right]\right\rangle = \left\langle R^{(1)}_{\mu\nu}\left[\epsilon^{0}\right]\right\rangle = \cdots = 0 \tag{2.24}$$

and hence

$$\left\langle R^{(1)}_{\mu\nu}(\gamma, h)\right\rangle = 0\,. \tag{2.25}$$

In Eq. (2.23) the part of the Ricci tensor quadratic in $h_{\mu\nu}$ and their derivatives has been taken to the right hand side and is seen as an effective source term due to the gravitational waves. It is important to note that Eq. (2.23) has the potential to lead to the description of a gravitational geon only by virtue of the high frequency approximation. Under the assumption that gravitational waves are weak but not of high frequency, Eqs. (2.12)–(2.14) would not hold and the two terms in Eq. (2.23) would have different orders. $R^{(2)}_{\alpha\beta} = \mathrm{O}(\epsilon^2)$ could never balance $R^{(0)}_{\alpha\beta}(\gamma) = \mathrm{O}(1)$ in this equation. This would prevent *a priori* the construction of a gravitational geon. This point can be understood physically by noting that the effective energy density associated with gravitational waves with amplitude $h << 1$ and frequency ω is roughly proportional to $(h\omega)^2$. This quantity can be of order unity only if $\omega \sim 1/h >> 1$. Therefore, it is clear that the high frequency approximation is a necessary condition for geon construction in the present context.

We shall designate as the "geon problem", the problem of finding a solution $(\gamma_{\mu\nu}, h_{\mu\nu})$ to the Einstein equations (2.21), (2.22) and (2.23) with the above mentioned properties and satisfying the boundary conditions describing asymptotic flatness [4]

$$h_{\mu\nu} \to 0 \qquad \text{as} \qquad r \to +\infty \; . \tag{2.26}$$

3 The BH analysis

To the authors' knowledge the only explicit attempt at gravitational geon construction was that of BH. In this Section we review their pioneering approach to the problem and critically analyze their work.

We follow BH in expressing the gravitational wave perturbations in terms of RW tensor spherical harmonics. For the sake of simplicity, as done by BH, we restrict ourselves to the case of odd modes with zero angular momentum along the z–axis (i.e. $m = 0$). The last assumption eliminates the φ–dependence from the $h_{\mu\nu}$ functions and considerably simplifies the Einstein equations. This can be seen from Eq. (2.7) and from the well–known form of the spherical harmonics that we present in Eqs. (3.4), (3.5) below. Thus, the metric perturbations are [5]

$$h_{\mu\nu}\,(t, r, \theta) = \mathcal{R}_{\mu\nu}(r)\,\Theta^l(\theta)\,\mathrm{e}^{-i\omega t} \; , \tag{3.1}$$

where

$$\mathcal{R}_{\mu\nu}(r) = h_0(r)\left(\delta^0_\mu\,\delta^3_\nu + \delta^3_\mu\,\delta^0_\nu\right) + h_1(r)\left(\delta^1_\mu\,\delta^3_\nu + \delta^3_\mu\,\delta^1_\nu\right) \; , \tag{3.2}$$

$$\Theta^l(\theta) = \sin\theta\,\frac{dY^{l0}}{d\theta} = \left(\frac{2l+1}{4\pi}\right)^{1/2}\sin\theta\,\frac{dP^l(\cos\theta)}{d\theta} \; . \tag{3.3}$$

[4] In principle one can impose that gravitational waves are confined to (and therefore the $h_{\mu\nu}$ have support in) a ball or a spherical shell. However, the less restrictive condition (2.26) is sufficient for our purposes.

[5] For ease of comparison with the BH paper, we use a complex exponential to describe the time–dependence of the metric perturbations in Eq. (3.1). This notation is adequate as long as linear quantities in $h_{\mu\nu}$ and their derivatives are considered, but clearly it is incorrect when the part of the Ricci tensor quadratic in $h_{\mu\nu}$ and their derivatives enters the discussion. For future reference, we use a function $\cos(\omega t)$ instead of a complex exponential in Eqs. (2.6), (2.7) and in our calculations of Sec. 4.

Here we use the expression of the spherical harmonics

$$Y^{lm}(\theta,\varphi) = C^{lm} e^{im\varphi} P^{lm}(\cos\theta) \qquad (m \geq 0) , \tag{3.4}$$

$$Y^{lm}(\theta,\varphi) = (-1)^m \left(Y^{l|m|}\right)^* \qquad (m < 0) , \tag{3.5}$$

with the normalization constants [19]

$$C^{lm} = (-1)^m \left[\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}\right]^{1/2} \tag{3.6}$$

which guarantee that $\int_{4\pi} d\Omega \left|Y^{lm}(\theta,\varphi)\right|^2 = 1$. Here * denotes complex conjugation and $P^{lm}(x)$ are the associated Legendre polynomials (which can be expressed in terms of the Legendre polynomials $P^l(x)$). Using the relation $P^{l0}(x) = P^l(x)$ we obtain

$$Y^{l0}(\theta) = \left(\frac{2l+1}{4\pi}\right)^{1/2} P^l(\cos\theta) , \tag{3.7}$$

from which Eq. (3.3) follows [6].

One can now insert the form (3.1)–(3.3) of the metric perturbations into the Einstein equations (2.18), obtaining equations for the unknown functions $h_0(r)$ and $h_1(r)$. Simultaneously solving Eqs. (2.21) and (2.23) for a pair $(\gamma_{\mu\nu}, h_{\mu\nu})$ then provides a solution to the geon problem.

The correct order of magnitude of the various terms in the Einstein equations is determined by Eqs. (2.11)–(2.14). The correct order of magnitude decomposition of the Einstein equations is absent in [11]. While the high frequency approximation was assumed in [11], it was not incorporated into the calculations. As a result, the authors did not obtain the two different orders $O(1/\epsilon)$ and $O(1)$ in the Einstein equations, using a parameter ϵ arising from the high frequency approximation. This is evident from the fact that their final equations (10a)–(10c) and (14) contain terms of different orders in the high frequency limit. In the remaining part of this Section we will show how the BH results can be reproduced and we will comment on their proposed geon model.

The BH equations can only be reproduced in the absence of high frequency waves. In terms of a parameter ϵ related to the weakness of the gravitational waves, Eqs. (2.11)–(2.14) must

[6]Note a misprint in the second of the equations (8) in [11], corresponding to our Eq. (3.2). Also to be noted is an inconsistency in the notation therein: the form (3.1)–(3.3) for the metric perturbations is assumed in [11], but the number m in the definition of the function Θ^{lm} corresponding to our Θ^l is retained. This is inappropriate since it is clear from Eqs. (8) and (9) in [11] that the intention was to set $m = 0$. Otherwise, the function Θ^{lm} would depend on both θ and φ, which is not the case, and the Einstein equations would be much more complicated.

be replaced by

$$\mathrm{O}\left(h_{\mu\nu}\right)=\mathrm{O}\left(\frac{\partial h_{\mu\nu}}{\partial x^{\alpha}}\right)=\mathrm{O}\left(\frac{\partial^{2}h_{\mu\nu}}{\partial x^{\alpha}\partial x^{\beta}}\right)=\mathrm{O}(\epsilon)\qquad\alpha\,,\,\beta=0,...,3\,. \tag{3.8}$$

As a consequence of these equations, the Ricci tensor has the form given by Eq. (2.15), where $R^{(0)}_{\mu\nu}(\gamma)=\mathrm{O}(1)$, $R^{(1)}_{\mu\nu}=\mathrm{O}(\epsilon)$ and $R^{(2)}_{\mu\nu}=\mathrm{O}\left(\epsilon^{2}\right)$. To order O(1) the Einstein equations give the well–known equations for a spherically symmetric, static background (see e.g. [25], p. 300) with vanishing energy–momentum tensor. As far as the order $\mathrm{O}(\epsilon)$ is concerned, only the (0, 3), (1, 3) and (2, 3) components of the Ricci tensor give nontrivial results. These components are

$$\begin{aligned}R^{(1)}_{03}&=-\frac{\mathrm{e}^{-\lambda}}{2}\left[\dot{h}_{13}\left(\frac{2}{r}-\frac{\lambda'}{2}-\frac{\nu'}{2}\right)+\frac{h'_{03}}{2}\left(\lambda'+\nu'\right)+\dot{h}'_{13}-h''_{03}-\frac{2\nu'}{r}\,h_{03}\right]\\&+\frac{1}{2r^{2}}\left(h_{03,22}-h_{03,2}\cot\theta\right)\ , \qquad (3.9)\\R^{(1)}_{13}&=-\frac{\mathrm{e}^{-\nu}}{2}\left(\ddot{h}_{13}-\dot{h}'_{03}+\frac{2\dot{h}_{03}}{r}\right)+\frac{\mathrm{e}^{-\lambda}}{r}\,h_{13}\left(\frac{\lambda'}{2}-\frac{\nu'}{2}-\frac{1}{r}\right)\\&+\frac{1}{2r^{2}}\left(h_{13,22}-h_{13,2}\cot\theta\right)\ , \qquad (3.10)\\R^{(1)}_{23}&=-\frac{\mathrm{e}^{-\lambda}}{2}\left[h'_{13,2}-2h'_{13}\cot\theta+h_{13}\left(\lambda'-\nu'\right)\cot\theta+\frac{h_{13,2}}{2}\left(\nu'-\lambda'\right)\right]\\&-\mathrm{e}^{-\nu}\left(\dot{h}_{03}\cot\theta-\frac{\dot{h}_{03,2}}{2}\right)\ , \qquad (3.11)\end{aligned}$$

where a dot and a prime denote differentiation with respect to t and r, respectively. We now insert the form of the metric perturbations (3.1)–(3.3) into the Einstein equations (2.21) and use the following property of the function Θ^{l} (see Appendix A):

$$\frac{d^{2}\Theta^{l}}{d\theta^{2}}-\cot\theta\,\frac{d\Theta^{l}}{d\theta}+l(l+1)\,\Theta^{l}=0\,. \tag{3.12}$$

After some manipulations we find [7]

$$i\omega\left[h'_{1}+h_{1}\left(\frac{2}{r}-\frac{\lambda'}{2}-\frac{\nu'}{2}\right)\right]-\frac{h'_{0}}{2}\left(\lambda'+\nu'\right)+h''_{0}-h_{0}\left[l(l+1)\,\frac{\mathrm{e}^{\lambda}}{r^{2}}-\frac{2\nu'}{r}\right]=0\ , \tag{3.13}$$

[7] Our Eq. (3.13) differs from Eq. (10c) of BH in the sign of the first term. Equation (3.14) differs from the BH Eq. (10a) in the sign of the second term in the first bracket, while Eq. (3.15) agrees with Eq. (10b) of BH. Note misprints in the BH Eq. (11) corresponding to our Eq. (3.16). One of the coefficients of Q in our Eq. (3.20) differs by a factor 1/2 from the corresponding one in BH Eq. (14). The sign of the right hand side of our Eq. (3.19) is opposite to that in the corresponding BH equation.

$$i\omega\, \mathrm{e}^{-\nu}\left(h_0' - \frac{2h_0}{r}\right) + h_1\left[\frac{l(l+1)}{r^2} - \omega^2 \mathrm{e}^{-\nu} + \frac{\mathrm{e}^{-\lambda}}{r}\left(\lambda' - \nu' - \frac{2}{r}\right)\right] = 0\;, \qquad (3.14)$$

$$i\omega\, \mathrm{e}^{-\nu} h_0 + \mathrm{e}^{-\lambda}\left[h_1' + \frac{h_1}{2}\left(\nu' - \lambda'\right)\right] = 0\;. \qquad (3.15)$$

Following BH we can now use Eq. (3.15) to eliminate h_0 from Eq. (3.14), obtaining the second order differential equation for $h_1(r)$:

$$h_1'' + h_1'\left[\frac{3}{2}\left(\nu' - \lambda'\right) - \frac{2}{r}\right] + h_1\left[\frac{1}{2}\left(\nu' - \lambda'\right)^2 + \frac{1}{2}\left(\nu'' - \lambda''\right) - l(l+1)\frac{\mathrm{e}^{\lambda}}{r^2} + \omega^2 \mathrm{e}^{\lambda-\nu} + \frac{2}{r^2}\right] = 0\;. \qquad (3.16)$$

We introduce the variable Q and the Regge–Wheeler coordinate r_* defined by

$$h_1 \equiv r\mathrm{e}^{(\lambda-\nu)/2}\, Q\;, \qquad (3.17)$$

$$dr_* = \mathrm{e}^{(\lambda-\nu)/2}\, dr\;. \qquad (3.18)$$

In terms of these quantities we have

$$h_0 = -\frac{1}{i\omega}\frac{d(rQ)}{dr_*} \qquad (3.19)$$

and[8]

$$\frac{d^2Q}{dr_*^2} + \left[\omega^2 + \frac{3}{2r}\left(\nu' - \lambda'\right)\mathrm{e}^{\nu-\lambda} - \frac{l(l+1)}{r^2}\mathrm{e}^{\nu}\right] Q = 0\;. \qquad (3.20)$$

This Schrödinger–like equation lends itself to the analogy with the dynamics of waves propagating in an effective potential [1, 10, 11].

At this point BH proceed with the specification of the background metric

$$\mathrm{e}^{\nu} = \begin{cases} 1/9 & \text{if} \quad r \leq a \\ 1 - 2M/r & \text{if} \quad r \geq a \end{cases}\;, \qquad (3.21)$$

$$\mathrm{e}^{\lambda} = \begin{cases} 1 & \text{if} \quad r < a \\ (1 - 2M/r)^{-1} & \text{if} \quad r > a \end{cases}\;, \qquad (3.22)$$

[8]An equation similar to Eq. (3.20) can be derived for the even modes with $m = 0$ [29].

where $a = 9M/4$ and M is the geon mass. This vacuum solution for the background metric implies that the effective energy density due to the gravitational waves vanishes for $r \neq a$. Since the effective energy is positive semi–definite, Eqs. (3.21), (3.22) imply that

$$h_{\mu\nu} = 0 \quad \text{for} \quad r \neq a \; . \tag{3.23}$$

Conversely, if the condition (3.23) is satisfied, the Birkhoff theorem guarantees that the metric is Minkowskian for $r < a$ and the Schwarzschild metric for $r > a$.

Therefore, in the BH model, gravitational waves are confined to a spherical shell, the thickness of which is exactly zero. Apparently, BH meant to build a geon model in which the gravitational waves are trapped in a spherical shell which has a nonvanishing thickness which is much smaller than its radius. However, their equations do not allow for this possibility. To be complete, we examine the viability of a geon with gravitational waves confined to a shell whose thickness is exactly zero. It is easy to see that such a model is physically meaningless and that the geon problem becomes mathematically ill–defined in this case. In fact, the solutions of the radial equations (3.13)–(3.16) cannot be ordinary functions but must be sought in some space of *distributions*. In Eq. (3.16), the coefficients proportional to $\nu' - \lambda'$ and $\nu'' - \lambda''$ are not ordinary functions and have a mathematical meaning only if they are regarded as distributions. The first of these two quantities can be expressed as

$$\nu' - \lambda' = 4Mr^{-2} \left(1 - \frac{2M}{r} \right)^{-1} \theta_H \left(r - a \right) \; , \tag{3.24}$$

where

$$\theta_H(x) \equiv \begin{cases} 0 & \text{if} \quad x < 0 \\ \\ 1 & \text{if} \quad x > 0 \end{cases} \tag{3.25}$$

is the Heaviside step function. Clearly, the radial derivative of $\nu' - \lambda'$ can be taken only in a distributional sense. Therefore the solutions of the Einstein equations are distributions and their domain is some space of test functions which must be specified in such a way that the coefficients and the operations involved in the Einstein equations are well defined. There is no indication as to the manner in which this functional space should be determined. It seems almost certain that, if a meaningful and unambigous mathematical formulation of the problem can be given, the distributional solutions $h_{\mu\nu}$ cannot be seen as locally integrable functions, but rather must have properties like a Dirac delta with support on $r = a$. Furthermore, the product of distributions is not defined and the Einstein equations involving the part of the Ricci tensor quadratic in $h_{\mu\nu}$ and its derivatives is mathematically meaningless in this case. This destroys the possibility of exploring one of the essential features of a gravitational geon. Moreover, if the $h_{\mu\nu}$ are allowed to be distributions, the whole meaning of the linearization around the background $\gamma_{\mu\nu}$, the condition $|h_{\mu\nu}| << 1$, and the estimates of the different orders of magnitude in the Einstein equations, become meaningless. The physical interpretation of a distributional metric and Riemann tensor is problematic. To appreciate this, one can consider the much simpler case of a metric which does not satisfy the appropriate junction conditions [27] on a spacelike or timelike hypersurface (this is the case of the metric $\gamma_{\mu\nu}$ given by Eqs. (3.21), (3.22) and the timelike hypersurface $r = a$ – see Appendix B). As suggested by Israel [28], and as can be seen from the computation of the Einstein tensor for the spherical metric specified by Eqs. (3.21), (3.22), a singular hypersurface S (in the sense [27] that the first, or the second fundamental form, or both are not continuous at S) is associated with nonvanishing $T_{\mu\nu}$, a source of stresses. The definition of a geon, a structure of pure gravitational waves in the absence of matter, excludes the use of a background metric which does not satisfy the proper junction conditions. If, in addition, the "perturbations" $h_{\mu\nu}$ are allowed to be distributions, the consideration of junction conditions loses its meaning, but the argument shows that delta–like sources of stresses are included in the problem. Thus, we exclude the case in which gravitational waves are confined to a shell, the thickness of which is exactly zero, as physically meaningless, mathematically ill–defined, and nonviable.

The only possible alternative for a geon model in which gravitational waves are confined to a spherical shell is the case in which the shell has a nonzero thickness. Apparently, BH meant to consider such a model, although this contradicts some of their equations. To be specific, let us consider a shell of radius a and thickness δa described by values of the radial coordinate in the range

$$a - \frac{\delta a}{2} \leq r \leq a + \frac{\delta a}{2} \, , \tag{3.26}$$

where $0 < \delta a << a$. In order for the geon to be a distribution of pure gravitational fields without matter, we must require that the metric tensor satisfies the appropriate junction conditions [27] at the two timelike hypersurfaces $S_{\pm} = \{(t, r, \theta, \varphi) : \;\; r = a \pm \delta a/2\}$. This guarantees the absence of a real (as opposed to "effective", i.e. generated by gravitational waves) stress–energy tensor $T_{\mu\nu}$ representing a matter distribution. In this model, the modified BH solution would be

$$\mathrm{e}^{\nu} = \left\{ \begin{array}{ccc} 1/9 & \mathrm{if} & r \leq a - \delta a/2 \\ & & \\ 1 - 2M/r & \mathrm{if} & r \geq a + \delta a/2 \end{array} \right. \, , \tag{3.27}$$

$$\mathrm{e}^{\lambda} = \left\{ \begin{array}{ccc} 1 & \mathrm{if} & r \leq a - \delta a/2 \\ & & \\ (1 - 2M/r)^{-1} & \mathrm{if} & r \geq a + \delta a/2 \end{array} \right. \, , \tag{3.28}$$

$$h_{\mu\nu} = 0 \quad \mathrm{if} \quad r < a - \frac{\delta a}{2}, \quad\quad r > a + \frac{\delta a}{2} \, . \tag{3.29}$$

The form of the background metric $\gamma_{\mu\nu}$ inside the spherical shell is not given by BH and must be determined by solving simultaneously the Einstein equations to the two lowest orders for a pair $(\gamma_{\mu\nu}, h_{\mu\nu})$ [26]. The proper orders of magnitude did not appear in [11] as a consequence of neglecting the high frequency approximation, despite the fact that this was introduced at the beginning of the paper in order to define time averages. This is the reason why there is only one set of equations in [11] mixing different orders and a complete solution to the geon problem is not provided. It is actually easy to see that the BH equations cannot be satisfied by a nontrivial solution, once the correct order of magnitude decomposition of the Einstein equation is performed. By using Eqs. (2.11)–(2.14) in Eq. (3.15) and discarding the higher order terms, one obtains

$$i\omega \mathrm{e}^{-\nu} h_0 + \mathrm{e}^{-\lambda} h_1' = 0 \;, \tag{3.30}$$

in which the real functions $h_0(r)$ and $h_1(r)$ are of order $\mathrm{O}(\epsilon^2)$, as can be deduced from Eq. (2.7). In conjunction with the boundary conditions (2.26), Eq. (3.30) gives

$$h_0 = h_1 = 0 \;. \tag{3.31}$$

It is natural to ask if such a solution based on a spherical shell of nonvanishing thickness is viable. This question will be answered in the next Section.

4 Resolving the geon problem

In this Section we study the geon problem assuming the high frequency approximation, as required, and we take into account the orders of magnitude accordingly. In what follows, we solve the geon problem in the case of a spherically symmetric, static and asymptotically flat background $\gamma_{\mu\nu}$. Apart from these assumptions and from the boundary conditions (2.26), we do not restrict ourselves to a spherical shell, nor do we require that the metric perturbations vanish outside a certain radius. At the end of this Section, the results will be generalized to $\gamma_{\mu\nu}$ being a time–dependent, slowly varying, spherically symmetric background metric. We consider separately odd and even modes. In addition, we will not restrict ourselves to a particular value of the number m.

4.1 Odd modes

The form of an odd RW mode is given by Eq. (2.7). The Ricci tensor is computed using Eq. (2.16) which, to the dominant order $\mathrm{O}(1/\epsilon)$ is simplified to (see Appendix C)

$$R^{(1)}_{\alpha\beta}\left[\epsilon^{-1}\right] = \frac{1}{2}\,\gamma^{\rho\tau}\left(h_{\alpha\beta,\rho\tau} - h_{\tau\alpha,\beta\rho} - h_{\tau\beta,\alpha\rho}\right) \;. \tag{4.1}$$

For our purposes it is sufficient to consider the Ricci component R_{22}. By taking into account the high frequency approximation and the orders of magnitude given by Eqs. (2.11)–(2.14), we find

$$R^{(1)}_{22}\left[\epsilon^{-1}\right] = \frac{1}{2}\left(\gamma^{00}h_{02,02} + \gamma^{11}h_{12,12}\right) \ . \tag{4.2}$$

Substitution of Eq. (2.7) into Eqs. (4.2) gives the simple equation

$$\omega \mathrm{e}^{-\nu} h_0 \sin(\omega t) + \mathrm{e}^{-\lambda} h_1' \cos(\omega t) = 0 \ . \tag{4.3}$$

The linear independence of the functions $\sin(\omega t)$ and $\cos(\omega t)$ and the boundary conditions (2.26) give

$$h_0(r) = h_1(r) = 0 \ . \tag{4.4}$$

These in turn imply that the effective source term $-\left\langle R^{(2)}_{\alpha\beta}\right\rangle$ of gravitational waves in Eq. (2.23) vanishes, leaving us with

$$R^{(0)}_{\alpha\beta}(\gamma) = 0 \ , \tag{4.5}$$

which has the Minkowski metric $\eta_{\mu\nu}$ as its only asymptotically flat solution. Therefore, we see that the geon problem has no solution in terms of odd waves. The difference with respect to the treatment of the previous Section arises solely from the high frequency approximation. We now consider the case of even modes.

4.2 Even modes

The form of an even RW tensor spherical harmonic is given by Eq. (2.6). Following the same process as for the odd waves, the relevant Einstein equations to order $\mathrm{O}(1/\epsilon)$ are

$$R^{(1)}_{01}\left[\epsilon^{-1}\right] = \frac{1}{2}\left(\gamma^{22}h_{22,01} + \gamma^{33}h_{33,01} + \gamma^{22}h_{01,22} + \gamma^{33}h_{01,33}\right) = 0 \ , \tag{4.6}$$

$$R^{(1)}_{02}\left[\epsilon^{-1}\right] = \frac{1}{2}\left(\gamma^{11}h_{11,02} + \gamma^{33}h_{33,02} - \gamma^{11}h_{01,12}\right) = 0 \ , \tag{4.7}$$

$$\begin{aligned}R^{(1)}_{11}\left[\epsilon^{-1}\right] = \frac{1}{2}\Big(&\gamma^{00}h_{00,11} + \gamma^{22}h_{22,11} + \gamma^{33}h_{33,11} + \gamma^{00}h_{11,00} + \gamma^{22}h_{11,22} + \gamma^{33}h_{11,33} \\ &-2\gamma^{00}h_{01,01}\Big) = 0 \ ,\end{aligned} \tag{4.8}$$

Insertion of Eq. (2.6) into Eqs. (4.6)–(4.8) gives

$$-2\omega K' \sin(\omega t) Y^{lm} + \frac{H_1}{r^2}\cos(\omega t)\left(\frac{\partial^2 Y^{lm}}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y^{lm}}{\partial\varphi^2}\right) = 0 \ , \tag{4.9}$$

$$\left[\omega\left(H_2 + K\right)\sin(\omega t) + \mathrm{e}^{-\lambda} H_1' \cos(\omega t)\right]\frac{\partial Y^{lm}}{\partial\theta} = 0 \ , \tag{4.10}$$

$$\left[\left(H_0'' + 2K'' + \omega^2 e^{\lambda-\nu} H_2\right) Y^{lm} + \frac{e^{\lambda} H_2}{r^2}\left(\frac{\partial^2 Y^{lm}}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y^{lm}}{\partial\varphi^2}\right)\right]\cos(\omega t)$$
$$-2\omega e^{-\nu} H_1' \sin(\omega t) = 0 \; . \tag{4.11}$$

The linear independence of the functions $\sin(\omega t)$, $\cos(\omega t)$ and the boundary condition (2.26) give

$$H_0 = H_1 = H_2 = K = 0 \; . \tag{4.12}$$

Again, we see that the only possible solution to the Einstein equations to the two lowest orders is the pair $(\gamma_{\mu\nu}, h_{\mu\nu}) = (\eta_{\mu\nu}, 0)$, i.e. the geon problem has no solutions also for the even modes case, as a consequence of the high frequency approximation.

4.3 The time–dependent and stationary cases

The previous results can be generalized to the case of a time–dependent, spherically symmetric background metric $\gamma_{\mu\nu}(t, r)$, under the assumption that its time variation occurs on a scale much larger than the period of the gravitational waves. In this case the high frequency approximation and Eqs. (2.11)–(2.14) remain valid. Equation (2.2) still holds, but Eq. (2.3) is replaced by

$$\lambda = \lambda(t, r) \; , \qquad \nu = \nu(t, r) \; . \tag{4.13}$$

As a consequence of the fact that the estimate of the orders of magnitude in the Einstein equations does not change, we find in this case the same equations that we presented above for the even and odd modes, and the same conclusions apply. If instead, the background metric $\gamma_{\mu\nu}(t, r)$ is allowed to vary on a time scale comparable to the period of the gravitational waves, the high frequency approximation does not hold and a gravitational geon cannot be constructed, as explained in Sec. 2. This remains valid for any time–dependent background metric $\gamma_{\mu\nu}(t, \vec{x})$ when symmetries are absent, due to the fact that our considerations based on Eq. (2.23) do not rely on the assumption of spherical symmetry. Apart from this argument, the realization of a geon with a rapidly varying background metric $\gamma_{\mu\nu}$ is problematic for another reason: If a spherically symmetric background is allowed to vary harmonically with frequency Ω comparable to the frequency of the gravitational waves, one expects a parametric resonance [30] for the modes with $\omega = n\Omega/2$, with $n = 1, 2, \cdots$. The strength of the resonance is a maximum for $n = 1$ and decreases rapidly as n increases. In the limit of a static background, the resonance phenomenon disappears. Accordingly, on the basis of studies of perturbations of black holes and relativistic stars [20], it is expected that in the case of a stationary axisymmetric background metric describing a rapidly rotating geon, the resonance phenomenon between the perturbations and the background metric occurs. In the general case of a time–dependent and rapidly varying background metric $\gamma_{\mu\nu}(t, \vec{x})$ without symmetries, it is not known how to decompose metric perturbations on a complete set playing the role of the tensor spherical harmonics in the spherical case, or even how to define frequencies in the strong curvature region. However, if such concepts can be given a meaning, it seems reasonable to expect some kind of resonance phenomena between the background metric and its gravitational wave perturbations. All these resonance phenomena certainly do not contribute to the realization of a stable configuration, but rather are associated with instabilities that tend to disrupt the system.

5 Discussion and conclusion

The results of the previous Section were derived by making use of some particular gauge conditions that RW imposed in order to set the metric perturbations in the form of Eqs. (2.6) and (2.7). However, it is clear from their very nature that our results are covariant and gauge–independent, since the solution $(\gamma_{\mu\nu}, h_{\mu\nu}) = (\eta_{\mu\nu}, 0)$ that we found has an invariant meaning (for example, the vanishing of the curvature tensor is a covariant concept).

Since a spherically symmetric gravitational geon cannot exist due to the fact that the high frequency approximation forces the elimination of gravitational waves, one might ask if it is possible to realize a gravitational geon in a configuration with less symmetry. We do not expect that such a geon can be constructed when the most primitive case is excluded. The main reason for this belief is that the key factor which leads to the non–existence of the spherical geon is not the spatial symmetry but rather the high frequency.

From a mathematical point of view, the main difference between our approach to the geon problem, as compared to that of BH, consists in our explicit use of the high frequency approximation. We have already seen in Sec. 2 that this is necessary for the geon problem to be meaningful. In Sec. 4 it was shown that the same approximation prevents the realization of a spherically symmetric geon.

An important point in the derivation of our results in Sec. 4 is the generality of our boundary conditions (2.26). These allow for a variety of geometries: spherical shells with finite thickness and $h_{\mu\nu} \neq 0$ only for $r \in (r_{in}, r_{out})$, degenerate shells with $r_{in} = 0$ (balls), or more general unbounded spherical configurations restricted only by the condition (2.26). In his papers on geons, Wheeler [1, 4, 7, 8] describes electromagnetic and neutrino geons as systems which are stable on a long time scale, but not absolutely stable, in the sense that they "leak" radiation to the exterior. The rate of the leaking is negligible, so that a geon is stable for a long time. However a secular instability is introduced, which seems unavoidable [7]. The BH model of a spherical shell with $h_{\mu\nu}$ exactly equal to zero outside a certain radius excludes such a possibility, and it could be conjectured that this might be the reason why their model is not viable, leaving a possibility open for the realization of physically more realistic "leaking" geons [9]. However,

[9] There is inconsistency in [11] at this point: in that paper it is required that $h_{\mu\nu}$ (and therefore Q) vanishes outside the spherical shell. However, the Schrödinger–like equation that is derived there for Q (our Eq. (3.20))

this possibility is excluded by our calculations. In fact our boundary conditions (2.26) allow for this possibility, which in turn is excluded by our results as well.

To provide additional intuitive physical insight, we recall our analogy of Sec. 2 between gravitational waves composing a geon and stars composing a galaxy. The high frequency approximation required in the geon case has a parallel in the case of a galaxy; it corresponds to the requirement that the individual stars have a very high velocity. It is clear that such stars would escape from the galaxy and would not be trapped by its potential well. A galaxy cannot be built exclusively from such stars in rapid motion. In other words, the system would not satisfy the virial theorem and would not be bounded. The difference with the gravitational geon case is that while one is not obliged to require that stars have a very high velocity when constructing a galactic model, the high frequency approximation is necessary for a geon and this, in turn, prevents its realization.

An independent argument to understand the impossibility of a gravitational geon is the following: it is well known that, in the limit of high frequencies, gravitational waves obey the geometric optics approximation [21, 24]. Spatially closed lightlike geodesics exist only inside black holes, which necessitate the existence of singularities. Thus, they are necessarily inconsistent with the definition of a geon. The null circular geodesic at $r = 3M$ in the Schwarzschild geometry is unstable. It is therefore hard to reconcile high frequency gravitational waves with stable trapped graviton trajectories in the absence of matter.

Traditionally, the geon was conceived as a structure of small–amplitude high–frequency gravitational waves compactified to the point where one could describe the resulting metric as the averaged "background" metric induced by the totality of the waves plus a small perturbation due to the local wave presence. This is what was analyzed in the present work. It is natural to consider also waves of "large" amplitude in which case linearization is no longer possible nor is it meaningful to envisage a splitting of the metric as before. In fact, to assign a measure to amplitude presupposes a standard for comparison and in the present work, the background metric served this role. To speak now of large amplitude is to consider waves for which there is no longer a discernable "background" and hence no standard for comparison of amplitude measure. This leads to the realm of exact solutions. One might ask whether an exact wave–like solution of the Einstein equations, singularity–free with localized curvature, asymptotically flat, could exist. Existing exact wave–like solutions such as the plane waves of Bondi, Pirani and Robinson or the cylindrical waves of Einstein and Rosen [34] are not localized and in the second case, are also not singularity–free. While it would appear doubtful that solutions with the geon–like properties can exist, to our knowledge they are not ruled out.

Implicit in the gravitational geon concept is the assumption that the gravitational field

implies a "leaking" geon, as is stated in [11]. In fact, the function Q has a nonvanishing tail for large values of the radius, due to the fact that the effective potential barrier is finite. This effect is analogous to the well–known tunnel effect in quantum mechanics.

has some particular essential features shared by other fields. Other fields, even in their pure states, carry energy. Energy has a mass equivalent and all masses gravitate. Thus, given a sufficient concentration of field energy, one could imagine a gravitated concentration into a spherical region with the effective mass displayed unambiguously by the coefficient of the $1/r$ part of the asymptotic static vacuum metric. The gravitational geon concept is built upon the assumption that the gravitational field itself, even in its pure state, will gravitate and thus have the potential to behave as other concentrations of matter or fields. Through the years, various authors such as Isaacson [21] have dwelt upon the similarities between the gravitational and other fields. For example, Isaacson has attempted to establish that there is a basis for considering a certain construct of the metric as an energy–momentum tensor of the gravitational field which is as substantial as a true energy–momentum tensor. However, this requires averaging and under the appropriate limits, his construct merges with the energy–momentum pseudotensor, the shortcomings of which epitomize the gravitational energy problem. If the gravitational field in its pure form really did have the properties which those authors have ascribed to it, then it would seem reasonable to expect that a gravitational geon could, at the very least in principle, be constructed. However, given the present results, it is worth considering alternative ideas.

Recently, one of the authors [31, 32] introduced a new hypothesis that gravitational energy is localized in regions of non–vanishing energy–momentum tensor. The motivation derived from the fact that the traditional means by which physicists have identified gravitational energy was through the covariant energy–momentum conservation laws. While those laws were extrapolated to produce energy–momentum pseudotensors, implying densities and fluxes even in vacuum, the fact is that the laws themselves are devoid of content in vacuum, producing the empty identity $0 = 0$. Given that there is a plethora of possible pseudotensors and, as their name implies, they are not really tensors, it was suggested [31] that the root of the ambiguity lies in the extrapolation of the conservation laws to regions in which they are without actual content. The hypothesis goes on to propose that the true expression of the gravitational contribution to energy is confined to regions of non–vanishing $T_{\mu\nu}$. In a sense this is the opposite of the Isaacson approach in that rather than being satisfied with a construct which reduces to the pseudotensor, the new hypothesis suggests that proper localization is realized when the pseudotensor is removed.

Clearly, the gravitational geon would negate the new hypothesis as it would provide an example of a space totally free of true energy–momentum tensor $T_{\mu\nu}$ yet exhibit an unambiguous energy content via its asymptotic metric. While one might propose exact plane gravitational wave solutions as counter–examples to the hypothesis, it is to be noted that these are unbounded fields with questionable relevance to physical situations and more directly, these wave solutions can be expressed in Kerr–Schild form for which the pseudotensor vanishes in its entirety [33]. The gravitational geon is a direct challenge to the hypothesis and if the geon cannot exist, the hypothesis has passed another test.

Acknowledgments

This research was supported, in part, by a grant from the Natural Sciences and Engineering Research Council of Canada.

Appendix A: Derivation of Eq. (3.12)

We start from the Legendre equation

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP^l(x)}{dx}\right]+l(l+1)P^l(x)=0 \tag{A.1}$$

and note that

$$\Theta^l(\theta)=\left(\frac{2l+1}{4\pi}\right)^{1/2}\sin\theta\,\frac{dP^l(\cos\theta)}{d\theta}=\left(\frac{2l+1}{4\pi}\right)^{1/2}\left(x^2-1\right)\frac{dP^l(x)}{dx}\,, \tag{A.2}$$

where $x=\cos\theta$. Using

$$\frac{d}{d\theta}=-\sin\theta\,\frac{d}{dx}\,, \tag{A.3}$$

$$\frac{d^2}{d\theta^2}=\sin^2\theta\,\frac{d^2}{dx^2}-\cos\theta\,\frac{d}{dx}\,, \tag{A.4}$$

and the Legendre equation (A.1), we find the relations

$$\frac{d\Theta^l}{d\theta}=-l(l+1)\left(\frac{2l+1}{4\pi}\right)^{1/2}\sin\theta\,P^l(x)\,, \tag{A.5}$$

$$\frac{d^2\Theta^l}{d\theta^2}=-l(l+1)\left(\frac{2l+1}{4\pi}\right)^{1/2}\left[xP^l(x)+\left(x^2-1\right)\frac{dP^l(x)}{dx}\right]. \tag{A.6}$$

Using Eqs. (A.5) and (A.2) in Eq. (A.6), Eq. (3.12) follows.

Appendix B: Junction conditions for the BH background metric

We consider the Darmois junction conditions [27] for the BH background metric on the timelike hypersurface $S\equiv\{(t,r,\theta,\varphi):\quad r=a\}$ separating the regions of the spacetime manifold $U\equiv\{(t,r,\theta,\varphi):\quad r<a\}$, $\bar{U}\equiv\{(t,r,\theta,\varphi):\quad r>a\}$. $\{x^\alpha\}=\{\bar{x}^\alpha\}=\{t,r,\theta,\varphi\}$ and $\{u^i\}_{i=0,2,3}=\{t,\theta,\varphi\}$ are coordinate systems in U, $\bar{U}$ and S, respectively (note that, in this Appendix, Latin indices assume the values 0, 2, 3 due to the timelike character of S). The unit normal to S is directed along the coordinate basis vector dual to dr and has components

$$n_\mu = \delta^1_\mu \, \mathrm{e}^{\lambda/2} \,. \tag{B.1}$$

The metric components $\gamma_{\mu\nu}$ in U and $\bar{\gamma}_{\mu\nu}$ in $\bar{U}$ are given by Eqs. (2.2), (3.21) and (3.22). The first fundamental form of S has components $\gamma_{ij} = \bar{\gamma}_{ij}$. The second fundamental form $K_{\mu\nu} \equiv n_{\mu;\nu}$ of any hypersurface $r =$constant has components

$$K_{ij} = n_{\alpha;\beta} \frac{\partial x^\alpha}{\partial u^i} \frac{\partial x^\beta}{\partial u^j} = -\Gamma^1_{ij} \, \mathrm{e}^{\lambda/2} \tag{B.2}$$

in coordinates $\{u^i\}$. Using the Christoffel symbols of a spherically symmetric metric (see e.g. [25]), we obtain the only nonvanishing components

$$K_{00} = -\frac{\nu'}{2} \, \mathrm{e}^{\nu-\lambda/2} \,, \tag{B.3}$$

$$K_{22} = r \, \mathrm{e}^{-\lambda/2} \,, \tag{B.4}$$

$$K_{33} = r \, \mathrm{e}^{-\lambda/2} \sin^2\theta \,. \tag{B.5}$$

The Darmois conditions [27] require the continuity of the first and second fundamental form across S. The first condition is trivially satisfied, while the second is violated. In fact, we have

$$\lim_{r\to a^-} K_{00} = 0 \neq \lim_{r\to a^+} K_{00} = -\frac{16}{27M} \,, \tag{B.6}$$

$$\lim_{r\to a^-} K_{22} = a \neq \lim_{r\to a^+} K_{22} = \frac{a}{3} \,, \tag{B.7}$$

$$\lim_{r\to a^-} K_{33} = a \sin^2\theta \neq \lim_{r\to a^+} K_{33} = \frac{a}{3} \sin^2\theta \,, \tag{B.8}$$

where the BH relation $a = 9M/4$ was used.

Appendix C: Dominant order in $R^{(1)}_{\alpha\beta}$

The second covariant derivatives appearing in Eq. (2.16) are

$$\begin{aligned} h_{\mu\nu;\alpha\beta} &= h_{\mu\nu,\alpha\beta} - \Gamma^\sigma_{\alpha\beta} h_{\mu\nu,\sigma} - \Gamma^\sigma_{\beta\mu} h_{\sigma\nu,\alpha} - \Gamma^\sigma_{\beta\nu} h_{\sigma\mu,\alpha} - \Gamma^\sigma_{\alpha\mu,\beta} h_{\sigma\nu} - \Gamma^\sigma_{\alpha\mu} h_{\sigma\nu,\beta} \\ &+ \Gamma^\sigma_{\alpha\beta}\Gamma^\rho_{\sigma\mu} h_{\rho\nu} + \Gamma^\sigma_{\beta\mu}\Gamma^\rho_{\alpha\sigma} h_{\rho\nu} + \Gamma^\sigma_{\beta\nu}\Gamma^\rho_{\alpha\mu} h_{\rho\sigma} - \Gamma^\sigma_{\alpha\nu,\beta} h_{\sigma\mu} \\ &- \Gamma^\sigma_{\alpha\nu} h_{\sigma\mu,\beta} + \Gamma^\sigma_{\alpha\beta}\Gamma^\rho_{\sigma\nu} h_{\rho\mu} + \Gamma^\sigma_{\beta\nu}\Gamma^\rho_{\alpha\sigma} h_{\rho\mu} + \Gamma^\sigma_{\beta\mu}\Gamma^\rho_{\alpha\nu} h_{\rho\sigma} \,. \end{aligned} \tag{C.1}$$

Symbolically, we express the various quantities in the last equation as follows:

$$\Gamma = \gamma\,\partial\gamma = \mathrm{O}(1)\,, \tag{C.2}$$

$$\gamma\,\partial h = \mathrm{O}(1)\,, \tag{C.3}$$

$$(\partial\gamma)h = \mathrm{O}(\epsilon)\,, \tag{C.4}$$

$$h\,\partial h = \mathrm{O}(\epsilon)\,, \tag{C.5}$$

$$\Gamma\,\partial h = \mathrm{O}(1)\,, \tag{C.6}$$

$$(\partial\Gamma)h = \mathrm{O}(\epsilon)\,, \tag{C.7}$$

$$\Gamma\,\Gamma\,h = \mathrm{O}(\epsilon)\,. \tag{C.8}$$

By using Eqs. (C.2)–(C.8) in (C.1) and then, in conjunction with Eq. (2.16), Eq. (4.1) follows. The quantity $(h_{\alpha\beta,\rho\tau} - h_{\tau\alpha,\beta\rho} - h_{\tau\beta,\alpha\rho})$ in Eq. (4.1) contains terms of order $\mathrm{O}(1/\epsilon)$ as well as terms of order $\mathrm{O}(1)$. We retain only the former ones in the linearized Einstein equations to order $\mathrm{O}(1/\epsilon)$.

References

[1] J. A. Wheeler, *Phys. Rev.* **97** 511 (1955).

[2] R. D. Sorkin, R. M. Wald and Z. Z. Jiu, *Gen. Rel. Grav.* **13** 1127 (1981).

[3] S. N. Sokolov, *Gen. Rel. Grav.* **24** 519 (1992).

[4] E. A. Power and J. A. Wheeler, *Rev. Mod. Phys.* **29** 480 (1957).

[5] R. D. Brill and J. A. Wheeler, *Rev. Mod. Phys.* **29** 465 (1957).

[6] F. J. Ernst, *Phys. Rev.* **105** 1662; 1665; *Rev. Mod. Phys.* **29** 496 (1957).

[7] J. A. Wheeler, *Rev. Mod. Phys.* **33** 63 (1961).

[8] J. A. Wheeler, *Geometrodynamics* (Academic Press, New York, 1962).

[9] D. R. Brill, in *Perspectives in Geometry and Relativity, Essays in honor of Václav Hlavatý*, ed. B. Hoffmann (Indiana Univ. Press, Bloomington, 1966) p. 38.

[10] T. Regge and J. A. Wheeler, *Phys. Rev. D* **108** 1063 (1957).

[11] R. D. Brill and J. B. Hartle, *Phys. Rev.* **135** B271 (1964).

[12] U. H. Gerlach, *Phys. Rev. Lett.* **25** 1771 (1970).

[13] J. M. Cohen and R. M. Wald, *J. Math. Phys.* **13** 543 (1972).

[14] S. Coleman and L. Smarr, *Comm. Math. Phys.* **56** 1 (1977).

[15] F. I. Cooperstock, V. Faraoni and G. P. Perry, *Mod. Phys. Lett. A* **10**, 359 (1995).

[16] F. K. Manasse, *J. Math. Phys.* **4** 746 (1963).

[17] F. J. Zerilli, *J. Math. Phys.* **11** 2203 (1970).

[18] V. D. Sandberg, *J. Math. Phys.* **19** 2441 (1978).

[19] K. S. Thorne, *Rev. Mod. Phys.* **52** 299 (1980).

[20] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Clarendon Press, Oxford, 1983).

[21] R. A. Isaacson, *Phys. Rev.* **166**, 1263; 1272 (1968).

[22] J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton Univ. Press, Princeton, 1987).

[23] M. A. H. MacCallum and A. H. Taub, *Comm. Math. Phys.* **30**, 153 (1973).

[24] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (Freeman, S. Francisco, 1973).

[25] L. D. Landau and E. M. Lifschitz, *The Classical Theory of Fields*, 4th edition (Pergamon Press, Oxford, 1975).

[26] M. E. de Araujo, *Gen. Rel. Grav.* **18**, 219 (1986); *Gen. Rel. Grav.* **21**, 323 (1989).

[27] W. B. Bonnor and P. A. Vickers *Gen. Rel. Grav.* **13** 29 (1981); G. Darmois *Mémorial des Sciences Mathematiques* (Gauthier–Villars, Paris, 1927), Fasc. 25; S. O'Brien and J. L. Synge *Jump Conditions at Discontinuities in General Relativity* (Dublin Institute for Advanced Studies, 1952); A. Lichnerowicz *Théories Relativistes de la Gravitation et de l'Électromagnétisme* (Masson, Paris, 1955).

[28] W. Israel Nuovo Cimento B **44** 1 (1966).

[29] F. J. Zerilli, *Phys. Rev. Lett.* **24** 737 (1970).

[30] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer–Verlag, New York, 1978).

[31] F. I. Cooperstock, *Found. Phys.* **22** 1011 (1992).

[32] F. I. Cooperstock, in *Topics in Quantum Gravity and Beyond*, ed F. Mansouri and J. J. Scanio (World Scientific, Singapore, 1993) p. 201.

[33] M. Gürses and F. Gürsey, *J. Math. Phys.* **16** 2385 (1975).

[34] D. Kramer, H. Stephani, M. MacCallum and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge Univ. Press, Cambridge, 1980).