however, some of the hypotheses assumed by this author are somewhat restrictive for our purposes; altogether, the example of $SO(3)$ is included in his classification.
36 S. Sternberg, *Lectures on Differential Geometry* (Prentice-Hall, Englewood Cliffs, N.J., 1964).
37 N. Ja. Vilenkin, *Fonctions spéciales et théorie de la representations des groupes* (Dunod, Paris, 1969), Chap. III, Sec. 6. 4.
38 The present scheme of assumptions has a provisional character; in particular, the relative independence of the various hypotheses, as stated here, is rather dubious.
39 E. Onofri and M. Pauri, University of Parma, Preprint IFPR-T-010, 1970, J. Math. Phys. (to be published)
40 L. P. Eisenhart, *Continuous Groups of Transformations* (Dover, New York, 1961).
41 S. A. Dunne, J. Math. Phys. **10**, 860 (1969).
42 R. F. Streater, Commun. Math. Phys. **4**, 217 (1967).
43 R. L. Ingraham, ICTP, Preprint ICTP, no. 9, 1967.
44 V. Fock, Z. Phys. **98**, 145 (1935).
45 H. Bacry, H. Ruegg, and J. M. Souriau, Commun. Math. Phys. 3, 323 (1966).
46 E. Onofri and M. Pauri, Lett. Nuovo Cimento **1**, 607 (1967).
47 M. Bander and C. Itzykson, Rev. Mod. Phys. **38**, 330, 346 (1966).
48 H. Bacry, CERN Report TH 579, Geneva, 1965.
49 C. Fronsdal, Phys. Rev. **156**, 1665 (1967).
50 H. M. Kleinert, in *Lectures in Theoretical Physics*, edited by W. E. Brittin and A. O. Barut (Gordon and Breach, New York, 1968); Fortschr. Physik **16**, 1 (1968).
51 B. Vitale, "'Invariance' and 'Noninvariance' Dynamical Groups," lecture notes, Institute of Mathematical Sciences, Madras, 1966.
52 N. Mukunda, L. O'Raifeartaigh, and E. C. G. Sudarshan, Phys. Rev. Letters **15**, 1041 (1965).
53 G. Cicogna, Nuovo Cimento **49**, 291 (1967).
54 F. Duimio and M. Pauri, Nuovo Cimento **51A**, 1141 (1967).
55 S. Helgason, *Differential Geometry and Symmetric Spaces* (Academic, New York, 1962).
56 The other homogeneous invariants are all zero, so that the realization satisfies condition (β).
57 For the meaning of these parameters, see Eq. (20).
58 I. T. Todorov, "Derivation and Solution of an Infinite-Component Wave Equation for the Relativistic Coulomb Problem," in *Group Representations in Mathematics and Physics*, Battelle Seattle 1969 Rencontres, edited by V. Bargmann (Springer-Verlag, Berlin, 1970).

Note on the Angular Momentum and Mass of Gravitational Geons

Jeffrey M. Cohen
Physics Department, University of Pennsylvania, Philadelphia, Pennsylvania, 19104

and

Robert M. Wald
Physics Department, Princeton University, Princeton, New Jersey 08540

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It is shown that (1) the angular momentum of a gravitational geon must be zero if it is axisymmetric and (2) the mass of a gravitational geon must be zero if it is stationary, i.e., if the space-time possesses a Killing vector which is timelike at infinity. Here angular momentum and mass are defined in terms of the asymptotic form of the metric at large distances; they are physical quantities which can be experimentally measured by distant observers. Since the gravitational geons previously considered are highly dynamical on a small scale, our result on the vanishing mass of a stationary geon does not conflict with previous analyses showing that gravitational geons can have mass. Similarly, our results do not exclude the possibility of gravitational geons having nonvanishing angular momentum if they are not strictly axisymmetric.

1. INTRODUCTION

A gravitational geon may be described physically as a localized region of pure space–time curvature. More precisely, we define a gravitational geon to be a solution of the vacuum Einstein field equations,

$$G_{\mu\nu} = 0 \tag{1}$$

which is (1) nonsingular, (2) topologically Euclidean, (3) asymptotically flat, i.e., there exist coordinates x^μ such that on the hypersurfaces $x^0 = \text{const}$ the metric takes the form $g_{\mu\nu} = \eta_{\mu\nu} + O(1/r)$ at large distances, where $\eta_{\mu\nu} = \text{diag}\,(-1, 1, 1, 1)$ and r is a radial parameter, and (4) approximately stationary in the asymtotically flat region, i.e., for sufficiently large r, derivatives of the metric with respect to x^0 can be neglected compared with derivatives with respect to the spacelike coordinates x^i. Gravitational geons as well as electromagnetic and neutrino geons have been studied as models for material bodies free from the uncertainty about any equations of state.[1]

In this paper, we prove that a gravitational geon cannot have a nonvanishing angular momentum if it is axisymmetric. We also show that the mass of a gravitational geon must vanish if it is stationary.

In Sec. 2 we review the definition of angular momentum and mass used in this paper. We obtain expressions for these quantities in Sec. 3 which are used in Sec. 4 to prove our results on gravitational geons.

2. DEFINITION OF ANGULAR MOMENTUM AND MASS

The discussion of this section follows closely that of Misner, Thorne, and Wheeler.[2]

The space–time metric of any asymptotically flat solution of Einstein's equations which is approximately stationary in the asymptotically flat region can be put in the following form[2] for large r:

$$\begin{aligned} dS^2 = &-\left(1 - \frac{2m}{r} + \frac{2m^2}{r^2}\right)dt^2 \\ &- 4\epsilon_{jkl} J^k\left(\frac{x^l}{r^3}\right) dt dx^j + \left(1 + \frac{2m}{r}\right. \\ &\left.+ \frac{3m^2}{2r^2}\right)\delta_{jk} dx^j dx^k + O\left(\frac{1}{r^3}\right) dx^\mu dx^\nu. \end{aligned} \tag{2}$$

Here Roman indices run from 1 to 3, Greek indices run from 0 to 3, and ϵ_{jkl} is the completely antisymmetric tensor. The parameters m and $J = ((J^1)^2 + (J^2)^2 + (J^3)^2)^{1/2}$ of a space–time are uniquely defined by Eq. (2), i.e., their values cannot be changed by a coordinate transformation which preserves the form, Eq. (2), of the metric. If the gravitational field is weak throughout the space–time, the linearized theory of gravity yields the following expressions[2] for m and J^k:

$$m = \int T^{oo} d^3x^i, \tag{3}$$

$$J^k = \epsilon_{kln} \int (x^l T^{no} - x^n T^{lo}) d^3x^i, \tag{4}$$

where $T^{\mu\nu}$ is the stress–energy tensor of matter. Thus, in the weak field limit, m and J may be identified, respectively, as the total mass and angular momentum. In the strong field case, Eqs. (3) and (4) are, of course, no longer valid, but the expansion of the metric, Eq. (2), still holds in the asymptotically flat region. In the strong field case, we *define* the total (active gravitational) mass to be m and the total angular momentum to be J. Both m and J have direct physical significance: A distant observer can measure m by a study of Keplerian orbits and can measure J by observation of gyroscope precession resulting from the dragging of inertial frames. Transforming from the symptotically Minkowskian coordinates of Eq. (2) to asymptotically spherical polar coordinates and aligning the z axis in the direction of $\mathbf{J}$, we put the metric of Eq. (2) into the following form which is more useful for our purposes:

$$g_{\mu\nu} = \begin{array}{c} \begin{array}{cccc} \underline{t} & \underline{\varphi} & \underline{r} & \underline{\theta} \end{array} \\ \begin{pmatrix} -[1-2m/r+O(1/r^2)] & [(-2J\sin^2\theta)/r+O(1/r^2)] & O(1/r^3) & O(1/r^2) \\ & r^2\sin^2\theta(1+O(1/r)) & O(1/r^2) & O(1/r) \\ \text{SYM} & & 1+O(1/r) & O(1/r^2) \\ & & & r^2[1+O(1/r)] \end{pmatrix} \end{array} \tag{5}$$

Note that, comparing Eq. (5) with the Kerr metric,

$$dS^2 = -\left(1-\frac{2mr}{\Sigma}\right)dt^2 - \frac{4mar\sin^2\theta}{\Sigma}\,dt\,d\varphi + \left((r^2+a^2)\sin^2\theta + \frac{2ma^2r\sin^4\theta}{\Sigma}\right)d\varphi^2 + \Sigma\left(d\theta^2 + \frac{dr^2}{\Delta}\right), \tag{6}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \tag{7}$$

$$\Delta = r^2 - 2mr + a^2, \tag{8}$$

we can immediately see that the angular momentum of the Kerr metric is given by[3]

$$J = ma. \tag{9}$$

3. FORMULAS FOR J AND m

We now obtain formulas for J and m which will be used in Sec. 4.

Let ξ^μ be any vector field which reduces to the vector field $\partial/\partial\varphi$ at large distances, where φ is the angular coordinate defined at large distances by the form of the metric, Eq. (5). (The definition of ξ^μ is left arbitrary in the nonasymptotically flat region.) Then, we have[3]

$$J = \frac{1}{16\pi}\lim_{r\to\infty}\int_{r,t\,\text{const}} {}^*d\xi = \frac{1}{16\pi}\lim_{r\to\infty}\int_{r,t\,\text{const}}(-g)^{1/2}(\xi^{r;t} - \xi^{t;r})\,d\theta\,d\varphi, \tag{10}$$

where t, φ, r, θ are the coordinates of Eq. (5). To prove (10), we note that since ξ^μ agrees with $\partial/\partial\varphi$ for large r, we obtain by direct calculation from the metric of Eq. (5) that

$$\xi^{r;t} = g^{t\alpha}\Gamma^r_{\varphi\alpha} = 3J\sin^2\theta/r^2 + O(1/r^3), \tag{11}$$

$$\xi^{t;r} = g^{r\alpha}\Gamma^t_{\varphi\alpha} = -3J\sin^2\theta/r^2 + O(1/r^3), \tag{12}$$

$$(-g)^{1/2} = r^2\sin\theta\,[1 + O(1/r)]. \tag{13}$$

Equation (10) then follows immediately.

A similar calculation establishes the following formula for m. Let ψ^μ be any vector field which agrees with $\partial/\partial t$ for large values of r. Then we have[4]

$$m = -\frac{1}{8\pi}\lim_{r\to\infty}\int_{r,t\,\text{const}} {}^*d\psi = \frac{1}{8\pi}\int(-g)^{1/2}(\psi^{t;r} - \psi^{r;t})\,d\theta\,d\varphi \tag{14}$$

4. APPLICATION TO GRAVITATIONAL GEONS

The results of this paper now follow from Eqs. (10) and (14).

The assumption that the space–time is topologically Euclidean implies that the 2-surface of constant r and t over which the integral, Eq. (10), is to be taken is the boundary of the interior part of the hypersurface, t = const. Hence, the divergence theorem implies

$$J = \frac{1}{16\pi}\int_{t=\text{const}} d{}^*d\xi = \frac{1}{16\pi}\int_{t=\text{const}}(-g)^{1/2}(\xi^{\mu;t} - \xi^{t;\mu})_{;\mu}\,dr\,d\theta\,d\varphi. \tag{15}$$

If the space–time is axisymmetric, we may take ξ^μ to be the axisymmetric Killing vector. Now, for a Killing vector, we have

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0. \tag{16}$$

Hence, we have

$$(\xi^{\mu;t} - \xi^{t;\mu})_{;\mu} = 2\xi^{\mu;t}{}_{;\mu}. \tag{17}$$

In addition,

$$\xi^\mu{}_{;\mu} = 0, \tag{18}$$

and by the Ricci identity we thus obtain

$$\xi^{\mu;\nu}{}_{;\mu} = \xi^{\mu;\nu}{}_{;\mu} - \xi^\mu{}_{;\mu}{}^{;\nu} = R_\mu{}^\nu\xi^\mu. \tag{19}$$

Thus, for an axisymmetric, topologically Euclidean space–time, we have

$$J = \frac{1}{8\pi} \int_{t=\text{const}} (-g)^{1/2} R_\varphi{}^t d^3x^i. \tag{20}$$

[Note that, in Eq. (20), φ is fixed by the symmetry but the choice of t is arbitrary except in the asymptotically flat region.] For a gravitational geon $R_\mu{}^\nu$ is zero by virtue of the field equation (1), and so for an axisymmetric gravitational geon Eq. (20) yields

$$J = 0. \tag{21}$$

An identical calculation starting from Eq. (14) shows that for a stationary, topologically Euclidean space-time,

$$m = -\frac{1}{4\pi} \int_{t=\text{const}} (-g)^{1/2} R_t{}^t d^3x^i. \tag{22}$$

(Here t is fixed throughout the space-time by the stationary symmetry.) Hence, for a stationary gravitational geon,

$$m = 0. \tag{23}$$

Since $G_\varphi{}^t = R_\varphi{}^t$, it follows from Eq. (20) that J is a conserved quantity associated with axial symmetry and arising from the conservation law $(G_\mu{}^\nu \xi^\mu)_{;\nu} = 0$. This quantity was used in Ref. 3 to define the angular momentum of an axisymmetric space-time. However, since $G_t{}^t = R_t{}^t - \frac{1}{2}R$, we see from Eq. (22) that in the stationary case m is *not* (in general) equal to the conserved quantity arising from $(G_\mu{}^\nu \psi^\mu)_{;\nu} = 0$, except for space-times with vanishing scalar curvature R (e.g., electiovac space-times).

Note that if matter is present in the interior, one obtains $m > 0$ for a stationary, topologically Euclidean space-time if $R_t{}^t = T_t{}^t - \frac{1}{2}T < 0$. For perfect fluid matter this condition becomes $\rho + 3p > 0$.

The proof of Eqs. (21) and (23) does not apply to black holes because for black holes the spacelike hypersurfaces either contain a singularity or have non-Euclidean ("wormhole") topology. In either case, Eq. (15) does not follow from Eq. (10). For further discussion see, e.g., Ref. 5 and the references cited there.

It should be emphasized that, in the proof of Eq. (21), it is required that the geon be strictly axisymmetric, i.e., axisymmetric on a small scale, not merely approximately axisymmetric when averaged over some region. Similarly, in the proof of Eq. (23), the geon must be strictly stationary. Since the gravitational geons previously considered are highly dynamical on a small scale, our results do not conflict with analyses which find them to have positive mass.[1,6] Nor do our results exclude the existence of geons having angular momentum which are not axisymmetric on a small scale, e.g., on account of gravitational waves traveling in the φ direction.

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[1] J. A. Wheeler, *Geometrodynamics* (Academic, New York, 1962).

[2] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, (to be published). For alternative approaches, see, e.g., L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1962), E. Lubkin, Rosen Festschrift (1968), A. Trautman, in *Gravitation*, edited by L. Witten (Wiley, New York, 1962), L. Tamburino and J. Winicour, Phys. Rev. **150**, 1039 (1966).

[3] J. M. Cohen, J. Math. Phys. **9**, 905 (1968). Note that for an axisymmetric space-time the definition of angular momentum used in this reference is equivalent to the one given here. In Footnotes 5 and 9 a coordinate independent procedure is given.

[4] A. Komar, Phys. Rev. **127**, 1411 (1962). For a discussion of Komar's approach to the definition of energy, see C. W. Misner, Phys. Rev. **130**, 1590.

[5] J. M. Cohen, Rosen Festschrift (1968), notes published in *Relativity and Gravitation*, edited by G. Kuper and A. Peres (Gordon and Breach, New York, 1971).

[6] D. Brill and J. Hartle, Phys. Rev. **135B**, 271 (1964).