Gravitational Collapse of Rotating Bodies

JEFFREY M. COHEN
Institute for Space Studies, Goddard Space Flight Center, NASA, New York, New York
(Received 15 December 1967)

We investigate the effect of gravitational collapse of rotating bodies on the induced rotation of inertial frames. In particular, it is shown that the angular velocity of the inertial frames, within an adiabatically collapsing, slowly rotating mass shell supported by elastic stresses, approaches that of the shell as the shell radius approaches the Schwarzschild radius. Even when this relative angular velocity approaches zero, the angular momentum (a conserved quantity) does not vanish; it remains constant during the collapse. On the other hand, an observer within a slowly rotating dust shell (at the point of maximum expansion) does not see the angular velocity of the inertial frames approach that of the shell as the radius approaches the Schwarzschild radius. This difference between the two situations is shown to be in accordance with Mach's principle. The effect of rotation on gravitational collapse is also considered. This is done to shed some light on an important question in astrophysics: Does rotation stop collapse, or does collapse crush rotation? A spherical shell of dust supported by "centrifrigal forces" is considered. It is shown that rotation cannot stop collapse unless the shell radius is equal to or larger than (9/8)×(Schwarzschild radius). This happens even though the velocity of the particles in the shell is allowed to approach that of light.

I. INTRODUCTION

According to Einstein,[1] Mach's principle implies that a "rotating hollow body must generate inside of itself a 'Coriolis field,' which deflects moving bodies in the sense of a rotation, . . ." A weak effect of this sort was found in 1918 by Thirring,[2] who showed that a slowly rotating mass shell (producing a weak gravitational field) drags along the inertial frames within it. Because of the approximations used, Thirring's result is valid only when the induced rotation is small compared to the rotation rate of the shell.

Subsequent to Thirring's work, various authors[3,4] have stressed the importance of obtaining a strong-field solution in order to see how Mach's principle enters into general relativity. In particular, Dicke[3] has argued that, if Mach's principle is contained in general relativity, there should be a limit in which the angular velocity of the inertial frames within the shell approaches that of the shell.

Recently, it has been shown[5,6] that such a limit does exist in general relativity. Within a slowly rotating mass shell whose geometric radius approaches its gravitational radius, the induced rotation of the inertial frames approaches that of the shell. The elastic stresses, necessary to keep the shell from collapsing, become large as the shell radius approaches the gravitational radius.

Unfortunately, from the above results it was not clear *why* the induced rotation of the inertial frames approaches that of the shell. Thus, it seemed reasonable to ask: Is the effect only a consequence of the nonphysical mass distribution of the thin shell? The answer is no. Such induced rotation effects manifest themselves also for more physical configurations of matter, e.g., a slowly rotating sphere of perfect fluid. The induced rotation of the inertial frames is exhibited throughout the interior of the fluid sphere,[7] as well as exterior to the sphere where the metric is the same as that for the spherical shell. When the geometric radius of the sphere approaches (9/8)×(Schwarzschild radius), the induced rotation rate at the center approaches that of the fluid.

Unfortunately, the problem is not yet completely solved since, at the center of the fluid sphere, both the pressure and red shift (from the center to an observer at infinity) become large as the radius approaches (9/8)×(Schwarzschild radius). In Sec. III, we consider a situation in which the pressure does not become large but the red shift does. The adiabatic gravitational collapse[8] of rotating configurations is investigated in Sec. II in order to examine the effects of collapse on rotation.

These induced rotation effects have raised questions of importance in astrophysics.[9] Does rotation stop gravitational collapse or does collapse crush rotation? In any event, angular momentum,[10] a conserved quantity, cannot be crushed. Since in the strong-field case, the gravitational field makes a non-negligible contribution to the angular momentum, the "angular momentum of a body" is not a well-defined concept in general relativity as it is in Newtonian mechanics.

---

[1] A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, N. J., 1956).
[2] H. Thirring, Physik. Z. 19, 33 (1918).
[3] R. H. Dicke, Am. J. Sci. 47, 25 (1959).
[4] J. A. Wheeler, *Relativity, Groups, and Topology*, edited by B. S. DeWitt and C. M. DeWitt (Gordon and Breach, Science Publishers, Inc., New York, 1964).
[5] J. M. Cohen, *Lectures in Applied Mathematics*, edited by J. Ehlers (American Mathematical Society, Providence, R. I., 1967), Vol. 8.
[6] D. Brill and J. Cohen, Phys. Rev. 143, 1011 (1966).
[7] J. Cohen, Middlestates Relativity Seminar, 1966 (unpublished); J. Cohen and D. Brill, Nuovo Cimento (to be published).
[8] For a discussion of the adiabatic collapse of nonrotating spheres, see, e.g., H. Bondi, *1964 Lectures on General Relativity* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1965).
[9] A. G. W. Camerson (private communication); see, e.g., F. Hoyle, W. A. Fowler, G. R. Burbridge, and E. M. Burbridge, Astrophys. J. 139, 909 (1964).
[10] See, e.g., J. M. Cohen, J. Math. Phys. 8, 1477 (1967); 9, 905 (1968), and the references cited therein.

Because of this, conservation of momentum does not *a priori* imply that rotation stops collapse or vice versa. This question is discussed in Sec. V.

II. ADIABATIC GRAVITATIONAL COLLAPSE OF ROTATING ELASTIC MASS SHELL

A noncollapsing slowly rotating mass shell supported by elastic stresses drags along the inertial frames within it. The induced rotation rate of these inertial frames approaches that of the shell when the gravitational radius r_{grav} approaches the shell radius r_0.[1,2] If a similar limit exists for a gravitationally collapsing shell, then collapse will crush the rotation[9] as $r_{\text{grav}} \to r_0$.

On the other hand, in Newtonian mechanics rotation often stops collapse because "centrifugal forces" build up since angular momentum is conserved. For an axially symmetrical system in general relativity, there is a Killing vector ξ_μ which generates an isometry group. Corresponding to this Killing vector is a conserved quantity, angular momentum, which reduces to the Newtonian expression in the weak-field limit.[5,6] In view of the existence of this conserved angular momentum in general relativity, it seems reasonable to ask if rotation will stop collapse in general relativity. To get some insight into this question, the adiabatic collapse[8] of a slowly rotating elastic mass shell is treated in this section. The elastic stress in this shell would be sufficient to maintain neutral equilibrium if the shell were not collapsing.[8] Other situations are treated later.

The metric associated with a slowly rotating shell which collapses adiabatically is the same as that for a noncollapsing shell supported by elastic stresses except for the time dependence[5,6]:

$$ds^2 = -V^2 dt^2 + \psi^4[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta (d\varphi - \Omega dt)^2], \quad (1)$$

where

$$\begin{aligned} V &= V_0 = (r_0 - \alpha)/(r_0 + \alpha), \\ \psi &= \psi_0 = 1 + \alpha r_0^{-1}, \\ \Omega &= \Omega_0, \quad \text{for} \quad r < r_0; \end{aligned} \quad (2)$$

$$\begin{aligned} V &= (r - \alpha)/(r + \alpha), \\ \psi &= 1 + \alpha r^{-1}, \\ \Omega &= (r_0 \psi_0^2 / r \psi^2)^3 \Omega_0, \quad \text{for} \quad r > r_0. \end{aligned} \quad (3)$$

Here,

$$\Omega_0 = \omega_s / \{1 + [3(r_0 - \alpha)/8\alpha(1 + \beta_0)]\}, \quad (4)$$

$$\beta_0 = \alpha / 2(r_0 - \alpha), \quad (5)$$

2α is the mass of the shell m, and ω_s is the angular velocity of the shell. Note that all quantities with subscript 0 change slowly with time as the shell collapses.

There is an important difference between the collapsing and noncollapsing cases. In the noncollapsing case, the angular momentum J is just a quantity associated with the rotating body. Once r_0 and ω_s are given, J is determined via

$$J = \tfrac{2}{3} m (1 + \beta_0) r_0^2 \psi_0^5 (\omega_s - \Omega_0) / V_0. \quad (6)$$

However, in the collapsing case, the requirement that angular momentum be conserved gives a relation (6) between the radius and angular velocity of the body which must be satisfied during the entire collapse. This relation (6) is completely determined once the initial values are given.

Substitution of Eq. (4) into Eq. (6) yields

$$2J = (r_0 \psi_0^2)^3 \Omega_0. \quad (7)$$

From Eq. (4), it can be seen that $\Omega_0 \to \omega_s$ as $r_0 \to \alpha$, i.e., the angular velocity of the shell relative to the local inertial frames within it approaches zero as the shell radius approaches the Schwarzschild radius. On the other hand, the angular momentum approaches

$$J = \tfrac{1}{2} (4\alpha)^3 \omega_s, \quad (8)$$

a nonzero value. Thus, the adiabatic gravitational collapse of a slowly rotating mass shell crushes the rotation but leaves the angular momentum unchanged.

This may seem to be a paradoxical result if one asks the question: How can collapse crush rotation but not angular momentum? However, from the point of view of the various observers, nothing strange takes place. For example, an inertial observer at infinity sees a rotating shell generating angular momentum. He measures this angular momentum via the metric in his asymptotically flat region using the relation[10]

$$8\pi J = \int_{\partial\Sigma} \xi_\mu (P^{\mu\nu} - \eta^{\mu\nu} P) d\sigma_\nu. \quad (9)$$

Here, $P^{\mu\nu}$ is the second fundamental form, $\partial\Sigma$ denotes the two-dimensional boundary of the spacelike surface, $d\sigma_\nu$ is a two-dimensional area element of $\partial\Sigma$, and ξ_μ is a Killing vector associated with the axial symmetry.

Meanwhile, an inertial observer within the shell sees a shell whose angular velocity approaches zero as the shell radius approaches its gravitational radius. He cannot measure the angular momentum of the shell unless he communicates with another observer outside the shell.

At this point the reader may be wondering about the observer just outside the shell. If the observer can build a closed surface around the rotating shell, he can measure a nonzero angular momentum. Thus, if the angular velocity of his local inertial frames relative to the shell vanishes everywhere, we indeed have a paradox.

What should we expect from Mach's principle? An observer inside the shell sees each element of mass in the shell moving in the same direction. Thus we might expect from Mach's principle that every element of mass in the shell will drag along the inertial frames within the shell in the same direction. The same is true along the rotation axes even outside the shell.

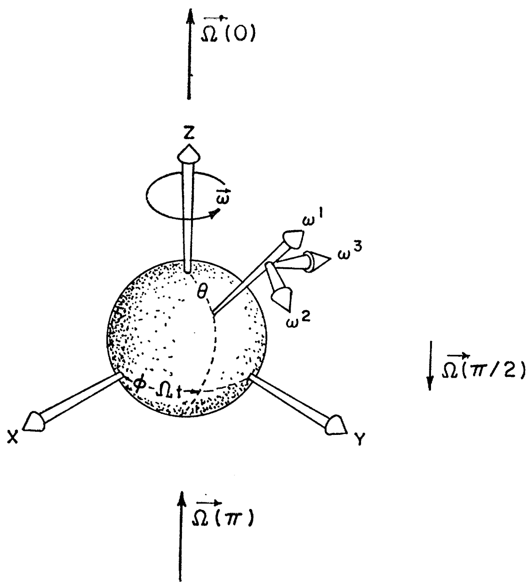

FIG. 1. Direction of induced-rotation rate $\mathbf{\Omega}$ of inertial frames outside of slowly rotating spherical mass. The vectors ω^1, ω^2, ω^3 denote orthonormal Cartan frames and $\boldsymbol{\omega}$ denotes the angular velocity of the rotating body.

Thus, one might expect the angular velocity of the inertial frames within the shell to be the same as those just outside the shell, along the rotation axes.

At other points outside the shell, the situation is more complicated. For example, an observer in the equatorial plane does *not* see all elements of the shell rotating in the same direction. Relative to him, the closest elements of the shell rotate in a direction opposite to the rotation direction of the shell. If we assume that the effect on inertial frames of an element of the shell falls off with distance, then we might expect the angular velocity of this observer's inertial frames to be opposite in direction from that of the frames within the shell. The direction of the rotation of inertial frames outside the shell, relative to the shell, is shown in Fig. 1.

To see if such Machian effects are actually contained in general relativity, we compute the angular velocity of the inertial frames.[11] Inside the shell, the angular velocity of the inertial frames is given by

$$V\mathbf{\Omega}=\Omega_0(\cos\theta\,\omega^1-\sin\theta\,\omega^2)\,. \qquad (10)$$

Thus an observer at infinity sees the inertial frames (within the shell) rotating with angular velocity Ω_0 about the rotation axis of the shell. Outside the shell, the expression for the angular velocity of the inertial frames becomes[12]

$$V\mathbf{\Omega}=\Omega[\cos\theta\,\omega^1+(1/2V)\sin\theta\,\omega^2]\,. \qquad (11)$$

Along the axis of rotation ($\theta=0$ or π), the magnitude of the angular velocity of the inertial frames is Ω and its direction is the same as the angular velocity vector of the shell's rotation. Along this axis and just outside the shell, the magnitude and direction of the angular velocity (of the inertial frames) is the same as that within the shell. In the equatorial plane ($\theta=\frac{1}{2}\pi$), the angular velocity of the inertial frames is opposite in direction from the shell's angular velocity.[13] These results are in agreement with the above conjecture based on Mach's principle.

Returning now to the question of the angular momentum, we note that the angular velocity of the inertial frames just outside the shell relative to the shell does not vanish everywhere. Consequently, it is not surprising that the angular momentum does not vanish. As mentioned in Sec. I, the gravitational field also carries angular momentum.

III. COLLAPSE OF A ROTATING SHELL OF DUST

Consider a body momentarily not expanding or collapsing but rotating. If the initial-value equations[5]

---

[11] The calculations can be carried out using Gödel's formula or that of L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison Wesley Publishing Co., Inc., Reading, Mass., 1962). Gödel's formula expressed in coordinate-free notation is

$$\mathbf{\Omega}=(\tfrac{1}{2})^*(dU\wedge U).$$

Here U is the four-velocity of a test particle on which the observer is sitting, dU is the exterior derivative of U, $\wedge$ denotes the exterior product, and * denotes the duality operation. $\mathbf{\Omega}$ is the angular velocity of the inertial frames relative to an observer sitting on the test particle. An observer at infinity will see this angular velocity red-shifted. For a discussion of this formula see, e.g., L. C. Shepley, thesis, Princeton University, 1965 (unpublished). For an introduction to exterior calculus see, e.g., D. Brill and J. Cohen, J. Math. Phys. **7**, 238 (1966); C. W. Misner and J. A. Wheeler, Ann. Phys. **2**, 525 (1957).

The formula of Landau and Lifshitz was derived for a stationary metric. In the notation of exterior calculation this formula takes the form

$$\mathbf{\Omega}=\tfrac{1}{2}(-g_{00})^{1/2}{}^*d(g_\alpha dx^\alpha),$$

where

$$g_\alpha=-g_{0\alpha}/g_{00}.$$

For the axially symmetric stationary metric

$$ds^2=-A^2dt^2+B^2dr^2+C^2d\theta^2+E^2(d\theta-\Omega dt)^2,$$

where the metric coefficients are functions of r and θ only, both formulas for $\mathbf{\Omega}$ give the same result

$$\mathbf{\Omega}=(Af^2/2CE)(E^2\Omega/A^2f^2)_\theta\omega^1-(Af^2/2BE)(E^2\Omega/A^2f^2)_r\omega^2,$$

where $f^2=1-E^2\Omega^2A^{-2}$ and ω^1 and ω^2 are unit vectors along the dr and $d\theta$ directions, respectively. These vectors are shown in Fig. 1.

[12] In the weak-field limit, this result (11) agrees with that of R. H. Boyer and T. G. Price, Proc. Cambridge Phil. Soc. **61**, 531 (1965). Their result (11) differs in sign from that of L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962).

[13] Since the induced rotation of inertial frames is a Machian effect, this result helps to explain, in terms of Mach's principle, the results of Boyer and Price, Ref. 12, and of J. Lense and H. Thirring, Phys. Z. **19**, 156 (1918). These authors note that, in the case of a satellite orbiting a rotating central body in its equatorial plane, the advance of the perihelion of the satellite orbit is decreased if both rotations (rotational and orbital) have the same sense. We see that this is caused by a retrograde rotation of the inertial frames (in the equatorial plane) induced by the rotating central body.

can be solved at this instant, then the solution can be continued out of the initial spacelike surface via the other six Einstein equations. In this way the collapse can be followed. Here we will consider only the initial-value problem.[5]

At this instant, the stationary metric associated with a rotating spherical shell of dust is very similar to that of the previous section.[5,6] The only difference is that the *pressure vanishes* (i.e., $\beta_0=0$) *for the case of dust.* Elsewhere,[5,6] it is shown how the pressure enters the initial-value equations and the full set of Einstein equations via the stress-energy tensor. Of course, after this instant the dust will not collapse adiabatically.[14]

To an observer at infinity, the momentarily stationary metric associated with a shell of dust is very much like an adiabatically collapsing shell. In each case, for $r\rightarrow\alpha$, the angular velocity of the shell (relative to the local inertial frames within it) approaches zero, while the angular momentum J does not.

On the other hand, an inertial observer inside the dust shell sees something different. For him the *angular velocity of the shell relative to his local inertial frames* approaches $\frac{3}{4}\Omega_0$ as the shell radius approaches the gravitational radius. This follows directly from Eq. (4), with $\beta_0=0$, if one keeps in mind the relationship between the proper time, τ, of an observer within the shell and the proper time, t, in the asymptotically flat region far from the shell, i.e., $d\tau=V_0dt$. Denoting the angular velocity seen by an observer within the shell by barred quantities, one can reduce Eq. (4) to

$$V_0(\bar{\omega}_s-\bar{\Omega}_0)=3(r_0-\alpha)\Omega_0/8\alpha.$$

In the limit as r_0 approaches α, this reduces to

$$\bar{\omega}_s-\bar{\Omega}_0=\tfrac{3}{4}\Omega_0.$$

Thus, in this limit, some observers note that the angular velocity of the dust shell relative to the local inertial frames within the shell does not approach zero.

These results raise a number of new questions: Why should the observers inside and outside the adiabatically collapsing mass shell both see perfect dragging along of the inertial frames within the shell, while only the observer outside the dust shell sees perfect dragging? Are these results in accordance with Mach's principle?[15] These questions will be discussed in the Sec. IV.

IV. DISCUSSION OF MACHIAN EFFECTS

Far from the rotating source, the rotating metric (1) approaches the Schwarzschild metric. If the Schwarzschild metric in this region is patched to a positive-curvature Friedmann metric,[16] we obtain a closed universe. The rotating source can than be interpreted as one of the many particles generating the Friedmann geometry. From Mach's principle, one expects the distribution of stress energy in the universe to influence the motion of inertial frames. If the stress energy of one body in the universe becomes large compared to that of the rest of the universe, an observer would not be surprised to find that this had the predominant effect on the local inertial frames in its vicinity.

Such an effect indeed manifests itself as the shell radius approaches the Schwarzschild radius. The elastic stress in the shell becomes very large, representing more nearly all the stress energy in the universe than that of the shell considered by Thirring.[2]

The stress energy of the dust shell does not become large compared to that of the rest of the universe as $r_0\rightarrow\alpha$. The observer within the dust shell, consequently, sees nothing strange in this limit. The shell rotates relative to the local inertial frames within it. However, the relative rotation is less than it would be if the shell did not drag along the inertial frames.

The observer far from the source notes that the stress energy of the shell is sufficient to make its gravitational radius approach its geometrical radius. Consequently, this observer is not surprised to see the angular velocity of the dust shell relative to the inertial frames within it approach zero as $r_0\rightarrow\alpha$.

On the other hand, the rotating shell represents only one of the many small dust particles generating the Friedmann geometry. Consequently, the rotating source has a negligible effect on the inertial frames in the Friedmann region. This is in accord with Mach's idea that the inertial properties of space are influenced by the mass distribution.

The situation with the angular momentum is also Machian. If a body rotates relative to the other bodies in the universe, Mach's principle implies that one cannot tell whether the body is rotating and the other masses are stationary, or if the body is nonrotating and the other masses are revolving about it.

A body which rotates relative to the other masses in the universe induces a rotation of the inertial frames relative to the other masses in the universe. Consequently, the other masses in the universe rotate in the opposite direction as the rotating source relative to the inertial frames. One might think that the question of which mass is actually rotating might be resolved by measuring the angular momentum. However, such a procedure is not as straightforward as it might seem, since there is not only a contribution from the rotating mass but also an opposite contribution from the other masses in the universe (because they rotate in the opposite direction relative to the inertial frames). This latter rotation rate is small while the mass is large.

[14] For a discussion of the collapse of a nonrotating shell of dust see W. Isreal, Nuovo Cimento 44, 1 (1966).

[15] E. Mach, *The Science of Mechanics* (Open Court Publishing Co., La Salle, Ind., 1902).

[16] See, e.g., J. M. Cohen, Intern. J. Theoret. Phys. (to be published), and references cited therein, for the details of patching the Schwarzschild metric to the Friedmann metric.

If these results are to be in accordance with Mach's principle, only relative rotation must be measurable. Thus, Mach's principle implies that both contributions to the angular momentum cancel. Then one cannot tell which mass is "actually" rotating.

Such an integral form of Mach's principle exists for closed universes in general relativity. Using the generalized form of Stokes's theorem,[10]

$$\int_{\partial\sigma} \omega = \int_{\sigma} d\omega\,, \tag{12}$$

one can show that the angular momentum J, defined on any closed simply connected spacelike surface, vanishes since the manifold has no boundary. Thus, in a closed universe, only relative rotation can be measured.

V. DOES ROTATION STOP COLLAPSE?

In order to shed some light on this question, we will consider a spherical shell of dust. Each dust particle moves in a circular orbit in the gravitational field of all the other particles. If a large number of particles with randomly oriented orbits is considered, the mass distribution is spherical and the resultant angular momentum vanishes. The angular velocity of an individual particle is limited only by the requirement that the particle's world line remain in the forward light cone. Because the total angular momentum vanishes, the inertial frames are not dragged along as they are by rotating bodies. Thus, since the rotation of the particles relative to the inertial frames cannot be stopped, we have a situation in which we have isolated the problem: Does rotation stop collapse?

The stress-energy tensor for particles of rest mass m_0 and density n_0 in the rest system is

$$T^{\mu\nu} = m_0 n_0 U^\mu U^\nu\,, \tag{13}$$

where U^μ is the four-velocity. If all the particles move in spherical orbits (with the same center and radius r_0) which are oriented randomly relative to each other, the nonvanishing components of the stress-energy tensor are related via

$$T^{22} = T^{33} = \tfrac{1}{2}(r_0\psi_0{}^2)^2\bar{\omega}^2 T^{00}. \tag{14}$$

Here T^{00} is the energy density (including both rest energy and kinetic energy) and $\bar{\omega}$ is the angular velocity of the particles.[17] The factor $\frac{1}{2}$ arises because all particles contribute equally to T^{00} but not to T^{22} and T^{33}. A particle in the xy plane contributes to T^{33} but not to T^{22}, while one in the xz plane contributes to T^{22} and not to T^{33}; particles in orbits with other orientations can contribute to both components, the magnitude of the contribution depending on the orientation of the orbit. When all contributions are added, we obtain the factor of $\frac{1}{2}$. The T^{0i} terms vanish since the average velocity of the particles vanishes. The T^{22} and T^{33} terms do not vanish, since they involve the square of the velocity.

In Newtonian mechanics, one finds that the stronger the gravitational attraction, the greater must be the angular velocity if the "centrifugal force" is to balance the gravitational force. A similar condition can be obtained in general relativity by substituting the stress-energy tensor into Einstein's field equations and requiring that the self-consistent solution be static. In general relativity, however, there is an important difference. The absolute value of the dust-particle velocity cannot exceed that of light. This leads to a constraint on the stress-energy tensor,

$$T^{22} = T^{33} \leq \tfrac{1}{2} T^{00}. \tag{15}$$

Such a constraint on the "centrifugal force" does not exist in Newtonian mechanics.

The problem is most easily solved by reducing it to another problem which has already been solved,[6] a shell of matter supported by elastic stresses in the shell [the metric is the same as that of Eq. (1) but is not time-dependent]. Without loss of generality let

$$\beta_0 = \tfrac{1}{2}(r_0\psi_0{}^2)^2\bar{\omega}^2\,; \tag{16}$$

then the stress-energy tensor (13) takes the form

$$\begin{aligned} T^{00} &= \rho\,, \\ T^{22} = T^{33} &= \rho\beta_0\,, \end{aligned} \tag{17}$$

the same as in the previous problem.[5,6] One cannot tell from the self-consistent metric or stress-energy tensor whether the shell is supported by elastic stress or by "centrifugal forces."

Thus, β_0 is given by Eq. (5). Equating Eqs. (5) and (16), the two expressions for β_0, yields

$$\alpha = (r_0 - \alpha)(r_0\psi_0{}^2)^2\bar{\omega}^2\,, \tag{18}$$

a relation between the angular velocity of the dust particles and the radius of the dust shell—a modified "Kepler's law." Because there is a *limit on the velocity of the dust particles*, the stresses which the particles can produce are constrained by Eq. (15), or by $\beta_0 \leq \frac{1}{2}$. The components of the stress-energy tensor can satisfy this relation for

$$r_0 \geq 2\alpha\,. \tag{19}$$

If we redefine our radial coordinate in such a way that the radius of the shell is $(2\pi)^{-1}\times$(circumference of a great circle on the shell), i.e., let $R_0 = r_0\psi_0{}^2$, the relation

[17] For a discussion of similar problems see A. Einstein, Ann. Math., **40**, No. 4, 922 (1939).

(19) becomes[18]

$$R_0 \geq (9/8) \times (\text{Schwarzschild radius}). \tag{20}$$

When the shell radius is less than this value, the dust particles cannot generate a large enough stress to maintain equilibrium[19]; the gravitational attraction overpowers the centrifugal force and the body collapses.

Although the "centrifugal force" increases as the particle velocity increases, the kinetic energy increases also, thereby increasing the gravitational attraction (i.e., the kinetic energy contributes to the Schwarzschild mass). This, plus the requirement that the local velocity of the particles cannot exceed that of light, means that the body must have a radius equal to or greater than $(9/8)\times$(Schwarzschild radius) if it is not to collapse.

VI. CONCLUSIONS

The angular velocity of inertial frames within an adiabatically collapsing and slowly rotating mass shell approaches the angular velocity of the shell as the shell's radius approaches its Schwarzschild radius. All observers come to this conclusion, since the elastic stress (necessary to maintain adiabatic collapse) becomes large compared to the stress energy of other bodies in the universe.

On the other hand, a slowly rotating shell of dust is not supported by elastic stress. Consequently, its stress energy never becomes large compared to the other masses in the universe, and the observers within the dust shell do not see the angular velocity of the inertial frames within the shell approach that of the shell. Such effects occur not only when one considers thin shells, but also manifest themselves when one treats, e.g., slowly rotating spheres of perfect fluid. For a closed universe, at least, these effects are found to be in accordance with Mach's principle.

An important question in astrophysics is: Does rotation stop gravitational collapse or does collapse crush rotation? To shed some light on this question we have considered a noncollapsing shell of dust particles supported only by "centrifugal forces." The shell is held together by the self-consistent gravitational field of all of the particles which move in randomly oriented circular orbits.

Collisions were assumed to be negligible, and all particles to have the same mass and move at the same velocity. It was found that if the radius of the shell exceeds $(9/8)\times$(Schwarzschild radius), rotation can stop collapse; but for a shell radius between this value and the Schwarzschild radius, the *rotation cannot stop collapse*. Two reasons for this are: (1) The particle world lines must remain within the future light cone; (2) large particle velocities imply large "kinetic energies" as well as large "centrifugal forces." The "kinetic energy" contributes to the Schwarzschild mass; thus, the gravitational attraction is greater than it would be for stationary particles.

ACKNOWLEDGMENTS

For helpful discussion I am indebted to Dr. R. H. Boyer, Dr. D. R. Brill, Dr. A. G. W. Cameron, Dr. R. O'Connell, W. J. Sarill, and Dr. S. Vila. I should also like to thank Dr. John C. Moench for his helpful advice during the writing of this paper. Part of this research was done while the author held a NAS-NRC Research Associateship at the NASA Institute for Space Studies, New York.

[18] This quantity, $(9/8)\times$(Schwarzschild radius), appears in other problems in general relativity; e.g., the radius of a static sphere of perfect fluid must be equal to or greater than this value, while the equilibrium radius of a spherical electromagnetic or gravitational geon is equal to this value [J. A. Wheeler, Phys. Rev. **97**, 511 (1955); D. Brill and J. Hartle, *ibid.* **135**, B271 (1964)]. The connection, if any, between the factor 9/8 of this paper and that for a fluid sphere is not known, but the results of this paper and those for the geon are closely connected. As the shell radius approaches $(9/8)\times$(Schwarzschild radius), the velocity of the particles in the shell approaches that of light, whereas the above geons contain zero-rest-mass particles moving at the velocity of light. Thus, the results agree in this limit as they should.

It may be of interest to note that if a photon leaves the vicinity of the surface of a body with this radius and travels radially to infinity, it experiences a red shift of 2. This is a common red shift observed in quasars. If the photon is emitted by one of the rapidly revolving particles, the red shift will be even larger; however, if it is emitted during a collision of two particles, the red shift may be about 2. For a discussion of the collision problems for two stars, see, e.g., A. G. W. Cameron and F. Seidel (to be published).

[19] The results obtained here differ from those of Einstein (Ref. 17) because a different velocity distribution is used here. We consider a thin shell containing particles moving at the same velocity, whereas Einstein considered a shell containing particles whose velocity increases with radius. (The results differ since the problems differ.) Because of this, the minimum radius obtained here is less than that obtained by Einstein. However, the minimum radius in both cases is greater than the Schwarzschild radius. Thus Einstein's discussion and conclusions are still valid.