Kerr Geometry as Space-Time Structure of the Dirac Electron

A.Burinskii[1][†],

(1) *NSI Russian Academy of Sciences*

Abstract

The combined Dirac-Kerr model of electron is suggested, in which electron has extended space-time structure of Kerr geometry, and the Dirac equation plays the role of a master equation controlling polarization of the Kerr congruence. The source contains a spinning disk bounded by a closed singular string of Compton size. It is conjectured that this Compton structure may be observed for polarized electrons under a very soft coherent scattering.

Introduction. The Kerr-Newman solution has gyromagnetic ratio $g = 2$, as that of the Dirac electron (Carter 1968) and represents a classical model of extended electron in general relativity (for references see [1,2,3,4]). There appears a natural question, what is the relation between the Dirac equation and the Kerr-Newman solution?

The answer is related with the problem of coordinate description of the Dirac electron which cannot be localized inside the Compton region. Absence of the clear space-time description prevents the consistent incorporation of gravity. Similar, in the multi-particle QED theory, the "dressed" electron is smeared over the Compton region. However, its coordinate description is again very obscure and main results are obtained by calculations in the momentum space. As a result, there appears some extreme point of view that the subsequent relativistic theory has to refuse from the wave function in coordinate representation at all [5]. These facts hinder from natural incorporation of gravity. On the other hand, ignorance of gravity in the Dirac theory and QED is justified by weeakness of the gravitational field of electron. However, electron has the extremely large spin/mass ratio (about 10^{44} in the units $\hbar = c = G = 1$,) which shows that gravitational effects have to be estimated on the base of the Kerr-Newman solution. The extremely high spin leads to the very strong polarization of space-time and to the corresponding very strong deformation of electromagnetic (em-) field which has to be aligned with the Kerr congruence. However, the *em-field of electron cannot be considered as small*, and the resulting influence turns out to be very essential! In particular, the em-field turns out to be singular at the Kerr ring which is a closed string of the Compton size $a = J/m$. The space-time acquires two folds with a branch line along the Kerr string and represents a very non-trivial background for electromagnetic processes.

In this work we obtain an exact correspondence between the wave function of the Dirac equation and the spinor (twistorial) structure of the Kerr geometry. It allows us to assume that the Kerr-Newman geometry reflects the specific space-time structure of electron, and electron contains really the Kerr-Newman circular string of Compton size. We suggest a combined Dirac-Kerr model of an electron, in which electron acquires the coordinate description of the Kerr geometry, while the Dirac equation plays the role of a *master equation* which controls the position and dynamics of the Kerr string and the

related twistorial polarization of the Kerr space-time. Dynamics of this combined Dirac-Kerr model in the external electromagnetic fields turns out to be indistinguishable from the behavior of the Dirac electron.

Real structure of the Kerr geometry Angular momentum of electron $J = \hbar/2$ is extremely high with respect to the mass, and the black hole *horizons disappear* opening the naked Kerr singular ring which represents a closed string [3], excitations of which generate spin and mass of the extended particle-like object - "microgeon" [1]. Singular ring may be regularized by Higgs field. If the Kerr string acquire tension T, $m = E = Ta$, the Kerr relation $J = ma$ yields the Regge behavior $J = \frac{1}{T}m^2$.

The Kerr principal null congruence is a twisted family of the lightlike rays – twistors. Frame of the Kerr geometry is formed by null vector field $k^\mu(x)$ which is tangent to the Kerr congruence. The Kerr-Schild form of metric is

$$g^{\mu\nu} = \eta^{\mu\nu} + 2H(x)k^\mu k^n, \tag{1}$$

where $\eta^{\mu\nu}$ is metric of an auxiliary Minkowski space-time M^4 and H is a real function, $x^\mu = (t, x, y, z)$.

Vector potential of the Kerr-Newman solution is aligned with this congruence

$$A_\mu = \mathcal{A}(x)k_\mu, \tag{2}$$

and the Kerr singular ring represents its caustic, see Fig. 1. **The Kerr theorem** determines the Kerr congruence via a holomorphic surface in the projective twistor space which has coordinates

$$Y, \quad \lambda_1 = \zeta - Yv, \quad \lambda_2 = u + Y\bar\zeta, \tag{3}$$

where $2^{\frac{1}{2}}\zeta = x + iy$, $2^{\frac{1}{2}}\bar\zeta = x - iy$, $2^{\frac{1}{2}}u = z - t$, $2^{\frac{1}{2}}v = z + t$ are the null Cartesian coordinates. Such congruences lead to solutions of the Einstein-Maxwell field equations with metric (1) and em-field in the form (2). Congruence of the Kerr solution is built of the straight null generators, twistors, which are (twisting) geodesic lines of photons. Therefore, for any holomorphic function F, the solution $Y(x^\mu)$ of the equation $F(Y, \lambda_1, \lambda_2) = 0$ determines congruence of null lines by the 1-form

$$e^3 = du + \bar{Y}d\zeta + Yd\bar\zeta - Y\bar{Y}dv. \tag{4}$$

The null vector field $k_\mu dx^\mu = P^{-1}e^3$ up to a normalizing factor P. Coordinate Y is a projective spinor $Y = \phi_2/\phi_1$, and in spinor form $k_\mu = \bar\phi_{\dot\alpha}\bar\sigma_\mu^{\dot\alpha\alpha}\phi_\alpha$.

Figure 1. The Kerr singular ring and congruence.

Complex representation of Kerr geometry. Complex source of Kerr geometry is obtained as a result of complex shift of the 'point-like' source of the Schwarzschild solution written in the Kerr-Schild form. Applying the complex shift $(x, y, z) \to (x, y, z + ia)$ to the singular source $(x_0, y_0, z_0) = (0, 0, 0)$ of the Coulomb solution q/r, Appel (in 1887 !) obtained the solution $\phi(x, y, z) = \Re e\, q/\tilde{r}$, where $\tilde{r} = \sqrt{x^2 + y^2 + (z - ia)^2}$ turns out to be complex. On the real slice (x, y, z), this solution acquires a singular ring corresponding to $\tilde{r} = 0$. It has radius a and lies in the plane $z = 0$. The solution is conveniently described in the oblate spheroidal coordinate system r, θ, where $\tilde{r} = r + ia\cos\theta$. The resulting real space is twofold having positive sheet $r > 0$, and negative one $r < 0$.
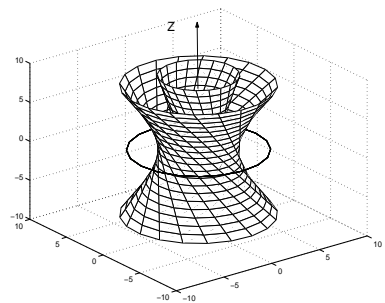
The Appel potential corresponds exactly to electromagnetic field of the Kerr-Newman solution written in the Kerr-Schild form [1]. The vector of complex shift $\vec{a}$ shows angular momentum of the Kerr solution $\vec{J} = m\vec{a}$. Newman and Lind suggested a description of the Kerr-Newman geometry in the form of a retarded-time construction, where its source is generated by a complex point-like source, propagating along a *complex world line* $X^\mu(\tau) \in CM^4$.

Figure 2. Singular ring and two singular half-strings.

In the rest frame of the Kerr particle, one can form two null 4-vectors $k_L = (1,0,0,1)$ and $k_R = (1,0,0,-1)$, and represent the 3-vector of complex shift $i\vec{a} = i\Im m X^\mu$ as the difference $i\vec{a} = \frac{ia}{2}\{k_L - k_R\}$. The straight complex world line corresponding to a free particle may be decomposed to the form $X^\mu(\tau) = X^\mu(0) + \tau u^\mu + \frac{ia}{2}\{k_L - k_R\}$, where the time-like 4-vector of velocity $u^\mu = (1,0,0,0)$ can also be represented via vectors k_L and k_R $u^\mu = \partial_t \Re e X^\mu(\tau) = \frac{1}{2}\{k_L + k_R\}$. One can form two complex world lines related to the complex Kerr source,

$$X_+^\mu(t+ia) = \Re e X^\mu(\tau) + ia k_L^\mu, \quad X_-^\mu(t-ia) = \Re e X^\mu(\tau) - ia k_R^\mu, \qquad (5)$$

which allows us to match the Kerr geometry to the solutions of the Dirac equation.
Complex Kerr string. The complex world line $X^\mu(\tau)$ is parametrized by complex time $\tau = t + i\sigma$ and represents the world sheet of a very specific string extended along imaginary time parameter $\sigma \in [-a, a]$. The Kerr congruence, gravitational and em- fields are obtained from this stringy source by a retarded-time construction which is based on the complex null cones, emanated from the worldsheet of this complex string [3,4]. The complex retarded time equation $\tau = t - r + ia\cos\theta$ sets the relation $\sigma = a\cos\theta$ between the complex points $X^\mu(t,\sigma)$ and angular directions θ of the real twistor lines. One sees that this string is open with the end points $\cos\theta = \pm 1$ which correspond to $X_\pm^\mu = X^\mu(t \pm ia)$. By analogue with the real strings, the end points may be attached to quarks. The complex light cones adjoined to the end points have a *real slice in the form of two especial twistors* having the discussed above null directions k_L^μ and k_R^μ which determine momentum and spin-polarization of the Kerr solution. These twistors form two half-strings of opposite chirality, see Fig. 2.

Chirons and excitations of the Kerr singular ring. The twistor coordinate Y is also the projective angular coordinate $Y = e^{i\phi}\tan\theta$ covering the celestial sphere, $Y \in CP^1 = S^2$. The exact Kerr-Schild solutions have em-field which is determined by arbitrary analytical function $\mathcal{A}(Y)$, in particular $\mathcal{A} = eY^{-n}$. The simplest case $n = 0$ gives the Kerr-Newman solution. The case $n = 1$ leads to an axial singular line along the positive semi-axis z. Due to factor $e^{i\phi}$, em-field of this solution has winding number n=1 around axial singularity. Since there is also pole at singular ring, $\sim (r + ia\cos\theta)^{-1}$, the em-field has also a winding of phase along the Kerr ring. Solution with $n = -1$ has opposite chirality and singular line along the negative semi-axis z. These elementary exact solutions ('chirons') have also the wave generalizations $\mathcal{A} = eY^{-n}e^{i\omega\tau}$ acquiring the extra dependence from the complex retarded time τ. The wave chirons are asymptotically exact in the low-frequency limit and describe the waves propagating along the Kerr circular string and induced waves along axial half-strings [3,4]. By lorentz boost the axial half-strings acquire modulation by de Broglie periodicity [3,4].
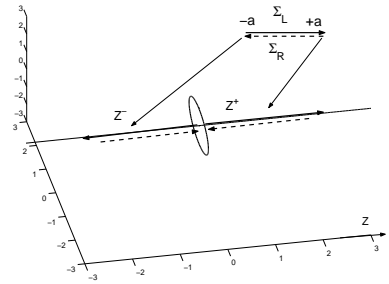
Dirac Equation in the Weyl Basis In the Weyl basis Dirac spinor has the form $\Psi = \begin{pmatrix} \phi_\alpha \\ \chi^{\dot\alpha} \end{pmatrix}$, and the Dirac equation splits into

$$\sigma^\mu_{\alpha\dot\alpha}(i\partial_\mu + eA_\mu)\chi^{\dot\alpha} = m\phi_\alpha, \quad \bar\sigma^{\mu\dot\alpha\alpha}(i\partial_\mu + eA_\mu)\phi_\alpha = m\chi^{\dot\alpha}. \tag{6}$$

The Dirac current $J_\mu = e(\bar\Psi\gamma_\mu\Psi) = e(\bar\chi\sigma_\mu\chi + \bar\phi\bar\sigma_\mu\phi)$, can be represented as a sum of two lightlike components of opposite chirality $J^\mu_L = e\bar\chi\sigma^\mu\chi$, $\quad J^\mu_R = e\bar\phi\bar\sigma^\mu\phi$. The corresponding null vectors

$$k^\mu_L = \bar\chi\sigma^\mu\chi, \quad k^\mu_R = \bar\phi\bar\sigma^\mu\phi, \tag{7}$$

determine the considered above directions of the lightlike half-strings. The momentum of the Dirac electron is $p^\mu = \frac{m}{2}(k^\mu_L + k^\mu_R)$, and the vector of polarization of electron in the state with a definite projection of spin on the axis of polarization is $n^\mu = \frac{1}{2}(k^\mu_L - k^\mu_R)$. In particular, in the rest frame and the axial z-symmetry, we have $k_L = (1, \vec{k}_L) = (1,0,0,1)$ and $k_R = (1, \vec{k}_R) = (1,0,0,-1)$, which gives $p^\mu = m(1,0,0,0)$, and $n^\mu = (0,0,0,1)$, which corresponds to transverse polarization of electron, $\vec{n}\vec{p} = 0$. The Dirac wave wave function sets also synchronization of the null tetrad in the surrounding space-time, playing the role of an 'order parameter'.

Dirac Equation as a Master Equation Controlling Twistorial Polarization. Em-field of the Kerr-Schild solutions $F_{\mu\nu}$ is to be aligned with the Kerr congruence, obeying the constraint $F_{\mu\nu}k^\mu = 0$. Therefore, twistorial structure of the Kerr-Schild solutions determines strong polarization of the em field. In particular, the elementary em-excitations on the Kerr background lead to the waves propagating along the Kerr circular string. Virtual photons are also concentrated near this string, forming its excitation. There is exact correspondence between two null vectors (7) obtained from the Dirac wave function and similar vectors k_L and k_R related to the ends of the complex Kerr string, Fig. 2. It allows us to unify the Dirac and Kerr structures, considering the Dirac equation as a master equation controlling twistorial polarization of the Kerr space-time.

Scattering. Contradiction between the discussed Compton size of electron and the results obtained for the deep inelastic scattering is seeming and has simple explanation. Relativistic boosts lead to asymmetry: $p_L << p_R$ or $p_L >> p_R$ which determines the sign of helicity. As a result one of the axial half-strings turns out to be strongly dominant. It allows one to use perturbative twistor-string model [6,7] which is based on a reduced description in terms of the lightlike momentum and helicity, and amplitude of scattering is determined *only by one* of the axial half-strings. We conjecture that *the Compton size of the Dirac-Kerr electron may be observed for polarized electrons under a very soft resonance scattering.*

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