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Singularities in Bootstrap Gravitational Geons*

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The solution to the empty-space, time-symmetric Einstein initial-value problem recently given by Komar as representing a bootstrap gravitational geon is shown to display singular behavior along portions of an axis in the regions in which the solution deviates from the standard Schwarzschild solution. This difficulty is in addition to the problems explicitly studied by Komar related to the jumps in the derivatives of the metric and seems to correspond more closely to the essential, delta-function type of singularity in the Schwarzschild solution at the origin. A direct analysis in terms of Cartesian coordinates seems to cast doubt on the likelihood that Komar's metric is even C^0 in any topologically trivial manifold.

I. INTRODUCTION

RECENTLY, Komar[1] has discussed the possibility of a solution to the initial-value Einstein equations, without explicit source term, corresponding to a topologically trivial, singularity-free initial surface for which the metric in an exterior region is precisely the Schwarzschild initial metric. Such solutions were named "bootstrap gravitational geons" by Komar since they could be interpreted as describing situations in which the gravitational radiation alone provides the source for the external Schwarzschild field, as electromagnetic energy does for Wheeler's geons.[2]

Previous studies of the Schwarzschild solution had seemed to indicate that the production of a nontrivial external field of precisely the Schwarzschild form requires at least one of the following conditions: (1) an explicit nongravitational energy-source term T_{ij} in the field equations; (2) a topologically nontrivial space in which certain closed surfaces are not the boundaries of volumes (wormholes); (3) an intrinsic singularity in the field. This is precisely the situation for the Maxwell electromagnetic field equations in which

$$\nabla\cdot\mathbf{E}=0 \tag{1}$$

implies that the total charge within any closed surface is zero unless either condition (2) or (3) above is satisfied.[3] Nevertheless, the well-known nonlinear, "self-contributing" aspects of the gravitational field distinguish it in many ways from the electromagnetic field. The existence of bootstrap gravitational geons would provide another important example.

However, the purpose of this paper is to point out that the solution described by Komar possesses anomalous differentiable structure of precisely the same sort as exhibited by the standard Schwarzschild solution at $r=0$. This is completely distinct from the "jumps" of the derivatives of the metric across certain spherical surfaces for which Komar was careful to show that no delta-function-type behavior in the Ricci tensor was generated. It would seem that such a demonstration would be even more necessary (and difficult) for the new singularity.

The existence of such additional problems is first discussed in Sec. II in terms of the second-order differential invariants of the Riemann tensor in the regions in which Komar's metric is apparently C^∞. A more direct analysis in Sec. III then shows that the components of Komar's metric cannot even be C^0 when expressed in the Cartesian coordinates naturally defined by the spherical coordinates in which Komar originally expressed his metric. There is, of course, still the possibility that Komar's coordinate system is simply not admissible in terms of differential structure of the manifold. However, this alternative still leaves unanswered many important questions.

II. INVARIANTS OF KOMAR'S SOLUTION

The initial surface metric described by Komar[1] is of the form

$$ds^2=dr^2+B(r)d\theta^2+C(r)\sin^2\theta d\varphi^2, \tag{2}$$

in which B and C are functions of r only.

An essential part of Komar's argument is that his metric is to be given on a manifold which is topologically Euclidean. Consequently, it would seem that his r, θ, φ should be interpreted as spherical coordinates in the usual sense so that the surfaces $r=\text{constant}>0$ are to be topological 2-spheres while $r=0$ is a point. Further, to ensure a complete manifold, the z axis, given by $\theta=0$, $\theta=\pi$, must be included.

In discussing Komar's metric Eq. (2) it is necessary to distinguish two possible sources of difficulty. First is the fact that he has chosen spherical coordinates, which are not globally admissible, even in the flat-space case, $B=C=r^2$, because of the presence of the term $\sin^2\theta$ along the z axis. Consequently, any discussion concerning which coordinate systems are to be admissible can only begin after the choice of one such is made, by relating it to the r, θ, φ.

The importance of specifying this differentiable structure of the manifold cannot be emphasized too greatly. It is not adequate merely to write an expression such as Eq. (2) in order to specify a geometry on a manifold. Rather, the precise relationship between this

* Supported by a grant from the National Science Foundation.
[1] A. Komar, Phys. Rev. **137**, B462 (1965).
[2] J. Wheeler, Phys. Rev. **97**, 511 (1955).
[3] C. Misner and J. Wheeler, Ann. Phys. (N. Y.) **2**, 525 (1957).

coordinate system and the admissible coordinate systems associated with the definition of the manifold must be clearly stated. For a clear statement of these points and a complete definition of the basic geometric object—differentiable manifold—dealt with in differential geometry, see the first few pages of the book by Lichnerowicz.[4]

In the second place, in breaking with the stringent analyticity of the Kruskal expression of the Schwarzschild solution, Komar considers metrics which are C^0. Of course, in doing so care must be exercised to ensure that the discontinuities in the derivatives of the metric do not give rise to a delta-function-type of behavior in the Ricci tensor. For a treatment of such questions see the work of Papapetrou and Treder.[5] The general philosophy in dealing with such problems seems to be that the metric will be of higher order of differentiability, at least C^2, except for surfaces of discontinuity across which the derivatives of the metric may "jump." However, the ruling criterion for determining the admissibility of such a jump is that it can be regarded as a limiting case of a situation in which the metric is at least C^2 and satisfies the Einstein equations across a region of width ϵ surrounding the surface, as ϵ goes to zero. Consequently, in such a situation the metric is C^2 on either side of the jump surface and the one-sided limits of the Riemann tensor invariants exist and are finite, although they may be unequal.

This is of course the case for the spherical surfaces of discontinuity of the metric derivatives carefully analyzed by Komar. Thus, in Eq. (2), B and C are C^∞ functions of r except for a finite set of values for r. Of course, it is certainly true that an attempt to evaluate the second-order invariants precisely at these surfaces would yield infinite results, but nevertheless the one-sided limits exist and are finite from each side.

In this section we will show that another anomalous region exists which was not considered by Komar, namely, the portion of the z axis along which his solution differs from the Schwarzschild solution but in which B and C are C^∞. Further, this behavior along this axis is of a much more serious nature than the simple discontinuities of the invariants since it corresponds to an actual divergence to infinity of these invariants, in precisely the same fashion as do the same invariants for the Schwarzschild solution as $r\rightarrow 0$. Hence, although Komar's solution avoids this singularity at the origin, which is generally regarded as essential and irremovable if the manifold is to be topologically Euclidean, it does contain another region of singularity of precisely the same sort.

The fact that the divergence of the invariants of the Riemann tensor would indeed give rise to difficulties in the metric is easily seen in terms of normal coordinates, in which the Riemann tensor is immediately seen to influence the behavior of the metric.[6] Any differential geometry in which normal coordinates are not regarded as admissible would of course be extremely unusual. Finally, it might be pointed out that this relationship between the behavior of the metric and that of the invariants of the Riemann tensor certainly does not hold for any arbitrarily constructed second order differential invariants, such as $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}/R_{\mu\nu}R^{\mu\nu}$, since such do not enter into the definition of intrinsically admissible coordinates such as the Riemann invariants do for normal coordinates.

It should again be emphasized that the following discussion concerns those regions in which the functions B and C are C^∞ and thus is totally unrelated to the jumps of the derivatives of these functions across the spherical surfaces described by Komar. For the present purposes, it is adequate to consider the invariants associated with the Ricci tensor (for the three-dimensional case these are equivalent to those of the full Riemann tensor). These invariants will be the eigenvalues of the mixed components, R_{ij}, solving the equation

$$\det(R_i{}^j-\lambda\delta_i{}^j)=0. \tag{3}$$

A simple calculation shows that the $R_i{}^j$ for Eq. (2) form the array

$$R_i{}^j=\begin{pmatrix} \alpha(r) & \beta(r)\cot\theta & 0 \\ B\beta(r)\cot\theta & \gamma(r) & 0 \\ 0 & 0 & \epsilon(r) \end{pmatrix}, \tag{4}$$

in which α, γ, ϵ, depend on B, C and their first two derivatives and, denoting r derivatives by primes,

$$\beta(r)=(B'C-C'B)/(2CB^2), \tag{5}$$

from Eqs. (3) and (4) it follows that two of the eigenvalues will be λ_+, λ_-

$$\lambda_\pm=f(r)\pm[g(r)-B\beta^2(r)\cot^2\theta]^{1/2}. \tag{6}$$

Thus, unless $\beta=0$, the metric in Eq. (2) will not be truly spherically symmetric, as pointed out by Komar. More important, however, is the fact that unless $\beta=0$ the metric cannot be continued to $\theta=0$ or $\theta=\pi$ without an essential singularity in its invariants. Consequently, in no admissible coordinate system which includes the line $\theta=0$, $\theta=\pi$ will the components of the metric be C^2 along the z axis.

Even more serious, however, is the fact that the invariants actually diverge to infinity along this line, precisely imitating that behavior of the Schwarzschild solution at the origin which bootstrap gravitational geons were to have eliminated. Hence, it appears entirely possible that this singularity in bootstrap geons acts as a delta-function source in the Einstein equations precisely as the singularity at the origin does in the

[4] A. Lichnerowicz, *Théories Relativistes de la Gravitation et de L'Electromagnétisme* (Masson, Paris, 1955).

[5] A. Papapetrou and H. Treder, Nachr. Akad. Wiss. Goettingen, II. Math. Physik. Kl. **20**, 53 (1959).

[6] E. Cartan, *Leçons sur la Géométrie des Espaces de Riemann* (Gauthier-Villars, Paris, 1951), pp. 234–237.

Schwarzschild metric. At any rate, this point should be clarified.

Finally, as mentioned above, the use of normal coordinates actually seems to imply that such behavior will be inconsistent even with the possibility that the metric be C^0. This will be taken up further in the following section.

III. CARTESIAN COORDINATES

Since bootstrap gravitational geons were intended to eliminate not only the singularity of the Schwarzschild metric at the origin, but also the necessity for the conjecture that a topologically nontrivial space ($S^2\times R$) would be the only way to eliminate the singularity, Komar explicitly required that the manifold be topologically Euclidean. Thus, it might seem reasonable to require the existence of globally admissible Cartesian-like coordinates, x, y, z, with $-\infty<x<\infty$, $-\infty<y<\infty$, $-\infty<z<\infty$. The admissibility of these coordinates and the fact that the metric is to be C^0 would then imply that the components of the metric expressed in this coordinate system be C^0 functions.

However, due to Komar's failure to completely specify his manifold by prescribing which coordinates are to be admissible, it is not immediately obvious how Komar's r, θ, φ are to be related to the admissible coordinates. The most natural assumption would seem to be the following. Set

$$\begin{aligned} x&=f(r)\sin\theta\cos\varphi,\\ y&=f(r)\sin\theta\sin\varphi,\\ z&=f(r)\cos\theta, \end{aligned} \tag{7}$$

in which $f'>0$ for all r and $f(0)=0$. We might then consider x, y, z as globally admissible coordinates and regard the singularity of any component of the metric in them as intrinsic. This is precisely what is meant by saying that r, θ, φ are to be regarded as spherical coordinates in the usual sense.

It is convenient to distinguish three possibilities. First, if $B/C=1$ everywhere, choose $f(r)$ to solve

$$df/dr=f/(B)^{1/2}, \tag{8}$$

and using Eq. (7), Eq. (2) becomes the singularity-free, conformally flat metric

$$ds^2=(dr/df)^2(dx^2+dy^2+dz^2). \tag{9}$$

Next, assume $B/C=p^2$, with $p^2=$constant greater than zero. This is easily seen to reduce to the previous case upon replacing φ in Eq. (2) by $p\varphi$. This merely means that the scale of the original φ was inappropriate. Note that this cannot be done if B/C depends on r. In fact, in this case this transformation would introduce a term proportional to φ^2dr^2 which will yield a non-single-valued metric.

Finally, assume $d(B/C)dr\neq 0$. Set $F=B-C$ so that for some r, $F\neq 0$. Hence, Eq. (2) becomes

$$ds^2=ds_0{}^2+Fd\theta^2, \tag{10}$$

in which $ds_0{}^2$ is of the form Eq. (2) but with $B=C$. Hence, make the same transformation as above for $ds_0{}^2$. In terms of these x, y, z coordinates the coefficient of dy^2 is

$$g_{22}=g_{22}{}^0+(F/r^4)z^2y^2/(x^2+y^2), \tag{11}$$

in which $g_{22}{}^0$ is the contribution from $ds_0{}^2$ and is thus a nonsingular function. However, the second term on the right side of Eq. (11) cannot be continued as a continuous function over a region including the z axis if $F\neq 0$. In fact,

$$1=\lim_{y\to0}\lim_{x\to0} y^2/(x^2+y^2)\neq\lim_{x\to0}\lim_{y\to0} y^2/(x^2+y^2)=0, \tag{12}$$

whereas if g_{22} were to be continuous these two limits would have to be equal.

Hence, if Komar's metric is to be C^0 on a topologically trivial manifold, his r, θ, φ cannot be regarded as spherical coordinates in the usual sense.

If such a manifold does indeed exist, and $\bar{x}$, $\bar{y}$, $\bar{z}$ are globally admissible Cartesian coordinates on it, the partial derivatives of $\bar{x}$, $\bar{y}$, $\bar{z}$ with respect to the x and y described in Eq. (7) must have different limits as the z axis is approached from different directions. The only reasonable explanation for this behavior would seem to be that what is regarded as a line, $z=0$, in the x, y, z coordinate system is really a volume, Z: $-F_1(\bar{y},\bar{z})\leq\bar{x}\leq F_2(\bar{y},\bar{z})$; $-G_1(\bar{x},\bar{z})\leq\bar{y}\leq G_2(\bar{x},\bar{z})$ when viewed from the admissible $\bar{x}$, $\bar{y}$, $\bar{z}$ coordinates. Thus, the difference in the limits of the components of the metric in the x, y, z coordinates as the z axis is approached would be due to the fact that the approach is to different regions of a surface of Z, given by $\bar{x}=f(\bar{y},\bar{z})$. However, this would mean that Komar has exhibited his metric only over a part of the manifold and must explicitly demonstrate that it can be continued without inadmissible singularity over the remaining region Z in such a way as to still satisfy the initial condition $R_3=0$.

Finally, the divergence of the invariants as the surface of Z is approached would still remain. The possibility that these invariants converge to different values on this surface or that they be discontinuous across it would be relatively easy to understand. However, the lack of even one-sided limits for these invariants is a problem of a different magnitude that would have to be resolved before the manifold could be understood in standard differential geometric terms.

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