A new topology for curved space–time which incorporates the causal, differential, and conformal structures*

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A new topology is proposed for strongly causal space–times. Unlike the standard manifold topology
(which merely characterizes continuity properties), the new topology determines the causal, differential, and
conformal structures of space–time. The topology is more appealing, physical, and manageable than the
topology previously proposed by Zeeman for Minkowski space. It thus seems that many calculations
involving the above structures may be made purely topological.

1. INTRODUCTION

In 1964 Zeeman published a paper[1] showing that the causal structure of Minkowski space M, already implied its linear structure. Causality was defined by means of a partial ordering on M, and it was shown that the group of automorphisms of M preserving the ordering is generated by the inhomogeneous Lorentz group and dilatations. (This is the *homothecy* group H of M, comprising all affine automorphisms which preserve the Lorentz metric up to a *constant* factor.) He then proposed[2] that the conventional (positive definite) metric topology $/\!\!/\!\!|$ of M should be replaced by a new topology $\mathcal{J}$ (the *fine* topology) which is related to the causal structure. $\mathcal{J}$ has the following properties[2]:

(1) $\mathcal{J}$ is defined to be the finest topology on M which induces the one-dimensional Euclidean topology on every straight timelike line, and the three-dimensional Euclidean topology on every spacelike hyperplane. Thus $\mathcal{J}$ is finer (and, in fact, strictly finer) than $/\!\!/\!\!|$.

(2) $\mathcal{J}$ incorporates the (homothetic) Lorentz structure at the primitive level of topology (rather than, as is conventional, *after* imposing linear structure); the homeomorphism group of $\mathcal{J}$ is H.

(3) If the path of a particle is interpreted as a $\mathcal{J}$-continuous map γ of the unit interval I into M such γ preserves order, the image of γ is piecewise linear, consisting of a finite number of straight timelike line segments, like the path of a free particle undergoing a finite number of collisions.

(4) $\mathcal{J}$ is Hausdorff, connected and locally connected, but not normal, locally compact or first countable.

This new topology obviously has several advantages over the standard one, which merely characterizes the continuity of M. Its very definition (1) is more intuitively appealing than that of $/\!\!/\!\!|$, since it requires a set to be open when (a) every inertial observer "times" it to be open,[7] and (b) every section of time simultaneity intersects it in an open set. The definition of $/\!\!/\!\!|$ involves 4–spheres in space–time, which have no particular physical meaning. The idea of (2) incorporating causal,

linear and even homothetic structure already in a topology $/\!\!/\!\!|$ is certainly physically appealing, and the idea (3) that the requirement of *continuity* of a curve should already restrict it to be physically meaningful is fascinating. However, there are disadvantages:

(1*) A 3-dimensional section of simultaneity has no meaning in terms of physically possible experiments. Also, the use of *straight* timelike lines in defining $\mathcal{J}$ suggests that $\mathcal{J}$ from the beginning has been equipped with information involving *inertial* observers, so that the emergence of *linear* structure in (2) is less surprising. (Though in fact this is not the reason for its energence.)

(2*) While the *isometry* and *conformal* groups of M are certainly significant physical, it is not necessarily clear that this is so for the *homothecy* group of M.

(3*) The set of $\mathcal{J}$-continuous paths does not incorporate accelerating particles moving under forces in curved lines.[2]

(4*) $\mathcal{J}$ is technically complicated.[2] In particular, the fact that no point has a countable neighborhood basis makes $\mathcal{J}$ hard to calculate with.

Zeeman suggests[2] that criticism (3*) could be overcome by generalizing his theory to general relativity, where the image of γ should become piecewise geodesic (thus accounting for gravitational forces). This generalization has recently been carried out by Göbel[3] who, for strongly causal space–time manifolds, replaced (1) by replacing "time axis" and "spacelike hyperplane" by "timelike geodesic" and "spacelike hypersurface." He then proves that Zeeman's conjecture about the generalization of (3) is correct, and, with the help of a theorem of Hawking[4] relating causal to differential structure, that the homeomorphism group is again the homothety group.

However, even in general relativity, particles need not move along geodesics since, for example, they may be charged and an electromagnetic field may be present (and this applies in special relativity also).[8] Thus the generalization to general relativity of the topology $\mathcal{J}$

(we also call the generalizations $\mathcal{J}$) only partially answers (3*), and leaves the other criticisms (1*), (2*) and (4*) as before.

In this paper, we propose a new topology P for strongly causal space—times M which share the attractive features of the topologies $\mathcal{J}$, but which also answer some of the above criticisms and have additional attractive physical features. The topology P, the *path* topology, has the following properties:

(1') P is defined to be the finest topology on M which induces the Euclidean topology on *arbitrary* (not necessarily smooth) timelike curves.

(2') The topology P incorporates the causal, differential, and smooth conformal structure; the homeomorphism group of P is the *conformal* group.

(3') The set of P-continuous paths incorporates *all* timelike paths. In fact, the set consists of all "Feynman paths" (described below).

(4') P is still technically complicated, but less so than $\mathcal{J}$. Indeed, P is Hausdorff, connected and locally connected, but not normal or locally compact. However, every point has a countable neighborhood basis, and this makes P much easier to deal with than $\mathcal{J}$.

The definition (1') has an immediate physical interpretation which is more appealing than (1): a set is open whenever every observer (accelerated or not) "times" it to be open. No nonphysical experiments are required. Also, (1') does not require any smoothness properties to define P, so that the emergence of smooth and conformal structure is more surprising. Again, (2') answers criticism (2*), and (3') shows that *all* possible observers, accelerated or not, are described by P-continuous curves. The fact that P has a countable neighborhood basis at each point makes it much easier to deal with than $\mathcal{J}$. Thus the computation of the set of *all* P-continuous paths becomes relatively easy (Zeeman only finds the order preserving paths for $\mathcal{J}$).[9] These paths are (possibly accelerated) "Feynman paths" which zigzag with respect to time orientation, like the Feynman track of an (accelerated) electron. This same "basis" property makes it relatively easy to find the general properties of P. In fact, P is path connected and locally path connected, but not regular or paracompact, in addition to having the properties mentioned above. The relative ease of calculations with P suggest that it could be usefully applied to "practical" problems in general relativity. If one could show (using the basis property) that a space—time must admit a (local or global) P-homeomorphism, this would mean that the space—time admits a (local or global) conformal diffeomorphism.

Thus we suggest that, while Zeeman's topology[2] and its general relativistic analogs[3] represent a radical and fascinating departure from the conventional schemes, our topology P has all of the required features, but is more intuitively appealing, manageable, and physical. Section 2 is devoted to standard results and definitions. In Sec. 3, timelike paths (not necessarily smooth) and Feynman paths are defined, and the topology is described. In Sec. 4 it is shown that P is not comparable

to $\mathcal{J}$, and the important basis property is proved. This is used to show that the set of continuous paths is the set of Feynman paths, and to prove various general properties of P. In Sec. 5 it is shown that P carries the chronological structure of M, and in Sec. 6, that P carries the causal, differential, and conformal structure of M. The final theorem is that the homeomorphism group of P is the group of conformal diffeomorphisms of M.

The burden of the argument is as follows. First we show that P-homeomorphisms take timelike curves to timelike curves. Then we show that this implies that P-homeomorphisms preserve causal relations. This is used to show that they are diffeomorphisms preserving the null cones, that is, conformal diffeomorphisms.

2. STANDARD DEFINITIONS AND RESULTS[5]

Space—time is taken throughout to be a connected, Hausdorff, paracompact, C^∞ real four-dimensional manifold M without boundary, with a C^∞ Lorentz metric (only a few orders of differentiability will actually be needed) and associated pseudo-Riemannian connection. M is taken to be time orientable throughout (that is, M admits a nonvanishing timelike vector field). For subsets A and B of M, the *chronological future* $I^+(A, B)$ of A *relative to B* is the set of all points in B which can be reached from A by a future directed smooth (i.e., C^∞) timelike curve in B of finite extent. The *causal future* $J^+(A, B)$ of A *relative to B* is the union of $A \cap B$ with the set of all points in B which can be reached from A by a future directed smooth causal curve (i.e., nonspacelike curve) in B. The *future horismos* $E^+(A, B)$ of A *relative to B* is defined as $J^+(A, B) - I^+(A, B)$. These definitions have duals, often regarded as self evident, in which "future" is replaced by "past" and "+" by "−." If A is the singleton set $\{p\}$ for a point $p \in M$, we write $I^+(p, B)$ rather than $I^+(\{p\}, B)$, for example, and $I^+(p)$ for $I^+(p, M)$. The relations $p \in I^+(q)$, $p \in J^+(q)$, and $p \in E^+(q)$ will sometimes be written $q < p$, $q \leqslant p$, and $q \to p$, respectively. These last relations are respectively called, chronological, causal, and horismos relations. The definitions and results[5] below will be needed in this paper. They are only stated in the generality needed:

2.1 If V is an open set, $q \in I^+(p, V)$ implies $p \in I^-(q, V)$, and conversely. Similar results hold for J and E. $I^+(p, V)$ and $I^-(p, V)$ are open sets. In particular, these statements hold for $I^+(p)$ and $I^-(p)$.

2.2 If V is an open set, either

$$\left.\begin{array}{ll} q \in J^+(p, V), & r \in I^+(q, V) \\ q \in I^+(p, V), & r \in J^+(q, V) \end{array}\right\} \text{ imply } r \in I^+(p, V).$$

2.3 Let $T_p(M)$ denote the tangent space of $p \in M$, and exp: $T_p(M) \to M$, the exponential mapping. Then there is an open neighborhood N of the origin of $T_p(M)$ such that $U \equiv \exp(N)$ is an open convex normal neighborhood of $p \in M$. That is, every pair of points in U can be joined by a unique geodesic curve in U, and geodesics in U through p are the images of straight lines through the origin in $N \subset T_p(M)$. Further, p possesses a neighborhood basis of open convex normal neighborhoods.

Furthermore, the normal neighborhoods may be taken to be normal neighborhoods of every point in them. Let $\epsilon > 0$ be sufficiently small so that the Euclidean open ball B of radius ϵ, centered at the origin, is contained in N. We define $B_u(p, \epsilon) = \exp(B)$, and whenever such a set is referred to, it is assumed that $\epsilon > 0$ is sufficiently small.[10]

2.4 Denote the normal coordinates of $q \in U$ by $x^i(q)$, where $i = 0, 1, 2, 3$, and x^0 is the time coordinate, x^1, x^2, x^3 the space coordinates. Then $x^i(p) = 0$ and

$$J^+(p, U) = \{ q \in U \mid (x^0(q))^2 - (x^1(q))^2 - (x^2(q))^2 - (x^3(q))^2 \geq 0, \ x^0(q) \geq 0 \}.$$

$J^-(p, U)$ satisfies the same condition except that $x^0(q) \leq 0$, and $I^\pm(p, U)$ are defined analogously, except that all inequalities are strict. Define, for any open set V, the cones (possibly) with and without origin by

$$C(p, V) = I^+(p, V) \cup I^-(p, V),$$

$$K(p, V) = C(p, V) \cup \{p\},$$

and, for an open convex normal neighborhood U of p, define

$$L_u(p, \epsilon) = B_u(p, \epsilon) \cap K(p, U).$$

Note that $L_u(p, \epsilon) - \{p\}$ is nonempty [the point with coordinates $(\frac{1}{2}\epsilon, 0, 0, 0)$ belongs to $B_u(p, \epsilon) \cap I^+(p, U)$] as is $B_u(p, \epsilon) - L_u(p, \epsilon)$ [take coordinates $(0, 0, 0, \frac{1}{2}\epsilon)$].

2.5 A space—time M is *chronological*, if there is no closed smooth future (or past) directed timelike curves in M. Equivalently, M is chronological, if and only if, $I^+(p) \cap I^-(p) = \phi$ for all $p \in M$. Analogously, M is *causal*, if there are no future (or past) directed causal curves.

2.6 Denote the manifold topology of M by $\mathcal{M}$. Consider the collection of all sets of the form $I^+(p) \cap I^-(q)$ for $p, q \in M$. These sets are open and (together with, possibly, the empty set ϕ) clearly form a basis for a topology on M. The topology is called the *Alexandroff topology* $\mathcal{A}$ of M, and in general, is coarser than $\mathcal{M}$.

2.7 A *causality neighborhood* D of a point $p \in M$ is an $\mathcal{M}$-neighborhood of p such that, whenever $\gamma: F \to M$ is a smooth causal path, $\gamma^{-1}(D)$ is connected. (Here F is a connected interval of the real line R.) M is *strongly causal at p*, if and only if, p has a neighborhood basis $\{D_\alpha(p) \mid \alpha \in A\}$ such that, for each $\alpha \in A$, $D_\alpha(p)$ is a causality $\mathcal{M}$-neighborhood. M is *strongly causal*, if it is strongly causal at each point. Another useful characterization of this property is given by the following consequence:

M is strongly causal $\Longleftrightarrow M$ is $\mathcal{A}$-Hausdorff $\Longleftrightarrow \mathcal{A} = \mathcal{M}$.

2.8 If U is an open convex normal neighborhood of $p \in M$, $E^+(p, U)$ consists of future directed null geodesics in U from p, and $E^+(p, U) \cup E^-(p, U)$ is the image of the null cone $N \cap N_p \subset T_p(M)$ in the neighborhood N of the tangent space $T_p(M)$ under exp.

2.9 The metric g at $p \in M$ is determined up to a constant by the tangent null cone $N_p \subset T_p(M)$.

2.10 The *isometry*, *homothecy* and *conformal* groups of M are those groups of C^∞ diffeomorphisms of M which

preserve, respectively, the metric tensor, the metric tensor up to a *constant* factor, and the metric tensor up to a (possibly variable) factor.

3. THE TOPOLOGY $\mathcal{P}$

3.1 Let F be a connected interval of the real line R. (The singleton closed set $[x, x]$ is excluded. For our purposes we may take F to be bounded, for there is an order preserving C^∞ diffeomorphism $\tan^{-1} : R \to (-\pi/2, \pi/2)$. Thus F is a finite, closed, open, or half open half closed interval.) A map $\gamma: F \to M$ is called a *path*, and its image a *curve* (the same symbol γ often being used for either, it being clear from the context which is meant). A path γ is *continuous* if γ is continuous with respect to $\mathcal{M}$ and the topology on F induced from the standard one on R. A point $q \in M$ is said to be an *initial end point* of a continuous path $\gamma: F \to M$, if for every neighborhood N of q there is a $t_N \in F$, such that $t \in F$ and $t < t_N$ implies $\gamma(t) \in N$. If a continuous path γ has an initial end point $q \notin \gamma(F)$, one may find a new continuous path $\gamma': F \cup \{t_1\} \to M$ such that $\gamma' \mid_F = \gamma$ and $\gamma'(t_1) = q$ where t_1 is the greatest lower bound of F. We shall therefore assume without loss of generality that continuous paths contain both their initial and final endpoints if they have them.

3.2 A path $\gamma: F \to M$ is called[5] *future directed and timelike at $t_0 \in F$*, if and only if γ is continuous and there is a connected neighborhood U of t_0 in F, and an open convex normal neighborhood U of $p = \gamma(t_0)$ such that

$$t \in U \text{ and } t < t_0 \Longrightarrow \gamma(t) \in I^-(p, U);$$

$$t \in U \text{ and } t > t_0 \Longrightarrow \gamma(t) \in I^+(p, U).$$

A path is called *future directed and timelike*, if it is future directed and timelike at each $t_0 \in F$. Similar dual definitions hold for "past directed." A path is *timelike at t_0*, if it is either future or past directed and timelike at t_0, and *timelike*, if it is either everywhere future directed, or everywhere past directed and timelike. A curve is timelike, if it is the image of a timelike path.

Proposition 3.3: Let $\gamma: F \to M$ be a continuous path which is timelike at each $t_0 \in \text{int}F$. Then γ is a timelike path.

Proof: Suppose γ is future directed at $t_0 \in \text{int}F$. Let U and U be as above. Suppose there were a $t_1 \in U$ with $t_1 > t_0$ such that γ was past directed at t_1. The coordinate x^0 of $\gamma(t)$ will be a continuous function of t. Thus there will be some t_2 such that $x^0(\gamma(t_2))$ is the maximum value of $x^0(\gamma([t_0, t_1]))$. Since γ is future directed at t_0, t_2 must be greater than t_0. Similarly, t_2 would have to be less than t_1. Consider the point $q = \gamma(t_2)$. On $I^+(q, U)$ the coordinate x^0 would be greater than its value at q. This would mean that $\gamma([t_0, t_1])$ could not be timelike at t_2. This shows that γ must be future directed for all $t \in U$ with $t \geq t_0$. A similar argument shows that γ must be future directed for all $t \in U$ with $t \leq t_0$. Thus, the set of points at which γ is future directed is open in $\text{int}F$. Similarly, the set on which γ is past directed is open. Because F is connected, γ must be either everywhere future directed or everywhere past directed in $\text{int}F$. Assume, without loss of generality, that γ is future directed. Suppose that $q = \gamma(t_1)$ is an initial endpoint.

Let U be a connected neighborhood of t_1 in F and let U be a convex normal neighborhood of q. Let $t_2 \in U$ with $t_2 > t_1$. Then $\gamma((t_1, t_2)) \subset \Gamma^-(r, U)$, where $r = \gamma(t_2)$. Therefore, by continuity $q \in \overline{\Gamma^-(r, U)} = J^-(r, U)$. Thus, $r \in J^+(q, U)$. However, by a similar argument one can find a $t_3 \in (t_1, t_2)$ such that $\gamma(t_3) \in J^+(q, U) \cap \Gamma^-(r, U)$. Therefore, by 2.2, $r \in \Gamma^+(q, U)$. Hence γ is also future directed timelike at its endpoints. ∎ It follows from 2.2 and 3.2 that a timelike path is locally 1–1, that is, each $t_0 \in F$ has a neighborhood V such that $\gamma|_V$ is 1–1. Notice that timelike paths need not be smooth. The point of this definition is that P will be defined independently of smoothness properties, but nevertheless smooth structure will emerge from P (Theorem 5). Curiously enough, a path may be timelike and smooth without being timelike. Let $\gamma: R \to M$ be the path defined in Minkowski space (usual coordinates) by $\gamma(t) = (t, \sin t, 0, 0)$. Then γ is timelike and smooth, but not smooth timelike, since it is null at the points $t = n\pi$ for integral n. However, $\Gamma^+(A, B)$ and $\Gamma^-(A, B)$, as defined by smooth timelike curves, agree with $\Gamma^+(A, B)$ and $\Gamma^-(A, B)$ as defined by timelike curves.

3.4 We now define a class of paths which are similar to timelike paths, in that their curves are constrained to lie within local light cones, but which may zigzag with respect to time orientation. A path $\gamma: F \to M$ is a *Feynman path*, if γ is continuous and, for each $t_0 \in F$, there is an open connected neighborhood U of t_0, and an open convex normal neighborhood U of $p = \gamma(t_0)$ such that

$$\gamma(U) \subset K(p, U).$$

A locally 1–1 Feynman path will be called a *Feynman track*. Suppose γ is a Feynman track, 1–1 in a neighborhood V of t_0. Let W be an open connected neighborhood of t_0 in $U \cap V$. Then, using the fact that W is connected, and that $\gamma|_W$ is 1–1 and continuous, it is easily shown that γ is either timelike at t_0 or $\gamma(W) \subset \Gamma^+(p, U) \cup \{p\}$, or $\gamma(W) \subset \Gamma^-(p, U) \cup \{p\}$. Obviously timelike paths are Feynman tracks, but there are many nontimelike Feynman tracks.

3.5 Suppose γ_1 is a timelike curve with future endpoint q, and γ_2 is a timelike curve with past endpoint q. Evidently the union $\gamma_1 \cup \gamma_2$ is also a timelike curve, which may be parametrized as a future or past directed timelike path. Any such path will be called a *product path* $\gamma_1\gamma_2$, and qualified with "future directed" or "past directed," according to the choice of the direction of the parameter. If γ_1 is as before, but with γ_2 now with future endpoint q, we may similarly define product paths, denoted $\widehat{\gamma_1\gamma_2} = \gamma$, which are timelike everywhere at $\gamma^{-1}(q)$. However, $\widehat{\gamma_1\gamma_2}$ is always a Feynman path.

3.6 Define a new topology (the *path* topology) P of M by specifying the collection P of open sets of the topology as follows: P is the finest topology satisfying the requirement that the induced topology on every timelike curve coincides with the topology induced from M. Thus, if a set $E \subset M$ is P-open, for every timelike curve γ there is an $O \in M$ with

$$E \cap \gamma = O \cap \gamma.$$

Conversely, if E satisfies this condition, it is P-open, and P is the largest collection of such sets. Obviously

$O \in M$ implies $O \in P$, so P is finer than M. We shall see below that it is strictly finer, and that P is not comparable to the general relatively analog of J.

4. GENERAL PROPERTIES OF

Here we show that P is strictly finer than M, but not comparable to J, and find an explicit neighborhood basis for P. Then we show that the P-continuous paths are Feynman paths, and various general properties of P are proved.

Proposition 4.1: Let $\gamma: F \to M$ be a path. If γ is P-continuous, γ is M-continuous. If γ is timelike, γ is P-continuous.

Proof: The first assertion follows since P is finer than M. For the second, note first that $E \in P$ implies $E \cap \gamma = O \cap \gamma$ for some $O \in M$. Hence $\gamma^{-1}(E) = \gamma^{-1}(E \cap \gamma) = \gamma^{-1}(O \cap \gamma) = \gamma^{-1}(O)$. But γ is timelike and so, in particular, continuous. Therefore $\gamma^{-1}(E) = \gamma^{-1}(O)$ is open. ∎

Proposition 4.2: Sets of the form $K(p)$, $K(p, U)$, and $L_u(p, \epsilon)$ are P-open.

Proof: Let γ be any timelike curve. Suppose first that $p \in \gamma$. Then by definition, $\gamma \subset \Gamma^+(p) \cup \Gamma^-(p) \cup \{p\} = K(p)$, so $\gamma \cup K(p) = \gamma \cup M$. Suppose next that $p \notin \gamma$. Then $\gamma \cap K(p) = \gamma \cap (\Gamma^+(p) \cup \Gamma^-(p))$. In either case, $\gamma \cap K(p) = \gamma \cap O$ for some $O \in M$. The proof that $K(p, U)$ is P-open is similar (replace M by U), and the proof for $L_u(p, \epsilon)$ follows because $B_u(p, \epsilon)$, being M-open, is *a fortiori* P-open. ∎

This proposition shows that P is strictly finer than M, since, for example, $p \in K(p, U)$ has no M-neighborhood contained in $K(p, U)$. We will show further that P is not comparable to Zeeman-type topologies J. Define S to be the set $[B_u(p, \epsilon) - [E^+(p, U) \cup E^-(p, U)]]$, where $\epsilon > 0$ is smaller than $\frac{1}{2}$ [and also sufficiently small for $B_u(p, \epsilon)$ to make sense]. Let $\gamma \subset U$ be the curve defined, in normal coordinates, by the timelike path $\gamma = [-\epsilon, \epsilon] \to U$, with equation $\gamma(t) = (t, t^2, 0, 0)$. Consider the set $R = (S - \gamma) \cup \{p\}$. Then any timelike geodesic will have an open intersection with R, as will any spacelike hypersurface. Therefore R is J-open. However, R is not P-open since $R \cap \gamma = \{p\}$, which is closed. On the other hand, $K(p, U)$ is P-open but not J-open, since the intersection of any spacelike hypersurface containing p with $K(p, U)$ is $\{p\}$.

Theorem 1: Sets of the form $L_u(p, \epsilon)$ form a basis for the topology P.

Proof: We must show that, for any P-open set E and any $p \in E$, there is a P-open neighborhood of p of the form $L_u(p, \epsilon)$ contained entirely in E. Suppose this to be false. Then there is an open convex normal neighborhood U of p such that, for every ball $B_u(p, \epsilon) \subset U$ and corresponding $L_u(p, \epsilon)$, there is a $q \in L_u(p, \epsilon)$ with $q \notin E$. Fix such a set $L_u(p, \epsilon_1)$ and assume, without loss of generality, that it contains a q_1, not in E, with $q_1 \in \Gamma^+(p, U)$. [If there is no such q_1, all points of the required type lie in $\Gamma^-(p, U)$, and the proof is as before with Γ^+, Γ^- and "future," and "past" interchanged.] Since p belongs to the open set $\Gamma^-(q_1, U)$, we can find a $\delta > 0$ with $B_u(p, \delta) \subset \Gamma^-(p, U)$. Let ϵ_2 be any positive number satisfying $\epsilon_2 \leq \min(\epsilon, \frac{1}{2}\epsilon_1)$. There is a $p_2 \in L_u(p, \epsilon_2)$ with

$q_2 \notin E$. Assume without loss of generality that $q_2 \in I^*(p, U)$. [If all q_2 of the required type lie in $I^-(p, U)$, discard q_1 and start with q_2, interchanging I^*. I^- and "future," "past."] Construct $\epsilon_3 \leq \frac{1}{2}\epsilon_2$ and $q_3 \in L_U(p, \epsilon_3)$ analogously, and inductively construct $\epsilon_{n+1} \leq \frac{1}{2}\epsilon_n \leq 2^{-n}\epsilon_1$ and $q_{n+1} \in L_U(p, \epsilon_{n+1})$. This way, possibly after discarding a finite number of points, we obtain a sequence $S = \{q_n\}$ of distinct points $S \subset I^*(p, U)$ and $S \notin E$, with the property that consecutive points q_s and q_{s+1} can be joined by a (unique) timelike geodesic curve. Form a past directed timelike product path $\zeta \equiv (q_1 q_2)(q_2 q_3) \cdots$ as in 3.5. Because $\epsilon_{n+1} \leq 2^{-n}\epsilon_1$, $\{q_n\}$ converges to p in the topology $\mathcal{M}$, and thus ζ has past endpoint p. From 3.1 there is, therefore, a unique timelike curve $\gamma = \zeta \cup \{p\}$. Since E is P-open, $E \cap \gamma = O \cap \gamma$ for some $O \in \mathcal{M}$. Since $S \notin E$, $S \notin E \cap \gamma = O \cap \gamma$, therefore $S \subset M - (O \cap \gamma) = (M - O) \cup (M - \gamma)$. But $S \subset \gamma$, so $S \notin M - \gamma$; therefore, $S = \{q_n\} \subset M - O$. But $M - O$ is $\mathcal{M}$-closed and $\{q_n\}$ is $\mathcal{M}$-convergent to p, therefore $p \in M - O$, so $p \notin O$. Hence, $p \notin O \cap \gamma = E \cap \gamma$. But p belongs to both E and γ, so we have a contradiction, and the theorem is proved. ∎

This basis property, which has no analogue in the fine topologies $\mathcal{J}$, makes the path topology P-much more manageable. Below, it will be used to prove all the basic general properties of P. But first, we shall find the set of all P-continuous paths.

Theorem 2: A path $\gamma: F \to M$ is P-continuous if and only if it is a Feynman path.

Proof: Suppose first that γ is a Feynman path. We must show that, for each $t_0 \in F$, the inverse image of each P-neighborhood of $\gamma(t_0)$ is a neighborhood of t_0. Let U be the open convex normal neighborhood of $p = \gamma(t_0)$ used in the definition 3.2. Let $\{L_u(p, \epsilon)\}$ be a P-neighborhood basis of p for all sufficiently small $\epsilon > 0$. Then $\gamma^{-1}(L_u(p, \epsilon)) = \gamma^{-1}(K(p, U) \cap B_u(p, \epsilon)) = \gamma^{-1}(K(p, U)) \cap \gamma^{-1}(B_u(p, \epsilon))$. But $\gamma^{-1}(K(p, U))$ contains an open connected neighborhood U of t_0, and $\gamma^{-1}(B_u(p, \epsilon))$ is an open neighborhood of t_0 (because γ is $\mathcal{M}$-continuous). Thus γ is P-continuous.

Suppose next that γ is P-continuous, and let U be an open convex normal neighborhood of $p = \gamma(t_0)$. Then $K(p, U)$ is a P-neighborhood of p, so $\gamma^{-1}(K(p, U))$ is a neighborhood W of t_0. Let U be an open connected neighborhood of t_0 in W. Then $\gamma(U) \subset K(p, U)$, and γ, being also $\mathcal{M}$-continuous, is a Feynman path.

Obviously P is first countable, since a countable neighborhood base at $p \in M$ is given by $\{L_u(p, 1/n); n = N, N+1 \cdots\}$ for some integer N. Because M admits a countable covering $\{U_i\}$ by normal neighborhoods, P is also separable (take points with rational coordinates in U_i). Note that sets of the form $K(p, U)$, $I^*(p, U)$, $I^-(p, U)$ and $L_u(p, \epsilon)$ are P-path connected, since any pair of points in any of these sets can be joined by product curves $\gamma_1\gamma_2$ or $\widehat{\gamma_1\gamma_2}$ of the type discussed in 3.5. Thus P is locally path connected, because every P-neighborhood E of $P \in M$ contains a neighborhood of the form $L_U(p, \epsilon)$. Also it is not hard to show that P is path connected. Indeed, since M is $\mathcal{M}$-connected and M is a manifold, M is $\mathcal{M}$-path connected. If p, $q \in M$ are any pair of points, they may be joined by an $\mathcal{M}$-continuous path $\gamma: I \to M$ from the closed unit interval I into $\mathcal{M}$. It is

then easy to approximate $\gamma(I)$ by a Feynman path, showing that M is P-path connected. Before proving further general properties, we need:

Proposition 4.3: $\overline{L_u(p, \epsilon)}^P = L_u(p, \epsilon) - (\partial B_u(p, \epsilon) \cap \partial K(p, U))$. (Here the lefthand side denotes the P-closure, and all other topological symbols refer to $\mathcal{M}$.)

Proof: Certainly $\overline{L_u(p, \epsilon)}^P \subset L_u(p, \epsilon)$ since P is finer than $\mathcal{M}$. Now points $q \in \partial B_u(p, \epsilon) \cap \partial K(p, U)$, have P-open neighborhoods of the form $L_u(q, \frac{1}{2}\epsilon)$ which do not meet $L_u(p, \epsilon)$, so $L_u(p, \epsilon)^P \subset L_u(p, \epsilon) - (\partial B_u(p, \epsilon) \cap \partial K(p, U))$. If $r \in M$ is in this latter set, every P-neighborhood of r contains a P-neighborhood of the form $L_u(r, \delta)$ which is easily seen to meet $L_u(p, \epsilon)$. ∎

Theorem 3: P is first countable and, separable. P is Hausdorff, path connected and locally path connected (and so *a fortiori* connected and locally connected). However, P is not regular, normal, locally compact or paracompact.

Proof: The first sentence has been dealt with above. P is Hausdorff because P is finer than $\mathcal{M}$. The connectivity properties have been dealt with above. To show that P is not regular, consider the P-neighborhood $L_U(p, \epsilon)$ and show that p has no P-closed neighborhood S contained in $L_U(p, \epsilon)$, using the basis property and 4.3. To show that P is not normal, consider the disjoint P-closed subsets $\overline{L_U(p, \epsilon)}^P$ and $[\partial B_U(p, \epsilon) \cap \partial K(p, U)]$, and show, using the basis property, that these sets cannot have disjoint P-open neighborhoods. To show that P is not locally compact, use the fact that closed subspaces of compact sets are compact, the basis property and 4.3, and note that $\overline{L_u(p, \epsilon)}^P$ is not $\mathcal{M}$-closed, so certainly not $\mathcal{M}$-compact, hence not P-compact, since P is finer than $\mathcal{M}$. P cannot be paracompact because paracompact spaces are normal.[11] ∎

5. P AND CHRONOLOGICAL STRUCTURE

We wish to prove P-homeomorphisms h, take timelike curves to timelike curves. Obviously h takes P-continuous curves to P-continuous curves, but, of course, P-continuous curves (Feynman paths) need not be timelike. We single out a subclass of P-continuous curves, by adding restrictions made only in terms of P. This subclass will coincide with timelike curves. This will enable us to prove that for strongly causal spacetimes P-homeomorphisms preserve or reverse causal relations.

Definition 5.1: A path $\gamma: F \to M$ is said to be *regular*, if and only if:

(A) γ is P-continuous and locally 1–1.

(B) For every $t_0 \in F$, there is a connected neighborhood U of t_0 and a P-neighborhood Π_0 of $p = \gamma(t_0)$, such that:

(1) $\gamma(U) \subset \Pi_0$

(2) Whenever $t_0 \in \text{int}F$ (the interior of F) and $a, b \in U$ satisfy $a < t_0 < b$, every P-continuous curve in Π_0 joining $\gamma(a)$ to $\gamma(b)$ contains $p = \gamma(t_0)$.

Proposition 5.1: A P-homeomorphism takes regular paths to regular paths.

Proof: The definition of regularity involves only set theoretical and P-topological notions. Since h is a P-homeomorphism, it preserves all required properties. ∎

Theorem 4: $\gamma : F \rightarrow M$ is timelike, if and only if, γ is regular.

Proof: Suppose first that γ is timelike and, for definiteness, future directed. By $\underline{4.1}$ γ is P-continuous, so satisfies (A) of the regularity condition. For $t_0 \in \text{int}F$ let U and U be as in $\underline{3.2}$. Set $\Pi_0 = K(p, U)$ where $p = \gamma(t_0)$. From $\underline{4.2}$, Π_0 is a P-neighborhood of $p = \gamma(t_0)$. Since γ is future directed and timelike, $\gamma(U) \subset \Pi_0$. Suppose next that $a, b \in U$ satisfy $a < t_0 < b$. Since γ is future directed $\gamma(a) \in \Gamma^-(p, U)$ and $\gamma(b) \in \Gamma^+(p, U)$. Let $\zeta = [r, s] \rightarrow \Pi_0$ be a P-continuous path with $\gamma(a) = \zeta(r)$ and $\gamma(b) = \zeta(s)$. Since ζ is P-continuous, ζ is, by $\underline{4.1}$, continuous, so $\zeta([r, s])$ is connected. If $p \notin \zeta([r, s])$, $\zeta([r, s]) \subset \Gamma^+(p, U) \cap \Gamma^-(p, U)$, then $\zeta([r, s])$ is contained in a disjoint union of open sets, and meets both. This contradicts the connectivity of $\zeta([r, s])$, so in fact, $p \in \zeta([r, s])$ and γ is regular. An analogous proof holds if γ is past directed and timelike, so the first half of the proposition is proved.

Suppose next that γ is regular, and that Π_0 is a P-neighborhood of $p = \gamma(t_0)$ satisfying the required conditions. Because γ is P-continuous, γ is continuous. From Theorem 1, p has a P-neighborhood of the form $L_{u'}(p, \epsilon)$ contained in Π_0. Choose an open convex normal neighborhood U of p with $U \subset B_{u'}(p, \epsilon)$. Then the P-neighborhood $K(p, U)$ of p is contained in $L_u(p, \epsilon)$. Since γ is P-continuous, $\gamma^{-1}(K(p, U))$ is a neighborhood of t_0. Let $U \subset \gamma^{-1}(K(p, U))$ be a connected neighborhood of t_0 such that $\gamma : U \rightarrow M$ is 1–1. Then $\gamma(U) \subset K(p, U)$. Assume now that $t_0 \in \text{int}F$, and choose $a, b \in U$ with $a < t_0 < b$. Since γ is 1–1 on U, both $\gamma(a)$ and $\gamma(b)$ lie in $\Gamma^+(p, U) \cup \Gamma^-(p, U)$. Assume for definiteness that $\gamma(a) \in \Gamma^-(p, U)$. Then $\gamma(b)$ cannot belong to $\Gamma^-(p, U)$ also, since otherwise, in view of the P-path connectivity of $\Gamma^-(p, U)$, there would be a P-continuous path in $\Gamma^-(p, U)$ which joins $\gamma(a)$ to $\gamma(b)$ but which does not contain p since $p \notin \Gamma^-(p, U)$. Therefore, $\gamma(a) \in \Gamma^-(p, U)$ and $\gamma(b) \in \Gamma^+(p, U)$. Let U_0^- and U_0^+ denote the disjoint connected intervals $\{t \in U \mid t < t_0\}$ and $\{t \in U \mid t > t_0\}$, respectively. Since γ is 1–1, $\gamma(U) \subset K(p, U)$, and $\gamma(U_0^-)$ and $\gamma(U_0^+)$ both belong to the disjoint union $\Gamma^-(p, U) \cup \Gamma^+(p, U)$ of open sets. But γ is P-continuous, hence continuous, therefore $\gamma(U_0^-)$ and $\gamma(U_0^+)$ are connected. However, we have just shown that $\gamma(U_0^-)$ meets $\Gamma^-(p, U)$ and $\gamma(U_0^+)$ meets $\Gamma^+(p, U)$, so in fact $\gamma(U_0^-) \subset \Gamma^-(p, U)$ and $\gamma(U_0^+) \subset \Gamma^+(p, U)$, therefore γ is future directed and timelike at $t_0 \in \text{int}F$ (or past directed and timelike at $t_0 \in \text{int}F$). Applying the same reasoning to each $t_0 \in \text{int}F$, we conclude that γ is 1–1 and timelike at each $t_0 \in \text{int}F$ and therefore, by proposition 3.3, γ is a timelike path ∎

Here and henceforth, M is always assumed chronological.

Proposition 5.3: A P-homeomorphism h takes cones $C(p)$ bijectively onto cones; $h(C(p)) = C(h(p))$.

Proof: There is a timelike path γ joining q to p if and only if $q \in C(p)$. By proposition $\underline{5.1}$ and Theorem 4, the image path $h_0\gamma$ is timelike, and it joins $h(q)$ to $h(p)$.

Therefore $h(q) \in C(h(p))$, and since h is bijective, it maps $C(p)$ bijectively onto $C(h(p))$.

Proposition 5.4: For a fixed $r \in M$, a P-homeomorphism h maps $\Gamma^+(r)$ [respectively $\Gamma^-(r)$] bijectively onto either $\Gamma^+(h(r))$ [respectively $\Gamma^-(h(r))$] or $\Gamma^-(h(r))$ [respectively $\Gamma^+(h(r))$].

Proof: Suppose first that there is a $p \in \Gamma^+(r)$ with $h(p) \in \Gamma^+(h(r))$ and let $q \in \Gamma^-(r)$. Join p to r and r to q by past directed timelike paths η, respectively; and form a past directed product path $\gamma = \eta\zeta$. The image curves $h_0\eta$, $h_0\zeta$, and $h_0\gamma$ are timelike, and since $h(p) \in \Gamma^+(h(r))$, $h_0\eta$ is past directed. Therefore $h_0\gamma$ is past directed and timelike. Hence, $h(q) \in \Gamma^-(h(r))$ for every $q \in \Gamma^-(r)$. A similar construction starting with a given $q \in \Gamma^-(r)$ shows that $h(s) \in \Gamma^+(h(r))$ for all $s \in \Gamma^+(r)$. This completes the "either" part of the proof. If there is no $p \in \Gamma^+(r)$ with $h(p) \in \Gamma^+(h(r))$, a similar construction gives the remainder of the proof. ∎

Proposition 5.4: The previous proposition holds with "a fixed $r \in M$" replaced by "every $r \in M$."

Proof: Suppose that, for a given $r \in M$, h preserved time orientation, $h(\Gamma^+(r)) = \Gamma^+(h(r))$ and $h(\Gamma^-(r)) = \Gamma^-(h(r))$. Let $A \subset M$ be the set of points at which h preserves time orientation. Then certainly $p \in A$ whenever $C(p) \cap C(r) \neq \emptyset$. We assert that A is M-open. Indeed, choose a point $s \in \Gamma^+(r)$. Then $r \in \Gamma^-(s)$ and since $\Gamma^-(s)$ is open, there is an open neighborhood U of r with $U \subset \Gamma^-(s)$. Then for any $q \in U$, $\Gamma^+(q) \cap \Gamma^+(r)$ contains s, so *a fortiori* $s \in C(q) \cap C(r)$. Therefore $q \in U \subset A$, and A is M-open. The set $M - A$ of points at which h is time orientation reversing is also open. But M is connected and $r \in A$, so $M = A$. If there is no $r \in M$ satisfying the above condition, every point satisfies the opposite condition. ∎

Proposition 5.5: A P-homeomorphism is an A-homeomorphism.

Proof: This follows immediately from 5.4 and the definition of the Alexandroff topology.

6. P AND CAUSAL, DIFFERENTIAL AND CONFORMAL STRUCTURE

The fact that P-homeomorphisms preserve or reverse chronological relations enable us to prove that, for strongly causal space—times, they locally preserve or reverse causal relations. In particular, they preserve null geodesics. It is then shown that P-homeomorphisms are diffeomorphisms and, since they preserve null cones, conformal diffeomorphisms.

Proposition 6.1: Suppose now and henceforth M is strongly causal. Then a P-homeomorphism h is M-homeomorphism, and h maps null geodesic curves to null geodesic curves.

Proof: Since M is strongly causal, it is *a fortiori* chronological, hence h is a A-homeomorphism. Strong causality also implies $A = M$, so h is an M-homeomorphism. Let D be a causality neighborhood of $r \in M$ and $U \subset D$ an open convex normal neighborhood of r. Then causality relations *restricted* to D, agree with causality relations *relative* to D. Now, in $U \subset D$ the horismos relations can be expressed in terms of chronology

relations since, for $p, q \in U$, $q \to p$ is and only if $[q \nleqslant p$ and $z < q \Rightarrow z < p]$. Since h is a M-homeomorphism, $h(U)$ and $h(D)$ are M-open. Let U', D' be respectively, an open convex normal neighborhood and a chronology neighborhood of $h(r)$ with $U' \subset D' \subset h(U)$. In $V \equiv U \cap h^{-1}(U')$, h will preserve chronology (and hence horismos) relations or reverse them. Suppose γ is the unique null geodesic curve joining $s, t \in U \cap h^{-1}(U')$. Then $t \in \gamma$ if and only if $r \to t \to s$. Therefore $h(t)$ satisfies $h(r) \to h(t) \to h(s)$ or $h(s) \to h(t) \to h(r)$, that is $h(t)$ lies on a null geodesic joining $h(r)$ to $h(s)$. Hence, $h(\gamma)$ is a null geodesic curve in $h(U) \cap U'$. ∎

Theorem 5: A M-homeomorphism $h: M \to M$, which takes null geodesic curves to null geodesic curves is a C^∞ diffeomorphism.

Proof: (This is a theorem of Hawking[4] which is given here in an improved from since it has never been published.) Let U be a convex normal neighborhood and $\gamma_i: F_i \to U (i = 1-4)$ be four C^∞ null geodesic paths such that:

(1) For each $t_1 \in F_1$, there is a unique null geodesic curve λ in U joining the point $\gamma_1(t_1)$ to the null geodesic curve γ_2.

(2) For each $t_3 \in F_3$ there is a unique point $q \in \lambda$, such that q and $\gamma_3(t_3)$ lie on a null geodesic curve in U.

(3) For each point $q \in \lambda$ there is a unique $t_4 \in F_4$, such that q and $\gamma_4(t_4)$ lie on a null geodesic curve in U.

(4) The map $\psi: F_1 \times F_3 \to F_4$ defined by $\psi(t_1, t_3) = t_4$, where t_1, t_3 and t_4 are as in (1)–(3) above is C^∞, and is such that $\partial \psi / \partial t_1$ and $\partial \psi / \partial t_3$ are nonzero.

For a sufficiently small neighborhood U, the metric differs by an arbitrarily small amount from that of Minkowski space. Comparison with Minkowski space shows that γ_i can be chosen to satisfy the above conditions. Condition (4) cannot be satisfied in less than three dimensions.

By the assumption of the theorem $h(\gamma_i)$ will be four null geodesic curves contained in $h(U)$. Thus one can find four C^∞ paths $\tilde{\gamma}_i: \tilde{F}_i \to h(U)$ which define the same null geodesic curves as $h(\gamma_i)$, but which may be parametrized differently. Let $\hat{h}_i: F_i \to \tilde{F}_i$ be defined by $\hat{h}_i = \tilde{\gamma}_i^{-1} h \gamma_i$. The maps $\hat{h}_i$ will be continuous and monotonic (because h preserves or reverses ordering). Therefore, by Lesbegue's theorem, they will be differentiable almost everywhere. Let $\tilde{\psi}: \tilde{F}_1 \times \tilde{F}_3 \to \tilde{F}_4$ be defined similarly to ψ. Then

$$\hat{h}_4(\psi(t_1, t_3)) = \tilde{\psi}(\hat{h}_1(t_1), \hat{h}_3(t_3)). \tag{I}$$

Differentiating (I) with respect to t_3 one has

$$\hat{h}_4'(\psi(t_1, t_3)) \frac{\partial \psi}{\partial t_3} = \frac{\partial \tilde{\psi}}{\partial \tilde{t}_3} \hat{h}_3'(t_3). \tag{II}$$

Because $\hat{h}_3$ is differentiable almost everywhere, it follows from property (4) that $\hat{h}_4'$ exists and is continuous. Similarly, by choosing different combinations of null geodesic paths one can show that each $\hat{h}_i$ is C^1. Now, differentiating (II) with respect to t_1 gives

$$\hat{h}_4''(\psi) \frac{\partial \psi}{\partial t_1} \frac{\partial \psi}{\partial t_3} + \hat{h}_4'(\psi) \frac{\partial^2 \psi}{\partial t_1 \partial t_3} = \frac{\partial^2 \tilde{\psi}}{\partial \tilde{t}_1 \partial \tilde{t}_3} \hat{h}_1' \hat{h}_3'.$$

Therefore, $\hat{h}_4$ is C^2. By repeating the above process, it may be shown that each $\hat{h}_i$ is C^∞. In other words, h maps a C^∞ parameter on a null geodesic curve to a C^∞ parameter.

Let $\gamma_i: F_i \to U$ be four C^∞ null geodesic curves, and $W \subset U$ be a neighborhood such that the map $\Gamma: W \to \{$an open set of $R^4\}$ defined by $\Gamma(q) = \gamma_i^{-1}(I'(q, U))$ is a C^∞ diffeomorphism. (Comparison with Minkwoski's space shows that this is possible for W sufficiently small.) Pairs of the form (W, Γ) form a C^∞ atlas for M which is preserved by h. Thus h is a C^∞ diffeomorphism. ∎

Corollary: A P-homeomorphism is a C^∞ diffeomorphism. ∎

Theorem 6: A P-homeomorphism h is a smooth conformal diffeomorphism.

Proof: Since h is a diffeomorphism which, locally, preserves null cones, and the metric g at $p \in M$ is determined up to a constant by the tangent null cone, h preserves the metric up to a constant factor which must, since h is smooth, be smooth. ∎

Theorem 7: The group Homeo (M, P) of P-homeomorphisms of M coincides with the group G of conformal diffeomorphisms of M.

Proof: By Theorem 6, Homeo $(M, P) \subset G$, and it remains to prove the opposite inclusion. Suppose $E \in P$, so that whenever γ is timelike, $E \cap \gamma = O \cap \gamma$ for some $O \in M$. Then if $g \in G$ $g(E) \cap g(\gamma) = gO \cap g\gamma$. But $g\gamma$ is timelike because h is conformal, and gO is M-open. Therefore, $gE \in P$, and g is P-open. Similarly, g is P-continuous so $g \in$ Homeo (M, P). ∎

It is instructive to give an example of a manifold for which G is strictly larger than the homothecy group. This is not the case for Minkowski space because, though the infinitesimal conformal group is larger than the infinitesimal homothesy group, the infinitesimal conformal group cannot be exponentiated to give a global action on Minkowski space. However, consider the manifold N obtained by removing the following closed set S from Minkowski space M.

$$S = \{q \in M \mid [x^0(q)]^2 - [x^1(q)]^2 - [x^2(q)]^2 - [x^3(q)]^2 \geq 0\}.$$

The conformal group of this manifold is generated by the homogeneous Lorentz group (including space, time, and space–time reversal), dilatations, and the inversion I given, in coordinates, by

$$x^\mu(I(q)) = \frac{x^\mu(q)}{[x^0(q)]^2 - [x^1(q)]^2 - [x^2(q)]^2 - [x^3(q)]^2},$$

$(\mu = 0, 1, 2, 3)$.

In fact, infinitesimal conformal diffeomorphisms which are not infinitesimal isometries are rather rare. DeFrise-Carter[6] has shown that the infinitesimal conformal diffeomorphisms of Lorentz manifolds are with two exceptions, essentially isometries. The exceptions are Minkowski space and the "plane wave" space–times. In the former, there are five linearly independent infinitesimal conformal transformations which are not isometries (the dilatations and "accelerations"),

and in the latter, only one (the dilations). Only the homothecy group acts globally on Minkowski space, but N admits global conformal transformations which are neither isometries nor homothesies.

ACKNOWLEDGMENTS

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[7] *Added in proof*: We are grateful to Dr. M. Dobson for pointing out that the inverted commas on "times" are essential. The observor does not measure the length of a time interval—many experiments are required to determine whether a set is open.
[8] *Added in proof*: Rüdiger Göbel informed us that he has a modification of the general relativity analog of $\mathcal{J}$ which allows the effects of a *fixed* electromagnetic field to be incorporated. We feel it is preferable to use $\mathcal{P}$, thus allowing *all* timelike curves to be continuous (not just geodesics or particles with a fixed charge in a fixed field).
[9] *Added in proof*: Actually the Zeeman topology, and Göbel's generalization admit spacelike curves as continuous curves.
[10] *Added in proof*: We may also assume $\mathcal{U}$ to be an open convex normal neighborhood of each of its points.
[11] *Added in proof*: It may also be of interest to note that $\mathcal{P}$ is not metrizable, since it is separable but not regular, and neither can $\mathcal{P}$ arise from a uniformity, since it is not regular, therefore certainly not completely regular.

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