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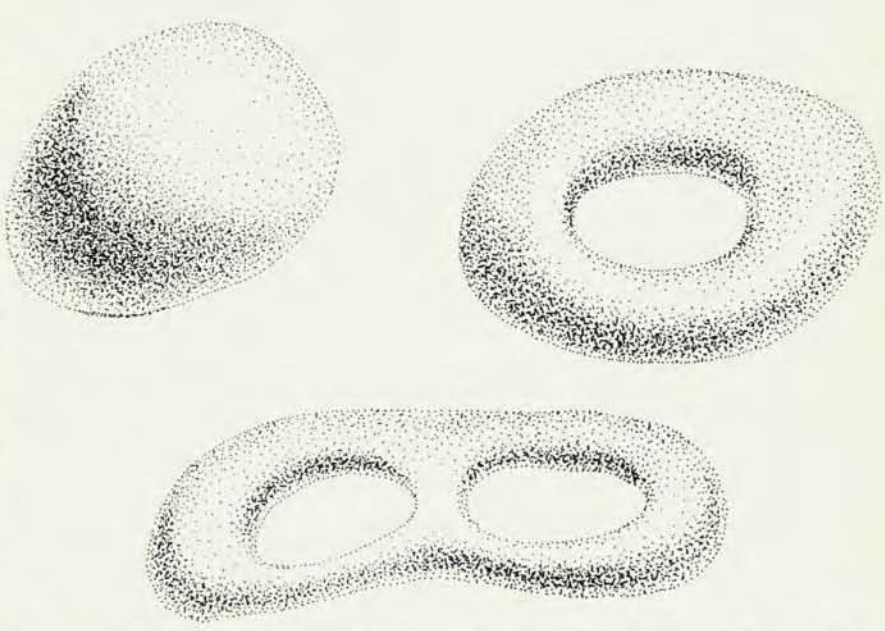
Among intellectual disciplines, mathematics occupies a unique position. It is in many respects an art, but it is also the language of science. Although a great deal of mathematics can be traced directly to external influences, much of its creativity is motivated internally: "pour la gloire de l'esprit humain," as Carl G. T. Jacobi put it. Often the mathematician lets the imagination soar, constrained only by logic, intrinsic structure, and a sense of historical continuity. Yet from time to time these abstract deliberations have important applications in other fields.

The general theory of relativity is one well-known example. In the first half of the nineteenth century, Karl Friedrich Gauss and his pupil Bernhard Riemann laid the foundations of a general theory of curved spaces of arbitrary dimensions. These ideas were taken up by several Italian mathematicians, including Curbastro Gregorio Ricci and his pupil Tullio Levi-Civitá; the tensor calculus they developed became the principal analytical tool of Riemannian geometry. These researches had no apparent connection with physical reality until some time after 1907, when Albert Einstein with his friend Marcel Grossmann recognized in them the appropriate framework for a relativistic theory of gravitation.

In the past few years, mathematicians and physicists may be witnessing a similarly miraculous confluence of ideas. On the one end are the physically motivated gauge theories, developed to deal in a unified way with electromagnetic, weak and strong interactions; on the other end is an internally motivated extension of Riemannian geometry, involving the notion of fiber bundles. It came as a revelation in the mid-1970s to many mathematicians and physicists alike that gauge theories used connections (vector potentials) on fiber bundles. One of the principal architects of gauge theories, Chen-Ning Yang, wrote, "I found it amazing that gauge fields are exactly connections on fiber bundles, which the mathematicians developed *without reference to the physical world*."



Topological structures with different integral curvatures. The integral curve k of a surface, which can be derived from purely local measurements, is a topological invariant, remaining constant no matter how the structure is locally distorted. For the sphere the value of k is 4π; for the torus it is zero; for the two-holed torus it is -4π. (Drawings for this article by Louis Fulgoni.)

Geometry and fiber bundles

To understand the concept of a fiber bundle and its role in modern global geometry it will be helpful first to review classical differential geometry.

Gauss made the pivotal discovery of a curvature at each point of a surface that can be calculated from suitable measurements—angles and lengths in triangles—in a small region. The notion of Gaussian curvature, in other words, is a purely "local" concept. For example, one can show the Earth is round without circumnavigating the globe and without photographs from outer space—as Eratosthenes did by comparing shadows in Alexandria and Syene. Circumnavigation or views from outer space bring out the overall

Isadore M. Singer is professor of mathematics at the University of California, Berkeley.

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or "global" structure of a surface, which is the concern of topology. Thus, for example, if a sphere is locally distorted by bumps and dips it remains globally a sphere. Topologically a sphere, a plane and a torus remain distinct, even when each is somewhat distorted.

The connection between topology and differential geometry is given by "global geometry," which tries to obtain information about the topology—the overall shape—of a space from local measurements made throughout the space. For example, we try to determine the shape of our universe from accessible measurements, without stepping out of the universe. The most stunning example in the theory of surfaces is the celebrated Gauss–Bonnet theorem, which says that the integral of the Gaussian curvature over an entire surface is a topological invariant, and is in fact an integer multiple of 2π. For a sphere, no matter how distorted, the integral curvature is 4π; for a torus it is zero; while for the "double-holed torus" shown on the previous page it is -4π.

Auxiliary spaces are useful in studying ordinary surfaces and their higher-dimensional analogs. One example is the space consisting of the tangent planes to a surface; others, shown in the illustration below, are the circles of unit radius in the tangent planes (on the left), and the lines normal to the surface (in the figure on the right).

Such spaces are called "fiber bundles." The "fibers" are the auxiliary spaces—the tangent planes, unit circles, normal lines or whatever—and the "bundle" is the totality of fibers as they fit together. The fibers can be complicated higher-dimensional surfaces, and the bundles can be just as complicated. For example, the dimension of the normal fibers to a surface depends on the dimension of the surrounding space: for a surface in a four-dimensional space the normals are planes and the fibers are thus two-dimensional.

Until the 1930s differential geometry was mainly concerned with tangent bundles and their associated tensor bundles. At that time, Hassler Whitney studied how surfaces sit in higher-dimensional spaces and was led to consider normal bundles and then more general bundles completely unrelated to the tangent bundle.

Fiber bundles have become so common in high-energy physics (grand unification schemes, symmetry breaking, dimensional reduction) that a short digression into their geometry is in order.

On the circle (a one-dimensional sphere, S^1) we can construct two fiber bundles whose fibers are circles; one of these is trivial; the other one is not. The first is a torus, which we can construct by taking a circle and bringing it around the base circle. As we close the torus we are effectively giving a way to connect the last circle to the first one: identify each point at an angle θ with the corresponding point on the other circle. The second fiber bundle is a Klein bottle, which we again construct by carrying a circle around the base circle, but now when the fiber circle is brought back to the starting point we make the identification $\theta \leftrightarrow -\theta$ instead of $\theta \leftrightarrow \theta$. (Essentially we are flipping the circle around before completing the space, in the same way that we flip a tape around before pasting its ends together to make a Möbius strip.)

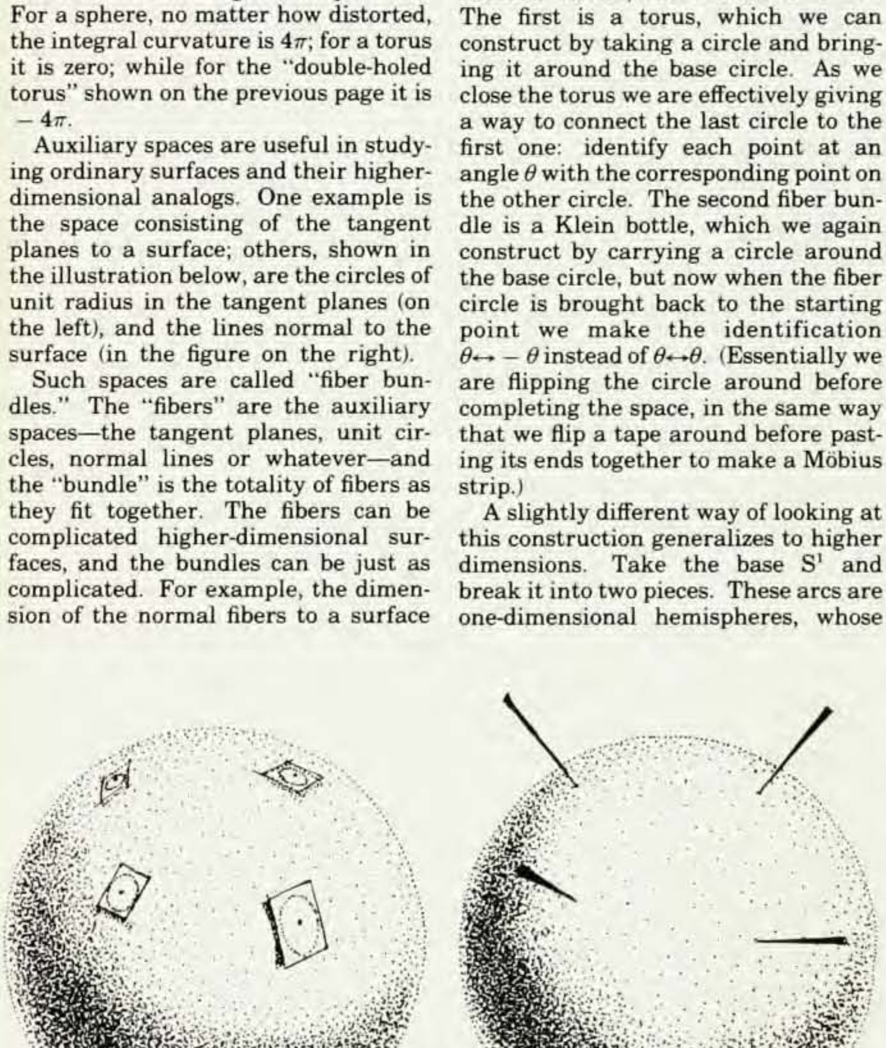
A slightly different way of looking at this construction generalizes to higher dimensions. Take the base S^1 and break it into two pieces. These arcs are one-dimensional hemispheres, whose "equator" consists of two points. We construct fiber bundles on each arc. To assemble a fiber bundle on the full circle the only question is how to patch the circles at the "equator." At one end we patch with the identity. At the other end we can patch either with $\theta \leftrightarrow \theta$, giving the torus, or with $\theta \leftrightarrow -\theta$, giving the Klein bottle.

We can construct circle bundles on a two-sphere S^2 (such as the surface of the Earth) in a similar manner. The upper and lower hemispheres have a common boundary, the equator, measured by the meridian angle θ. Place a circle at each point on the two hemispheres, and patch the circle from the upper hemisphere at longitude θ with the circle from the lower hemisphere by rotating through the angle θ.

Other circle bundles could be constructed by rotation through angles $k\theta$ for any whole number k. All the circle bundles can be made this way. For example the bundle of tangent circles for the two-sphere corresponds to $k = 2$. It is remarkable that Paul Dirac found this classification of circle bundles on the two-sphere while studying magnetic monopoles in 1932, outside the stream of mathematical development.

Dirac started with Maxwell's equations for the electromagnetic field $F_{\mu\nu}$. (The relativistic field $F_{\mu\nu}$ has as its components the electric and magnetic field vectors.) In a concise mathematical notation we can write Maxwell's equations as $dF=0$ and $d{*}F=j$, where j is the four-component charge and current density. The first of these equations allows us to define a vector potential, so that $F=dA$.

To obtain a magnetic monopole, Dirac interchanged the electric and magnetic fields to obtain $dF=\rho_m$ and $d{*}F=0$. He studied the static case for a point magnetic pole at the origin. Away from the origin—say on a two-sphere of positive radius—the magnetic pole density, ρ_m, vanishes, so that $dF=0$. It would be tempting to conclude that $F=dA$ on the entire two-sphere, but that cannot be done because the total magnetic flux through the sphere is not zero (as it would have to be if the field were the curl of a vector potential). However, on the upper and lower hemispheres separately we can write $F=dA$, with a potential A_+ for the upper and A_- for the lower hemisphere. Unfortunately A_+ and A_- will not agree along the equator. It is the pasting of circles along the equator described above that compensates for the mismatch of A_+ and A_- and gives a vector potential on the circle bundle.



Fiber bundles on the surface of a sphere: at left, the tangent planes and unit circles on the tangent planes, on the right, normal lines (only a few of the "fibers" are shown in each case). In a higher-dimensional space, the normal fibers would also have a higher dimensionality.

Differential geometry and physics

The classification of bundles and their invariants, known as characteristic classes, proceeded in the 1930s and

1940s with the work of Whitney, Eduard Stiefel, L. S. Pontrjagin, and Shiing Chern. Among other things, they found interesting integral formulas for the invariants of bundles, generalizing the Gauss–Bonnet formula.

In the 1970s, these global invariants cropped up in physics. To explain how, we need to explore the idea of curvature a bit further. An observer carrying a frame of reference or clock around a closed curve can compare his frame with that of an observer left behind. How much the frames differ measures the average curvature over a surface bounded by the curve. This curvature for space–time, according to Einstein, is the gravitational field.

The electromagnetic field can also be thought of as a curvature, one associated with a circle bundle. The observer carries a circle along with him and records the angle of rotation when he returns home, thus measuring the average field in a surface bounded by the curve. Specifically, the $\mu\nu$ component of the electromagnetic field tensor is the infinitesimal rotation an observer experiences on traversing an infinitesimal square along the μ and ν coordinate directions.

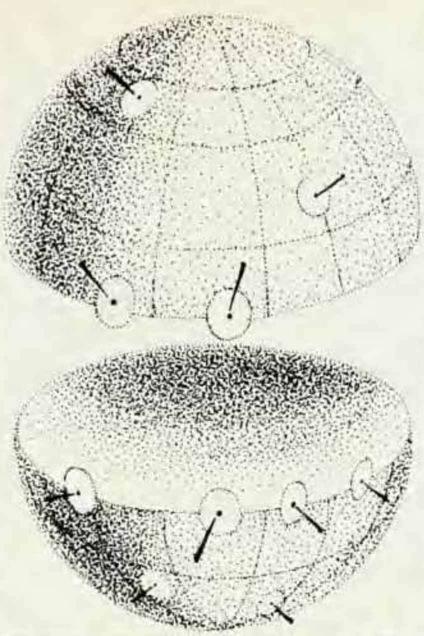
In a general gauge theory the circle of rotations in the plane representing phase shifts is replaced by more complicated symmetry groups of rotations in higher-dimensional planes. It is generally believed that the theories of the weak force and the strong force, and their unifications with electromagnetism will be gauge theories with an appropriate internal symmetry group. For example, Yang and Richard Mills introduced an SU(2) gauge theory in 1954 to study the symmetry ("isospin") of the two-component object of proton/neutron, ignoring the electric charge.

In gauge theories one studies all the ways of transporting a complicated symmetry group along curves in space–time and examines their associated curvature (or force) fields. Gauge theories break away from space–time in much the same way that fiber bundles break away from the tangent bundle. Space–time has global symmetries, the Poincaré group. But electromagnetism and the weak and strong forces have additional local internal symmetries giving extra degrees of freedom—the fibers in the bundle of local symmetries. It is natural, then, that fiber bundles are an appropriate framework for gauge theories.

The result of infinitesimal transport of an object with internal symmetries is given by a vector potential or Yang-Mills field

$$A_\mu(x) = A^a{}_\mu(x)T_a$$

where T_a are generators of a Lie algebra, say sU(N), the skew-adjoint $n \times n$ matrices of trace 0. (As is usual in



Circle bundles on two hemispheres. When the hemispheres are combined to form a full sphere the two equators—and the circle bundles on them—must be identified. Exactly how they are identified determines the topology of the circle bundle on the complete sphere.

these cases, we are using the summation convention, in which repeated indices are summed over.) In the case of electromagnetism the index has only a single value, and $T_a = ie/\hbar c$.

We can define the "curvature" field of the vector potential in a way that is analgous to the case of gravity: the curvature tensor or field strength is determined from parallel transport around infinitesimal rectangles. The result for the field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A^a{}_\mu T_a, A^a{}_\nu T_a]$$

(we have used the common notation ∂_μ for $\partial/\partial x_\mu$). Note that the commutator term makes the fields nonlinear functions of the vector potentials. All the trouble and all the interest stem from the nonlinear term. It is a feature of the non-Abelian Lie algebra.

For example, in electromagnetism a gauge transformation $\phi(x)$ (that is, a function with values in the symmetry group) transforms the vector potential A_μ to $A_\mu + \phi^{-1}\partial_\mu\phi$ and does not affect the field strength. For a non-Abelian group A_μ is changed to $\phi^{-1}A_\mu\phi + \phi^{-1}\partial_\mu\phi$ and $F_{\mu\nu}$ is changed to $\phi^{-1}F_{\mu\nu}\phi$.

To obtain a quantum field theory one starts with a classical Lagrangian density. For pure gauge theories (without matter fields) the Lagrangian action functional is the Lebesgue L^2 norm of the field strength:

$$S(A) = -1/4 \int d^4x \, tr(F_{\mu\nu}F_{\mu\nu})$$

a direct generalization of quantum electrodynamics. (The trace refer to the matrices T_a, which generate our Lie algebra.) Because the field strength is nonlinear in the potential, pure non-Abelian gauge theories are already far from trivial. The gauge fields divide themselves into topologically distinct disconnected classes determined by an integer k, often called the topological charge (a characteristic class mentioned above.) It is given by

$$k = (-1/8\pi^2) \int d^4x tr(\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta})$$

where ϵ is the completely skew-symmetric Levi-Civitá permutation symbol. Heretofore in physics, the discreteness in quantum mechanics, represented by quantum numbers, came from eigenvalues of operators and ultimately from symmetries of groups. There is now the intriguing possibility that some quantum numbers and conservation laws may be topological in nature. In any case, the physicist, in calculating physical quantities, integrates over all configurations of the gauge field. The fact that these fields split into distinct topological classes means the vacuum structure is nontrivial.

Many problems in quantum field theory can be formulated as an evaluation of the Feynman–Kac path integral:

$$\langle f(A) \rangle = \left[\int_A DAe^{-S(A)}\right]^{-1} \int_A DA\, f(A)e^{-S(A)}$$

The function f is to be invariant under gauge transformations and the integral is to be taken over all field configurations. In fact, one of the fundamental problems of quantum field theory is to make sense of the path integral: One really doesn't know what "integration over all fields" means. In any case, the major contributions to $\langle f \rangle$ will presumably come from the points where the action is stationary. The saddle-point method fixes on these stationary points. One writes the action as a quadratic term plus a remainder and uses a perturbative expansion for the remainder. This program emulates quantum electrodynamics where the method has been spectacularly successful because of the small coupling constant.

The stationary points of the action are determined by the Euler–Lagrange equations. In this case the equations of motion are

$$\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0$$

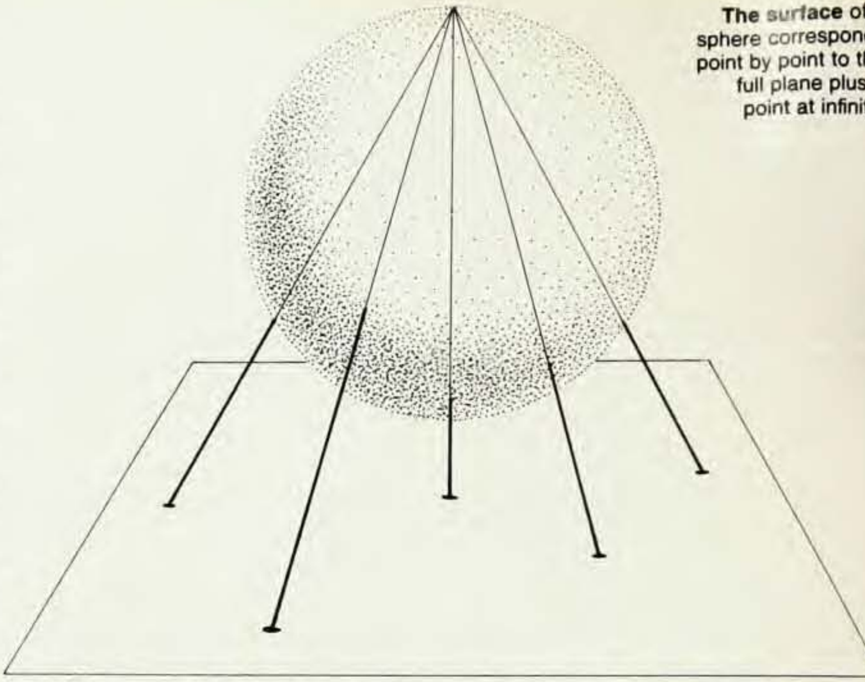
For quantum electrodynamics these, together with the automatic Bianchi identity, are just the free-space Maxwell equations. For the non-Abelian fields the equations are, of course, nonlinear. If perturbative methods are used to evaluate the integral, it is essential to find the critical, or stationary, points of the action.

A. Belavin, A. Polyakov, A. Schwartz, Y. Tyupkin and Y. I. Manin in the Soviet Union and Gerhard 't Hooft in Holland found special solutions giving local minima, of these nonlinear global partial differential equations on R^4, Euclidian four-dimensional space. These solutions have singularities at a single point and are called instantons or pseudoparticles. (An "ansatz" for them is given in terms of harmonic functions, say $\lambda/|x-p|^2$.) The singularities can be "gauged away" that is, one can find a phase transformation $\phi(x)$, smooth except at the singular point, p, such that the transformed vector potential, $\phi^{-1}A_\mu\phi + \phi^{-1}\partial_\mu\phi$ is not singular at p. The transformed vector potential is smooth near p, and A_μ is smooth everywhere else including infinity, but the two potentials have a mismatch on a three-sphere S^3 bounding a small ball around p. To compensate for this mismatch we have to construct a fiber bundle on S^4 (four-space plus infinity—see the figure at right) whose fibers have the symmetry SU(2). As in the case of the Dirac monopole, the bundle is constructed by pasting fibers along the boundary-sphere S^3 using the gauge transformation $\phi(x)$. In this way one obtains a vector potential on the entire fiber bundle that has no singularities.

More generally, Karen Uhlenbeck has shown that any solution with finite action and simple (meaning isolated) singularities really lives on some SU(2) fiber bundle over the four-sphere. If fiber bundles had not been invented earlier, they would have been, in the 1970s, to describe these instanton solutions (just as Dirac found the circle bundles on the two-sphere).

The solutions locally minimizing the action are called self-dual solutions, and their degrees of freedom can be calculated. It turns out that there are more self-dual solutions than there are different kinds of instantons. What do they look like? It was observed that the problem of finding self-dual solutions could be transformed into a problem in algebraic geometry. That problem amounts to finding all algebraic plane bundles over CP^3, complex projective three-space (the complex lines in complex four-space). This startling observation is based on the Penrose twistor program, a way of looking at massless particles in physics that is very different from the traditional view. Happily enough, algebraic geometers had independently been studying the classification problem of algebraic plane bundles over CP^3, and all the self-dual solutions have been found [by M.F. Atiyah, V. G. Drinfeld, N. J. Hitchin and Yu. I. Manin, Phys. Lett. **65A**, 185 (1978)].

It is always striking when developments in a branch of pure mathematics (in this case algebraic geometry, seemingly far removed from practical matters) give the solution to a problem



The surface of a sphere corresponds point by point to the full plane plus a point at infinity.

posed in some other field of science. Yet it happens often enough not to be surprising. But one must pause and contemplate "the unreasonable effectiveness of mathematics in the natural sciences." as Eugene Wigner put it. What is gratifying in this particular case is that the solutions can be put in the form of an "ansatz," or rule, that is usuable and easily understood without the language of algebraic geometry. One can check directly that the ansatz gives solutions with the right number of degrees of freedom. Now the physicist need appeal to the mathematics only to ensure that there are no other solutions than the ones displayed.

Here is the ansatz for topological charge k: Let $T(x)$ be a complex $N+2k$ by N matrix-valued function, with $T^\dagger T$ the $N\times N$ identity matrix. The columns of T are thus N orthogonal vectors with $N+2k$ entries. Choose $2k$ additional vectors that are orthogonal to the column-vectors of T and denote the $N+2k$ by $2k$ matrix of these vectors by Δ, so that $T^\dagger\Delta = 0$. Let $A_\mu = T^\dagger\partial_\mu T$; it is a U($N$) vector potential. Then A_μ is self-dual when $\Delta(x) = a + b{\cdot}x$ and the $2k\times 2k$ matrix $\Delta^\dagger\Delta$ commutes with right multiplication by quaternions. Here $a=(a_1,a_2)$ and $b=(b_1,b_2)$ with the a_i and b_i constant $N+2k$ by k matrices and x is the quaternion $x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$, represented by the 2×2 matrix

$$\begin{bmatrix} x_0 + x_1 i & x_2 + x_3 i \\ x_2 - x_3 i & x_0 - x_1 i \end{bmatrix}$$

All the self-dual solutions can be obtained in this way.

Global differential geometry and algebraic geometry have helped find the self-dual solutions, on the classical level, of the Lagrangian equation of motion for a non-Abelian gauge theory. Quantum field theory begins and builds on these classical solutions. How they and their configurations, particularly the new solutions, contribute to the nonperturbative properties of gauge theories is still unsettled. In fact, how to compute in continuous nonperturbative gauge theory is unknown. Because the geometry and topology of fiber bundles are so intimately connected with gauge theories, it may turn out that mathematical insights will help provide the key to the nonperturbative theory.

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