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Geometric Theory of Charge*

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The symmetry properties of elementary particles—in particular the number of linearly independent states with a fixed mass and momentum—can be explained by relativistic covariance alone, if the irreducible representation assigned to particles with antiparticles is chosen so that it contains $2(2s+1)$ linearly independent states for fixed mass and momentum. No nongeometric symmetry element such as charge conjugation is introduced. The existence of superselection rules within these irreducible representations is made necessary by the postulate that time-reversal invariance be directly verifiable. In this representation, the space-inversion operator has the same effect as the product PC in the usual representation. A charge operator Q is defined and it is shown that all physically realizable states are eigenstates of Q with integral eigenvalues. For "simple" particles, Q can have only the eigenvalues $0, \pm 1$.

I. INTRODUCTION

In the absence of a satisfactory detailed theory of elementary particles, it is desirable to exhaust the possible inferences from space-time symmetry principles. Indeed, the consequences of the invariance of natural laws under the full Lorentz group have been studied with great success by many authors. Unfortunately, it has not been possible to explain even such basic properties of elementary particles as the multiplicity of linearly independent states by relativistic invariance alone. In particular, the existence of antiparticles is explained by an added symmetry, charge conjugation, which is extraneous to the space-time mappings of the Lorentz group. The assignment of irreducible representations of the Lorentz group presently accepted for spinless particles, for instance, provides only one linearly independent state for a given momentum. The fact that two linearly independent states with precisely the same mass are observed is explained by the existence of two irreducible representations with the same mass, which are coupled by a nongeometric symmetry element, viz., the charge conjugation. The case of fermions is exactly analogous except for the doubling of the number of linearly independent states.

This disappointing outcome of Dirac's attempt to explain the basic properties of particles from Lorentz invariance and quantum mechanics alone, was further complicated by the observations on nonconservation of parity. It seems that either I, invariance under space-reflection, has to be completely dropped or that the product of I and charge conjugation alone can be maintained as a physical symmetry element.

On the other hand, it is just those observations which revive the hope for a satisfactory explanation on the basis of relativistic symmetry alone. If we *postulate* invariance of natural laws under the full Lorentz group, then the explanation of the experiments (e.g., decay of polarized neutrons) can only be that the inversion operator $U(I)$ does not convert a neutron into itself, i.e., does not leave the subspace of neutron states invariant. If the conjecture advanced by Landau[1] is correct (and there is, at least, no experimental evidence against it), then the antineutron would have the "inverted" decay pattern of the neutron. From the viewpoint of the postulated invariance under the Lorentz group alone, we would have to interpret this (anticipated) fact as indicating that $U(I)$ converts a neutron into an antineutron. It follows that the irreducible subspace must include both neutrons and antineutrons, and must therefore have 4 linearly independent states for a given momentum. One is led to suspect that the assignment of the irreducible representation was erroneous in the past, and that one has to look for an irreducible representation with 4 linearly independent states. In fact, such "doubled" representations (up to a factor) exist and have been enumerated by Wightman.[2] These considerations have motivated a renewed attempt to explain the basic properties of elementary particles on the basis of the full Lorentz invariance alone.

We adopt the tentative postulate that all rigorous degeneracies of elementary particles must be accounted for by the assignment of the appropriate irreducible representation of the Lorentz group. Thereby, we rule out the existence of two irreducible subspaces with the same spin and exactly equal masses ("Accidents don't happen").

A summary of the arguments and results will now be given. In Sec. II, it is pointed out that the necessity of giving an experimentally verifiable meaning to time-reversal invariance leads to a condition on state vectors which excludes some vectors in the "doubled" irreducible representations from the set of physically realizable states. As a mathematical consequence (Sec. III), two of the doubled irreducible representations are

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[1] L. Landau, Nuclear Phys. **3**, 127 (1957).
[2] A. S. Wightman, *Les Problèmes Mathématiques de la Théorie Quantique des Champs, Colloques Internationaux du Centre National de la Recherche Scientifique* (Centre National de la Recherche Scientifique, Paris, 1959).

found to be empty of states and are discarded, while the remaining representation receives a superselection rule. The two mutually orthogonal sets of states are interpreted (Sec. IV) as particles and antiparticles, and they are carried into each other by the space-inversion operator. In our representation, the space-inversion operator has the same effect as the product of space-inversion and charge-conjugation operators in the usual representation.

In Sec. V, the many-particle states of a system with one type of particle with "doubled" representation are investigated. The superselection rule on the one-particle subspace implies similar rules for many-particle states which may be expressed in terms of a self-adjoint operator Q which commutes with all observables, has integral eigenvalues, and separates subspaces of states. The operator Q may be interpreted as a charge or baryon-number operator.

In Sec. VI it is pointed out that in the presence of superselection rules, the choice of the representative of an element of the Lorentz group in a given irreducible representation space is ambiguous by more than a factor of unit modulus. As a consequence, at least two essentially different operators may be considered to represent time-and-space inversion. In the presence of several types of particles, many different operators $U^{(n)}(T)$ on the reducible part of Hilbert space may be defined.

In the general case of several types of particles, the implications of the superselection rules on many-particle states are doubtful. If the most natural rule for the selection of the separating operators $Q^{(n)}$ is adopted (namely, the rule that defines "simple" particles), it is found that the operators $Q^{(n)}$ can have only the eigenvalues 0 and ± 1. All rigorous degeneracies and absolute selection rules for known elementary particles can be accounted for by introducing two sets of operators $U^{(n)}$ and the corresponding $Q^{(n)}$ (charge and baryon number). This is perhaps the first nonarbitrary distinction between elementary and composite particles.

In Sec. VIII, field operators are defined whose transformation properties are induced by those of the functions in the one-particle subspaces. The simplest covariant equation of motion of the field operator in the presence of an external electromagnetic field confirms the interpretation of the two sets of states in the one-particle subspaces as positive and negative.

II. VERIFIABILITY OF SYMMETRY ASSUMPTIONS

The assumption of invariance of natural laws under the orthochronous relativity group can be given a form that can be directly compared with experiment[3]: Given a state represented by a vector Ψ in Hilbert space, there exists another state represented by the vector $U(L)\Psi$ which is obtained by repeating the preparation of state Ψ with all instruments involved in the preparation of Ψ being translated, rotated, or accelerated with respect to the original experiments. The kind of modification prescribed is implied in the geometric meaning of the element L of the group of length-preserving mappings of space-time into itself. In the case of space translations and rotations, the operational instructions implied in L are self-evident. In the case of Lorentz transformations, the instruments which produce the state $U(L)\Psi$ are to be given a velocity with respect to the laboratory. Finally, for elements L involving space inversion, all instruments used to prepare the state $U(L)\Psi$ are to be changed by interchanging the words "left" and "right" in the operational instructions. The verifiable prediction of the invariance statement is then contained in the statement that a scalar observable φ which represents a measurement made at the space-time point x will have the new expectation values[4]

$$\langle\varphi(Lx)\rangle' \equiv (U(L)\Psi, \varphi(Lx)U(L)\Psi) = \langle\varphi(x)\rangle \equiv (\Psi, \varphi(x)\Psi). \quad (2.1)$$

We have repeated with great explicitness the well-known physical interpretation of relativistic invariance, in order to point out that no such direct operational instruction for the verification of the time-inversion invariance exists.

One might attempt to introduce an analogous instruction for the preparation of the state $U(T)\Psi$ by "letting time run backward," but this is only a verbal exercise.[5] It is necessary therefore to replace the direct geometric meaning of $U(T)$ by an instruction which is obtained from experience.

Consider first the classical case. The statement is: If a state Ψ_i (specified by positions, momenta, and possibly intrinsic angular momenta of all particles at one time) evolves into a state Ψ_f after a time t, then there exists a state $T\Psi_f$ which evolves into a state $T\Psi_i$ during time t. The statement is empty unless an instruction for the preparation of any state $T\Psi$ corresponding to a given Ψ is provided. The statement would still be empty if this instruction had to rely on a specific theory, since symmetry statements should be used as requirements to be imposed on a theory, and not be deduced from it. In fact, the instruction is easy to formulate: $T\Psi$ is obtained from Ψ by reversing all momenta and angular momenta and leaving the positions unchanged. In this purely empirical form the statement is verifiable, and has been verified to a large extent. The essential point is that a directly verifiable statement of time-reversal invariance has to be made in terms of a geometric operation (180° rotation).

[3] R. Haag, Kgl. Danske Videnskab. Selskab, Math.-fys. Medd. **29**, 12 (1955).

[4] For nonscalar observables, the generalization of this statement can be given in an elegant way according to A. Wightman, Problèmes Mathématiques de la Théorie Quantique, University of Paris, 1958 (unpublished).

[5] This distinctive nature of the time-reversal element of the relativity group was pointed out by R. Haag (reference 3). For the classical case, we adopt the viewpoint of S. Watanabe, Phys. Rev. **84**, 1008 (1951).

For at least a restricted set of (improper) quantum mechanical states, the classical statement can be immediately adapted.

Postulate I: For a one-particle state which is a simultaneous eigenvector of momentum **P** *and of the projection of angular momentum* ($s=\mathbf{J}\cdot\mathbf{P}/|\mathbf{P}|$) *on it, the time-reversal operator is equivalent to a 180° rotation with respect to a direction normal to* **P**, i.e.,

$$U(T)\Psi_{\mathrm{p},s}=e^{i\alpha}U(R_{\mathrm{p}})\Psi_{\mathrm{p},s}=e^{i\beta}\Psi_{-\mathrm{p},s}. \tag{2.2}$$

In this form, the statement is in agreement with observations as well as with all theories of elementary particles considered in the past. In the case of the "doubled" representations, it will be seen that not all simultaneous eigenvectors of **P** and s satisfy Eq. (2), and therefore we have to impose it as a *condition on states*. The fact that not all vectors or rays in Hilbert space can be states, is well known.[6] Our particular superselection rule seems indispensable to give a verifiable meaning to time-reversal invariance in the "doubled" representations.

We state explicitly two more assumptions on superselection rules.

Postulate II: If a vector Ψ *is a state, then all vectors* $\sum a_i U(L_i)\Psi$ *are states, if the* L_i *are the elements of the proper Lorentz group.*

Indeed, it would be absurd if, for instance, the time development of a state could turn it into a vector which is not physically meaningful.

Postulate III: States form mutually orthogonal subspaces in the Hilbert space.

These are usually assumed properties of superselection rules, and it seems difficult to obtain reasonable physical results without them.

III. SELECTION OF PHYSICALLY ADMISSIBLE IRREDUCIBLE REPRESENTATIONS

The "doubled" irreducible representations for nonvanishing mass have been enumerated by Wightman.[2,7] The three "doubled" representations may be obtained from the usual one by doubling the number of components of the function in representation space. The operators may be written under the form of a direct product $V(L)\otimes U_0(L)$ of 2×2 matrices $V(I)$, $V(T)$, $V(IT)$, and operators $U_0(L)$ which act only within the subspace defined by the usual representation. The usual representations, which will be referred to as type I, may be realized either on sets of normalizable functions of $2s+1$ components[8,9] or on the normalizable solutions of Dirac-type equations.[10] In the first case, particles with spin $\frac{1}{2}$ are described by 4-component, in the second case by 8-component functions for the "doubled" representations. We will write the wave functions as

$$\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$$

where φ and ψ have the same number of components as in representation I, and both φ and ψ span subspaces which are the bases of irreducible representations of the proper Lorentz group. Since the operators U_0 and their generators have been described in great detail[8–10] we do not discuss them.

TABLE I. Matrices of the various types of representation.

Type	$[U(T)]^2$	$[U(IT)]^2$	$V(I)$	$V(T)$	$V(IT)$
I	±1	±1	1	1	1
II	±1	∓1	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$
III	∓1	±1	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$
IV	∓1	∓1	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$

The matrices V are tabulated in Table I[2,7] together with the squares of $U(T)$ and $U(IT)$. While the matrices V are, of course, determined only up to a factor of unit modulus, the squares of the antiunitary operators such as $V(T)U_0(T)$ are well-defined numbers which do not change with the phases of the operators. (See, e.g., reference 6.) The upper sign in Table I applies to integral, the lower to half-integral spin.

Consider first spinless particles in the representations III and IV. The eigenfunctions of the momentum operator in representation I may be chosen to be real, e.g., $\Psi_p=\delta(\mathbf{p}-\mathbf{p}')$. The general eigenfunctions of the momentum operator in representations III and IV are then of the form

$$\begin{pmatrix}a & \Psi_{\mathrm{p}}\\ b & \Psi_{\mathrm{p}}\end{pmatrix},$$

where a and b are complex numbers. To test such vectors for physical validity, we write according to Eq. (2.2)

$$T\begin{pmatrix}a\Psi_{\mathrm{p}}\\ b\Psi_{\mathrm{p}}\end{pmatrix}=\begin{pmatrix}+b^*\Psi_{-\mathrm{p}}\\ -a^*\Psi_{-\mathrm{p}}\end{pmatrix}\quad\text{or}\quad\begin{pmatrix}-b^*\Psi_{-\mathrm{p}}\\ +a^*\Psi_{-\mathrm{p}}\end{pmatrix}. \tag{3.1}$$

If these are to be equal to

$$e^{i\alpha}U(R_{\mathrm{p}})\begin{pmatrix}a\Psi_{\mathrm{p}}\\ b\Psi_{\mathrm{p}}\end{pmatrix}=e^{-i\delta}\begin{pmatrix}a\Psi_{-\mathrm{p}}\\ b\Psi_{-\mathrm{p}}\end{pmatrix}, \tag{3.2}$$

then we must have

$$a=e^{i\delta}b^*,\quad b=-e^{i\delta}a^*; \tag{3.3}$$

or

$$a=-e^{i\delta}b^*,\quad b=e^{i\delta}a^*$$

[6] G. C. Wick, A. S. Wightman, and E. P. Wigner, Phys. Rev. **88**, 101 (1952).

[7] L. Michel and A. S. Wightman, Princeton University Lecture Notes (unpublished).

[8] E. P. Wigner, Ann. Math. **40**, 149 (1939).

[9] Yu. M. Shirokov, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 861, 1196, 1208 (1957); **34**, 717 (1958) [translation: Soviet Phys.—JETP **6**, 664, 919, 929; **34**(**7**), 493 (1958)].

[10] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. (U.S.) **34**, 211 (1948).

for representations III and IV, respectively. Clearly, these equations have no nonzero solutions, and we conclude that the representations III and IV have no physical states at all.

For the general case of particles with spin, it is more convenient to use a unitary transformation to diagonalize $V(T)$. If there are physical states at all, then, according to our assumptions, there must exist a representation in which they have the form

$$\begin{pmatrix}\varphi\\0\end{pmatrix},\quad \begin{pmatrix}0\\\varphi\end{pmatrix},$$

where the functions φ form the basis of an irreducible representation of the proper Lorentz group. It is also easy to see that the matrix $V(T)$ must be diagonal in this representation.

However, $U(T)$ is an antilinear operator, and the diagonalization problem does not, as in the case of unitary operators, have a solution unique up to a reordering of the diagonal elements. It will be shown that in this case the problem may have either no solution or several essentially inequivalent ones. Since any antiunitary operator may be written as the product of a unitary operator R and the operator K of complex conjugation, the operator $U(T)$ has the form $V(T)KR_0(T)$, where R_0 leaves the subspaces invariant. Since we are not concerned with transformations within the subspaces, the problem requires the finding of a unitary 2×2 matrix M such that

$$MVKM^{-1}=DK, \tag{3.4}$$

or

$$MVKM^{-1}K=D, \tag{3.5}$$

where D is a diagonal matrix.

It is easy to verify that the problem has no solution for representations III and IV. Therefore, these representations must be discarded in general. The only remaining possibility is the representation II, which by the orthogonal transformation

$$(2^{-\frac{1}{2}})\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

is brought into the desired form

$$V(I)=\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix},\quad V(T)=\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix},\quad V(IT)=\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}. \tag{3.6}$$

The squares of the operators $U(T)$ and $U(IT)$ remain, of course, unchanged by the transformation.

Clearly, the vectors

$$\begin{pmatrix}\varphi\\0\end{pmatrix},\quad \begin{pmatrix}0\\\varphi\end{pmatrix}$$

are states, while vectors of the form

$$\begin{pmatrix}\Psi_{\mathbf{p}}\\\Psi_{\mathbf{p}}\end{pmatrix}$$

are not. However, the diagonalization problem in this case has many other solutions. Correspondingly, in the representation (6), there are vectors

$$\begin{pmatrix}\Psi_{\mathbf{p}}\\ i|a|\Psi_{\mathbf{p}}\end{pmatrix}$$

which would also, by our first requirement (Sec. II, Postulate I) alone, qualify as states. By our Postulate III, we must discard some of these vectors, and retain only two mutually orthogonal sets. We choose the set

$$\begin{pmatrix}\varphi\\0\end{pmatrix},\quad \begin{pmatrix}0\\\varphi\end{pmatrix}$$

and we omit the proof that another choice would lead to physically identical results.

A similar analysis shows that for particles with zero mass the "doubled" representations are empty of physical states.

IV. PHYSICAL INTERPRETATION OF THE IRREDUCIBLE REPRESENTATIONS

We have postulated in Sec. I that the multiplicity of linearly independent one-particle states with a given mass and momentum must be due to the structure of the irreducible representations of the Lorentz group alone, and not to the existence of additional symmetry principles. To all particles with antiparticles, we must therefore assign representation II, which has $2(2s+1)$ linearly independent states for a given momentum. We interpret the states

$$\begin{pmatrix}\varphi\\0\end{pmatrix}\quad\text{and}\quad \begin{pmatrix}0\\\varphi\end{pmatrix}$$

as particle and antiparticle states. We have three verifiable predictions from this assignment:

(1) There is a superselection between particle and antiparticle states. This result is in agreement with experience, and it has been conjectured previously by Wick, Wightman, and Wigner.[6]

(2) Particles with momentum $\mathbf{p}$ and projection s are converted into antiparticles with $(-\mathbf{p}, -s)$ by space inversion. This result, while not confirmed experimentally, is not in disagreement with experiment. It was conjectured by Landau.[1] In our assignment, the operator $U(I)$ has the same effect as the product of $U(I)$ and the charge conjugation operator C in the usual assignment.

(3) Since the time-reversal operator does not transform particles into antiparticles, the usual argument concerning the expectation value of a vector that is

invariant under time reversal leads to the conclusion that the electric dipole moment must vanish, but not the magnetic moment.

V. ONE TYPE OF CHARGED PARTICLE ONLY

To discuss the implications of the superselection principle on many-body states, we consider first a theory in which only one type of particle with antiparticle exists. We are going to refer to the two types of states as positive and negative, although the argument applies also to neutral particles with antiparticles. The implication of the superselection rule for one-particle states in the many-particle system may be formulated as follows.

If a physical state is defined by the addition of an extra particle to a pre-existing system, the added particle can be only purely positive or purely negative —not a linear superposition. In the general theory of scattering, the creation of physical particles at a finite time is described by quasi-localized particle-creation operators[11,12] **c** [called $Q(0)$ by Haag] which create rigorous one-particle states when applied to the physical vacuum

$$\mathbf{c}\Psi_0=\psi^{(1)}, \tag{5.1}$$

and have certain other restrictive properties which define their quasi-local nature.[12,13]

A many-particle state Ψ' of the above-mentioned type is then defined by

$$\Psi'=\sum \mathbf{c}_i\Psi, \tag{5.2}$$

where Ψ is any state. The superselection rule states that a vector obtained by adding a linear combination of a positive and of a negative particle, i.e.,

$$\Psi'=(a\mathbf{c}_+ + b\mathbf{c}_-)\Psi, \tag{5.3}$$

is not a state unless either $a=0$ or $b=0$.

It is known that every superselection rule implies restrictions on that class of self-adjoint operators which may be considered as observables.[6] It is therefore desirable to replace the restriction on states by a restriction on observables. The connection between states and observables is given by the remark that immediately after the measurement of a nondegenerate observable (a complete commuting set of observables), the state of the system is the eigenvector that corresponds to the observed values.[6] The superselection rule states therefore that a vector that is not a state cannot be an eigenvector of a nondegenerate observable.

In the irreducible subspace, the superselection rule can be expressed by the requirement that every nondegenerate observable $\mathcal{O}$ must commute with a set of matrices

$$\begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{-i\delta} \end{pmatrix}.$$

Clearly, the eigenvectors of such operators can be only of the form

$$\begin{pmatrix} \varphi \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \varphi \end{pmatrix},$$

but not

$$\begin{pmatrix} a\varphi \\ b\psi \end{pmatrix} \quad (a\neq 0,\ b\neq 0),$$

and this is just the statement of superselection. If all nondegenerate observables commute with a matrix, then the degenerate observables do so *a fortiori*. With the notation

$$W_\delta=\begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{-i\delta} \end{pmatrix}, \tag{5.4}$$

we have

$$[\mathcal{O},W_\delta]\psi^{(1)}=0, \tag{5.5}$$

where the $\psi^{(1)}$ are the one-particle states.

By Eq. (1), the operators W_δ have the property

$$W_\delta\mathbf{c}_\pm\Psi_0=e^{\pm i\delta}\mathbf{c}_\pm\Psi_0. \tag{5.6}$$

We now wish to extend the definition of W_δ from the one-particle state to the whole Hilbert space $\mathcal{H}$.

First, we define

$$W_\delta\Psi_0=\Psi_0.$$

Hence Eq. (6) may be written

$$W_\delta\mathbf{c}_\pm W_\delta^{-1}\Psi_0=e^{\pm i\delta}\mathbf{c}_\pm\Psi_0. \tag{5.7}$$

It is natural to define a set of unitary operators W_δ on $\mathcal{H}$ by generalizing this equation to

$$W_\delta\mathbf{c}_\pm W_\delta^{-1}=e^{\pm i\delta}\mathbf{c}_\pm, \tag{5.8}$$

and we may conjecture that the requirement that all observables commute with the operators thus defined will imply our superselection rule for many-body states.

We will show that a nondegenerate operator $\mathcal{O}$ cannot have a state of the form Ψ' [Eq. (3)] for eigenvector if

$$[\mathcal{O},W_\delta]=0. \tag{5.9}$$

If, contrary to our assertion, such a vector were an eigenvector of $\mathcal{O}$, we would have

$$\mathcal{O}\Psi'=\mathcal{O}(a\mathbf{c}_+ + b\mathbf{c}_-)\Psi=\lambda(a\mathbf{c}_+ + b\mathbf{c}_-)\Psi, \tag{5.10}$$

where $a\neq 0$, $b\neq 0$; and, because of Eq. (9),

$$\mathcal{O}(ae^{i\delta}\mathbf{c}_+ + be^{-i\delta}\mathbf{c}_-)W_\delta\Psi=\lambda(ae^{i\delta}\mathbf{c}_+ + be^{-i\delta}\mathbf{c}_-)\Psi. \tag{5.11}$$

Since $\mathcal{O}$ is nondegenerate, the vector on the right-hand side of Eq. (11) can differ only by a factor $e^{i\beta}$ from the vector Ψ'.

Since, by assumption, Ψ is a state, there exists another nondegenerate observable $\mathcal{O}'$ of which it is an eigenvector:

$$\mathcal{O}'\Psi=\mu\Psi, \tag{5.12}$$

$$W_\delta\mathcal{O}'W_\delta^{-1}W_\delta\Psi=\mathcal{O}'W_\delta\Psi=\mu W_\delta\Psi. \tag{5.13}$$

[11] H. Ekstein, Nuovo cimento **4**, 1017 (1956).
[12] R. Haag, Phys. Rev. **112**, 669 (1958).
[13] H. Ekstein, Phys. Rev. **117**, 1590 (1960).

Hence, $W_\delta\Psi$ can differ from Ψ only by a factor of unit modulus $e^{i\alpha}$, and we have

$$(a\mathbf{c}_+ + b\mathbf{c}_-)\Psi = e^{i(\alpha+\beta)}(ae^{i\delta}\mathbf{c}_+ + be^{-i\delta}\mathbf{c})\Psi. \quad (5.14)$$

But, since $\mathbf{c}_\pm$ is a creation operator which does not annul any vector, this equation is absurd, unless $a=0$ or $b=0$. This proves the assertion.

A state of the type

$$\prod_n^N \mathbf{c}_{+n} \prod_m^M \mathbf{c}_{-m}\Psi_0 = \Psi_{N,M} \quad (5.15)$$

is an eigenvector of the operator W_δ with eigenvalue $e^{i(N-M)\delta}$. We can define a self-adjoint operator Q by

$$W_\delta = e^{iQ\delta} \quad (5.16)$$

and infer that Q commutes with all observables, that its eigenvalues are integers, and that no two of its nondegenerate eigenvectors can be linearly combined to give states. These are properties of the charge and baryon-number operators.

The asymptotic states $\Psi^{(\pm)}$ are constructed from the states $\prod\mathbf{c}_+\prod\mathbf{c}_-\Psi_0$ by a well-known limiting process[11–15] which involves only the energy operator H. Since this must be an observable, it commutes with Q. We can therefore infer that the asymptotic particle-creation operators $A_\pm{}^\dagger$ are transformed by W_δ in the same manner as the operators $\mathbf{c}$, i.e.,

$$W_\delta A_\pm{}^\dagger W_\delta{}^{-1} = e^{\pm i\delta}A_\pm{}^\dagger \quad (5.17)$$

for both "in" and "out" operators.

It follows then that the S operator itself commutes with Q.

VI. RAY MAPPING IN THE PRESENCE OF SUPERSELECTION

It is well known[16] that the physical invariance principles do not require a proper representation of the group elements L by operators $U(L)$ on Hilbert space, but only a mapping of elements L on operator rays $\mathbf{U}(L)$ which consist of all operators $\tau(L)U(L)$ ($|\tau|=1$) such that

$$\mathbf{U}(L_1)\mathbf{U}(L_2) = \mathbf{U}(L_1L_2). \quad (6.1)$$

However, if Eq. (1) is to be satisfied for every ray $\mathbf{f}$ in Hilbert space, it is possible to select one representative operator $U(L)$ from each operator ray $\mathbf{U}(L)$ so that the mapping becomes a many-to-one mapping from elements L to representative operators $U(L)$.[16] Such a choice is essentially unique, because the only other possible operators which are representative for a given set $\mathbf{U}(L)$ are of the form $\tau(L)U(L)$ ($|\tau|=1$).

In presence of superselection rules, Eq. (1) must be modified so that its assertion is restricted to physical states only:

$$\mathbf{U}(L_1)\mathbf{U}(L_2)\mathbf{f} = \mathbf{U}(L_1L_2)\mathbf{f}, \quad (6.2)$$

where $\mathbf{f}$ is a ray which represents a physically realizable state. In this case, the choice of representative operators is not unique up to a factor of unit modulus.

By our Postulate II (Sec. II), the states $\mathbf{f}$ span subspaces $\mathcal{H}_i$ in Hilbert space. If a given representative $U(L)$ is replaced in its action on $\mathbf{f}_i(\mathbf{f}_i\epsilon\mathcal{H}_i)$ by $\tau_iU(L)$, i.e., if

$$U'(L)\mathbf{f}_i = \tau_i(L)U(L)\mathbf{f}_i, \quad (|\tau_i|=1, \mathbf{f}_i\epsilon\mathcal{H}_i), \quad (6.3)$$

then clearly $U'(L)$ is also a possible representative of the operator ray $\mathbf{U}(L)$. Therefore, there may be several inequivalent representatives $U^{(n)}(L)$ in a representation in the presence of superselection rules. This possibility was discussed by Wightman.[17]

In our representation II, it is possible to adopt the unit matrix for $V(T)$ in addition to the matrix

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

This is not the most general possibility, but it will be sufficient for our purpose.

For representation II, with the adopted superselection rule, we consider the two assignments given in Table II.

Of course, the matrices of class (b) can be reduced to give two representations of type I, but this is an unphysical operation, since the vectors in the two invariant subspaces are not states. The representation is mathematically but not physically reducible because of the superselection rule.

VII. SEVERAL TIME-REVERSAL OPERATORS IN THE COMPLETE HILBERT SPACE

We have seen that in an irreducible subspace of type II at least two essentially inequivalent time-reversal operators can be defined. The extension of a symmetry operator from the one-particle states to the many-body states is accomplished by the construction of direct products of the irreducible representations.[3] In the case where only one type of particle is present, this construction is unambiguous, and the introduction of the operators U_b in addition to U_a does not give any new results. However, in presence of several types of particles, one cannot know *a priori* which of the

TABLE II. Assignments for representation II.

	$V(I)$	$V(T)$	$V(IT)$	$[U(T)]^2$	$[U(IT)]^2$
a	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	±1	∓1
b	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	±1	±1

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operators $U^{(n)}$ (defined on the whole Hilbert space) will have the form U_a or U_b on a particular irreducible subspace.

If there are two types of particles with representation II, we could have, e.g., either of the schemes shown in Fig. 1. These diagrams symbolize the assignments

$$\begin{aligned} U^{(1)}\psi_{\alpha,\beta}=U_a\psi_{\alpha,\beta}, \quad & U^{(1)}\psi_\alpha=U_a\psi_\alpha, \\ & U^{(1)}\psi_\beta=U_b\psi_\beta, \\ U^{(2)}\psi_{\alpha,\beta}=U_b\psi_{\alpha,\beta}, \quad & U^{(2)}\psi_\alpha=U_b\psi_\alpha, \\ & U^{(2)}\psi_\beta=U_a\psi_\beta. \end{aligned} \tag{7.1}$$

The operators $U^{(n)}$ are the operators defined on the whole Hilbert space, while $U_{a,b}$ are the operators defined on the irreducible representations in Table II. Factors of unit modulus may be inserted in Eq. (7.1) for greater generality.

The existence of superselection rules in the one-particle subspaces again implies superselection rules for the many-particle states. In order to guarantee these, we can again define operators $W_\delta^{(n)}$ with the properties

$$\begin{aligned} W_\delta^{(1)}\mathbf{c}_{\alpha\pm}(W_\delta^{(1)})^{-1}=e^{\pm i\delta}\mathbf{c}_{\alpha\pm}, \\ W_\delta^{(2)}\mathbf{c}_{\beta\pm}(W^{(2)})^{-1}=e^{\pm i\delta}\mathbf{c}_{\beta\pm}, \end{aligned} \tag{7.2}$$

which commute with all observables. These operators guarantee the superselection rules for states which consist only of α or only of β particles, but the commutation rules of $W_\delta^{(1)}$ with $\mathbf{c}_\beta$ and of $W_\delta^{(2)}$ with $\mathbf{c}_\alpha$ are not given by any symmetry property. One could have, for instance

$$[W_\delta^{(1)},\mathbf{c}_\beta]=[W_\delta^{(2)},\mathbf{c}_\alpha]=0, \tag{7.3}$$

or

$$W_\delta^{(1)}=W_\delta^{(2)}. \tag{7.4}$$

The most natural assumption is that the operators $W_\delta^{(n)}$ are associated with the operators $U^{(n)}$ as follows. For any particle of type ν, the commutation rules are of the form

$$W_\delta^{(n)}\mathbf{c}_{\nu\pm}(W_\delta^{(n)})^{-1}=\exp(\pm i\delta_\nu^{(n)})\mathbf{c}_{\nu\pm}, \tag{7.5}$$

and $\delta_\nu^{(n)}=0$ for those particles ν on whose irreducible subspace $U^{(n)}\psi_\nu=U_b\psi_\nu$, i.e., where the operator $U^{(n)}$ does not imply any superselection rules on particle ν. Those particles which satisfy this assignment rule, will be called simple. Even with this assumption, the relation between two nonvanishing phases remains indeterminate at this point. For instance, in Fig. 1(2), we have

$$\begin{aligned} W_\delta^{(1)}\mathbf{c}_{\beta\pm}(W_\delta^{(1)})^{-1}=e^{\pm i\delta}\mathbf{c}_{\beta\pm}; \quad & W_\delta^{(2)}\mathbf{c}_{\beta\pm}(W_\delta^{(2)})^{-1}=e^{\pm i\delta}\mathbf{c}_{\beta\pm}, \\ W_\delta^{(1)}\mathbf{c}_{\alpha\pm}(W_\delta^{(1)})^{-1}=e^{\pm i\delta}\mathbf{c}_{\alpha\pm}; \quad & W_\delta^{(2)}\mathbf{c}_{\alpha\pm}(W_\delta^{(2)})^{-1}=\mathbf{c}_{\alpha\pm}, \end{aligned} \tag{7.6}$$

which is unambiguous, but in Fig. 1(1),

$$\begin{aligned} W_\delta^{(2)}\mathbf{c}_{\alpha,\beta\pm}(W_\delta^{(2)})^{-1}=\mathbf{c}_{\alpha,\beta\pm}, \\ W_\delta^{(1)}\mathbf{c}_{\alpha\pm}(W_\delta^{(1)})^{-1}=\exp(\pm i\delta_\alpha)\mathbf{c}_{\alpha\pm}, \\ W_\delta^{(1)}\mathbf{c}_{\beta\pm}(W_\delta^{(1)})^{-1}=\exp(\pm i\delta_\beta)\mathbf{c}_{\beta\pm}, \end{aligned} \tag{7.7}$$

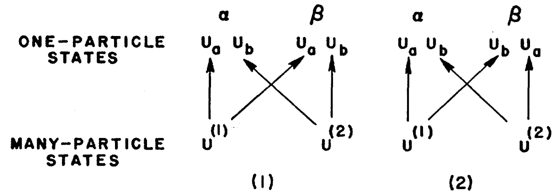

FIG. 1. Possible schemes for two types of particles with representation II.

where δ_β may be considered to be any nonvanishing function of δ_α. However, this ambiguity will be removed by further analysis.

By the argument given at the end of Sec. V, the scattering operator S commutes with the operators $W_\delta^{(n)}$. Now consider many-particle processes in which an initial state of the form

$$\prod_i^{n_+}A_{\alpha+}{}^{\dagger(-)}(k_i)\prod^{n_-}A_{\alpha-}{}^{\dagger(-)}(k_i)\prod^{m_+}A_{\beta+}{}^{\dagger(-)}(k_i)\prod^{m_-}A_{\beta-}{}^{\dagger(-)}(k_i)\Psi_0$$

is converted into a state of the same type, but with different particle numbers n_+', n_-', m_+', m_-'. The matrix element for this scattering process can be nonvanishing only if

$$(n_+-n_--n_+'+n_-')\delta_\alpha=(m_+'-m_-'-m_-+m_+)\delta_\beta.$$

Therefore, the phases δ_α and δ_β must be integral multiples of a constant q.

We have then, for the phases $\delta_\nu^{(n)}$ in Eq. (7),

$$\delta_\nu^{(n)}=q^{(n)}m_\nu\delta,$$

where m_ν is an integer and $q^{(n)}$ a real number.

We can now introduce, as in Sec. V, a self-adjoint operator $Q^{(n)}$ such that

$$W_\delta^{(n)}=\exp(iQ^{(n)}q^{(n)}\delta), \tag{7.9}$$

with

$$W_\delta^{(n)}\mathbf{c}_{\nu\pm}(W_\delta^{(n)})^{-1}=\exp(\pm i\delta m_\nu q^{(n)})\mathbf{c}_{\nu\pm}. \tag{7.10}$$

The operator $Q^{(n)}$ is diagonal with respect to all states, and has integral eigenvalues. This result supports our previous interpretation of $Q^{(n)}$ as the charge or baryon-number operator.

Further information about the properties of simple particles can be obtained by considering a reaction of two simple particles α and β which results in the creation of a new simple particle γ. Of course, a scattering process with only three particles would violate the laws of conservation of momentum and energy, but we may add a particle of the same type to the final and initial states. Since the presence of this extra particle does not charge the following argument, we will omit it for the sake of simplicity.

Consider two simple particles α and β, both having either a nonvanishing charge or baryon number. In our formalism, this means that there exists an operator $U^{(1)}$ such that $U^{(1)}\psi_{\alpha,\beta}=U_a\psi_{\alpha,\beta}$. We consider the two operators $[U^{(1)}(T)]^2$ and $[U^{(1)}(IT)]^2$. Since the S

operator commutes with both, an S-matrix element can be nonvanishing only if

$$([U^{(1)}(T)]^2\Psi_{\alpha\beta},S[U^{(1)}(T)]^2\Psi_\gamma)=(\Psi_{\alpha\beta},S\Psi_\gamma) =([U^{(1)}(IT)]^2\Psi_{\alpha\beta},S[U^{(1)}(IT)]^2\Psi_\gamma), \quad (7.11)$$

where $\Psi_{\alpha\beta}$ and Ψ_γ are the asymptotic "out" states. Since the asymptotic states transform like products of the corresponding one-particle states,[3] the quantities $[U^{(1)}]^2\Psi_{\alpha\beta}$ can be determined from Table I if the spins of the particles α and β are known. Assume first that α has integral spin, β half-integral. Then, by Table I,

$$[U^{(1)}(T)]^2\Psi_{\alpha\beta}=(+)\cdot(-)\Psi_{\alpha\beta}, \quad [U^{(1)}(IT)]^2\Psi_{\alpha\beta}=(-)\cdot(+)\Psi_{\alpha\beta}. \quad (7.12)$$

It follows from Eq. (7.1) that

$$[U^{(1)}(T)]^2\Psi_\gamma=[U^{(1)}(IT)]^2\Psi_\gamma=-\Psi_\gamma. \quad (7.13)$$

The reader will convince himself that for any choice of spins, the operators $[U^{(1)}(T)]^2$ and $[U^{(1)}(IT)]^2$ produce the same sign change on Ψ_γ. By Table I, we conclude that the operators $U^{(1)}$ act on the subspace γ as operators of type b. Therefore, by our definition of simple particles, the charge (or baryon number, according to the physical meaning of $U^{(1)}$) of the reaction product is zero. Since both particles α and β have nonzero eigenvalues of $Q^{(1)}$, this can be true only if the reacting particles α and β have equal and opposite eigenvalues of $Q^{(1)}$.

We can now consider other reactions of the three simple particles α, β, γ with other simple particles which also create simple particles. By repeating the argument given above, we conclude that the eigenvalue of the operators $Q^{(n)}q^{(n)}$ on the irreducible subspaces either are zero or are equal in absolute magnitude to a given constant. Since the eigenvalues of $Q^{(n)}$ on many-particle states can be only larger, in absolute value, we may take its value to be $q^{(n)}$, the smallest nonzero eigenvalue of $Q^{(n)}q^{(n)}$ according to Eq. (9). We have the general result that the eigenvalues of $Q^{(n)}$ on the one-particle subspaces of simple particles are 0 and ± 1.

These properties are common to those particles which are currently considered as elementary, and we are therefore led to a physical interpretation of elementary particles as being simple.

It was pointed out by Haag[3] that within the accepted framework of field theory there is no nonarbitrary difference between elementary and composite stable particles. Nevertheless, common sense imposes a distinction between what are currently considered elementary particles and stable composite "particles" such as chairs or black dwarfs. Our scheme of classification offers a nonarbitrary definition of elementary particles as those that are simple.[18]

The assignment of representations and superselection rules to elementary particles is now obvious. We define two sets of operators $U^{(1)}$, $U^{(2)}$ with corresponding $Q^{(1)}$, $Q^{(2)}$. They are given the physical meaning of charge and baryon number, respectively. On subspaces of elementary particles with charge (baryon number) 0, the operators $U^{(1)}(U^{(2)})$ have the form U_b; on those which have charge (baryon number) ± 1, it is U_a. For instance, the particles $\pi^\pm$, $(p,\bar{p})$, $(n,\bar{n})$ are characterized by Fig. 2.

There is no necessity to introduce a third operator for lepton number for the purpose of this scheme. For composite particles, the assignment is obtained from the reaction which creates them. For instance, the $(\alpha,\bar{\alpha})$ particle is obtained by the reaction

$$2p+2n\rightarrow\alpha,$$

or

$$2\bar{p}+2\bar{n}\rightarrow\bar{\alpha}.$$

Hence,

$$Q^{(1)}\psi_\alpha=\pm 2\psi_\alpha \quad Q^{(2)}\psi_\alpha=\pm 4\psi_\alpha.$$

The assignment of the representatives of the Lorentz group can be read off the reaction equation in conjunction with Table II. It is $U^{(1)}\psi_\alpha=U^{(2)}\psi_\alpha=U_b\psi_\alpha$. This example shows the difference between simple and composite particles. The spin of α is 0, hence the number of linearly independent states is 2. Both operators $U^{(1)}$, $U^{(2)}$ are of the form U_b, and therefore the representation would be reducible, except for the existence of a superselection rule which forbids the existence of linear combinations of states with charge $+2$ and -2 (the representation is physically irreducible). The difference between simple and composite particles can be seen as follows. Since there is no operator of the type a, the existence of the superselection rule would appear wholly unmotivated, by considering the subspace of $(\alpha,\bar{\alpha})$ alone. It is only motivated by the manner in which the α particle has been produced from simple particles. In other words, the assignment analysis must start from simple particles, if it is to be nonarbitrary.

A sensitive test of our scheme is the prediction that an elementary particle of zero charge and zero baryon number must belong to representation I, and must have only $(2s+1)$ linearly independent states for a given momentum (for $m\neq 0$). The assignment would be invalidated by the existence of two π^0 or K^0 particles with exactly equal mass.

For massless particles, our assignment provides no

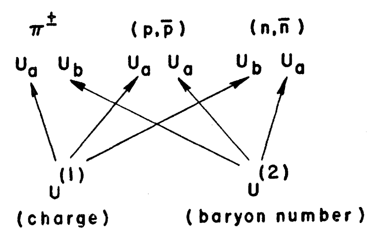

FIG. 2. Assignment of representations and superselection rules to elementary particles.

[18] But not only elementary particles are simple. The hydrogen atom, for instance, meets all requirements for simple particles.

charge and baryon number, since the "doubled" representations are unphysical.[19] The discovery of a massless particle with nonzero charge or baryon number or with more than two linearly independent states for a given spin would also invalidate our assignment.

VIII. INTERACTION WITH THE ELECTROMAGNETIC FIELD

We have justified our interpretation of the operator Q as charge operator by showing that it has some of the properties of the latter: its eigenvalues are integers, its eigenstates are separated by superselection, and its eigenvalues on one-particle states of simple particles are 0 and ± 1. The primary definition of charge by its interaction with the electromagnetic field cannot be deduced from mere symmetry considerations, but we will show that the most obvious type of coupling of particles with an external electromagnetic field is in agreement with our interpretation.

For this purpose, field operators must be defined whose transformation properties reflect those of the one-particle states. The general prescription for determining the properties of creation operators $A^\dagger$ is as follows.[3,20] If the representative $U(L)$ of an element L transforms a one-particle state ψ_1 into ψ_2, and if creation operators $A_1{}^\dagger$ and $A_2{}^\dagger$ are defined by

$$A_1{}^\dagger\Psi_0=\psi_1;\quad A_2{}^\dagger\Psi_0=\psi_2, \tag{8.1}$$

then

$$U(L)A_1{}^\dagger U(L)^{-1}=A_2{}^\dagger. \tag{8.2}$$

The creation operators are uniquely defined by the usual commutation or anticommutation relations, and by the requirement that their adjoints annul the vacuum state. The operators may be considered as creating either asymptotic "in" or "out" states.

For spinless particles of representation II, we define field operators

$$\varphi(x)=(2\pi)^{-\frac{3}{2}}\int d^3k\, e^{ikx}[A_+(\mathbf{k})+A_-{}^\dagger(-\mathbf{k})](2k_0)^{-\frac{1}{2}}. \tag{8.3}$$

One shows readily that under the operations of the restricted group it follows that

$$U(L)\varphi(x)U(L)^{-1}=\varphi(Lx). \tag{8.4}$$

From Eqs. (3.6), (8.1), and (8.2), one has

$$U(I)A_+{}^\dagger(\mathbf{k})U(I)^{-1}=A_-{}^\dagger(-\mathbf{k}). \tag{8.5}$$

Hence,

$$U(I)\varphi(\mathbf{x},x_0)U(I)^{-1}=\varphi^\dagger(-\mathbf{x},x_0). \tag{8.6}$$

We see again that the inversion operator has the same effect in our representation as its product with charge conjugation in the usual representation.

In an external field, the physical covariance requirement is changed to the statement: If an element of the Lorentz group induces a change $A_\mu\rightarrow A_\mu'$, the corresponding state is obtained from $\Psi\rightarrow\Psi'=U(L)\Psi$. To guarantee this convariance, one postulates equations of motion for the field operators which are invariant with respect to a simultaneous change $A_\mu\rightarrow A_\mu'$ (induced by L) and $\varphi\rightarrow U(L)\varphi(x)U(L)^{-1}$. The only such equation, linear and second order in φ and using only one coupling constant, is

$$(i\partial_\mu-eA_\mu)(i\partial_\mu-eA_\mu)\varphi(x)+m^2\varphi(x)=0. \tag{8.7}$$

In accordance with Landau's suggestion,[1] we assume that A_μ behaves as a pseudovector under space inversion. The simultaneous transformation $A_i\rightarrow A_i$, $(i=1, 2, 3)$, $A_0\rightarrow -A_0$, $\varphi(x)\rightarrow U(I)\varphi(x)U(I)^{-1}$ transforms Eq. (8.7) into

$$(i\partial_\mu+eA_\mu)(i\partial_\mu+eA_\mu)\varphi^\dagger(x)+m^2\varphi^\dagger(x)=0, \tag{8.8}$$

which is the Hermitean adjoint of Eq. (8.7). This proves the invariance of Eq. (8.7) under space inversion. To see the observable implications of this equation, we need only apply the operator to the vacuum, and observe that because of Eq. (8.3) and $A\Psi_0=0$, φ creates only negative, $\varphi^\dagger$ only positive particles. This establishes the connection between the two types of physical states in the irreducible representation and the electromagnetic field.

It is more usual to introduce the field operators with their transformation properties by considering $\varphi(x)$ first as numerical functions which realize the Hilbert space of the irreducible representation, and by determining the transformation properties of the field operators from those of the functions $\varphi(x)$. We have not done so because we would need a two-component function $\varphi_\nu(x)$ for the irreducible representation, while the field operator has only one component, and the corresponding change of the formalism by the transition from functions to operators would seem artificial.

The case of the particles with spin $\frac{1}{2}$ is analogous. If one wishes to realize the Hilbert space of the irreducible representation by functions ψ_μ which satisfy the Dirac equation, one needs eight components,[7] but the field operator defined by analogy with Eq. (3) has only $2(2s+1)=4$ components.

A draft of an article by V. Bargmann, A. S. Wightman, and E. P. Wigner, received after completion of this paper, gives a detailed derivation of the "doubled" representations and proposes similar assignments.

ACKNOWLEDGMENTS

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[19] From this viewpoint, the neutrino has no antiparticle, but two linearly independent states for a given momentum which are carried into each other by space inversion.

[20] D. Kastler, Ann. Univ. Saraviensis 5, 213, 153 (1956).