

A NEW APPROACH TO THE GEOMETRIZATION OF MATTER

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ABSTRACT: *We show that the sum over geometries in the Lorentzian 4-D state sum model proposed for quantum GR in [1] includes terms which correspond to geometries on manifolds with conical singularities. Natural approximations suggest that they can be interpreted as gauge bosons for the standard model, plus fermions, plus dark matter.*

1. Introduction

Over the last few years, a new model for the quantum theory of gravity has appeared. The model we are referring to is in the class of spin foam models; more specifically it is a Lorentzian categorical state sum model [1]. It is based on state sums on a triangulated manifold, rather than differential equations on a smooth manifold. The model has passed a number of preliminary mathematical hurdles; it is actually finite on any finite triangulation [2]. The biggest hurdle it still has to overcome is an explicit physical interpretation, or differently put, a classical limit.

The purpose of this paper is to outline a radically new way to include matter in this type of theory. While it may seem premature in light of the abovementioned hurdle, it is extremely natural in the setting of the model, and perhaps easier to find than the classical limit itself. If one accepts the approximate arguments we make, the bosonic part of the standard model, rather than any random collection of matter fields, is what appears. The approach yields a fermionic sector as well, but we do not yet understand it. Also a natural family of candidates for dark matter appears in it. The crucial point of departure for this paper is the observation that for a discrete state sum, unlike for a Lagrangian composed of continuum fields, there is no need for the spacetime to consist entirely of manifold points. We find that in investigating the kind of singular points which the model naturally allows a number of intriguing parallels to the standard model arise.

The realization that a specific possibility appears for including matter in the model came as two complimentary points of view on the construction of the model met. The first, the quantum geometric point of view, is an interpretation of the categorical state sum as a sum over Lorentzian discrete quantum geometries [1]. The second, the Group Field Theory picture [3], interprets the state sums on particular triangulations as Feynman diagrams for a quantum field theory on a group manifold. The cross fertilization of these two approaches, as discussed in [2] was an important motivation for the finiteness proof. The sum over Feynman diagrams in the GFT picture can be interpreted as a superposition of quantum geometries in the LCSS point of view.

However, there is an important discrepancy between the two pictures. Not every Feynman diagram in the GFT picture corresponds to a manifold. The most general diagram, as we shall discuss below, is a manifold with conical singularities over surfaces propagating along paths and connecting at vertex-like cobordisms of the surfaces.

Thus we are faced with a dilemma. We must take one of the following paths:

1. Restrict the class of Feynman diagrams we sum over in the GFT picture in a nonlocal and unnatural way,
2. Abandon the GFT picture altogether, and try to add matter to the LCSS picture; or
3. Reinterpret the conical singularities in the GFT picture as matter.

Thus, within the line of development we are pursuing, the new proposal for matter, namely that it results from geometric excitations at conical singularities, is actually the most parsimonious (as well, needless to say, as the most optimistic) possibility to consider.

Put differently, the GFT picture seems to be telling us to take the departure of the LCSS model, namely substituting a superposition of discrete geometries for a continuum picture, to its logical conclusion of including all simplicial complexes.

The development of the LCSS approach to quantum gravity has proceeded, rather surprisingly, at the mathematical level of rigor. The finiteness result cited above is a theorem. Regrettably, the proposal in this paper cannot be formulated as rigorously at this point. We are only able to progress by making approximations. However, it is at least possible to state the results as conjectures, which a careful study of some well defined integrals could in the future prove.

Since we are making a radical departure from existing lines of development towards a fundamental understanding of matter fields, we preface our proposal with a brief historical discussion, which shows that the new suggestion is not as far conceptually from other approaches as might at first appear. This is what motivates the phrase “geometrization of matter” in our title.

2. Matter and Space

Our current understanding of the physics of matter is rooted in the idea of symmetry. Fields in quantum field theories are determined by their quantum numbers, which index representations of the symmetries of the theory. Interactions (vertices in Feynmanology) are linear maps on state spaces which intertwine the action of the symmetries, or as physicists like to say “are not forbidden by the symmetries of the theory.”

Our ideas about symmetry are much older than quantum field theory and derive from our experience of space. Already in the nineteenth century, mathematicians had the idea that different types of geometry correspond to different types of symmetry.

When mathematicians and physicists have tried to understand the symmetries of quantum physics, they have invariably resorted to explaining matter fields in terms of one or another sort of geometry. Aside from the manifestly spacetime symmetries of spin and energy-momentum, every approach to find a fundamental explanation of the internal or gauge symmetries has invoked one or another geometric setting.

Thus gauge theory is formulated as the geometry of vector bundles, Kaluza Klein theories resort to higher dimensional spacetime, supergravity is based in superspace, string theory originally lived in the geometry of loop spaces, while its M theoretic offspring seem to be dwelling in bundles over manifolds of various dimensions again, perhaps with specified submanifolds as well.

One could also mention noncommutative geometry, which studies deformations both of families of symmetries and of the spaces they act on to noncommutative C^* algebras.

Nevertheless, at this point, we cannot say that any of these approaches have really succeeded. A particular difficulty in many of them has been the failure of the standard model to emerge from a limitless set of possibilities.

We want to propose that this historical survey suggests the following points:

1. Our understanding of symmetry is so rooted in geometry that if the fundamental theory of matter is not geometrical we will not find it anytime soon.

2, We need to try a different type of geometry, and hope to get lucky as regards the standard model.

Now we want to claim that the ideas of quantum geometry which have developed in the process of understanding the LCSS/GFT models point to a natural generalization of the geometry of spacetime, namely simplicial complexes. In this new geometric framework, it is plausible that the standard model emerges naturally. To reiterate, the shift from smooth manifolds to simplicial complexes is natural because we have substituted combinatorial state sums for differential equations.

3. The topology of a class of simplicial complexes.

The GFT picture is a generalization of the idea of a state sum attached to a triangulation of a 4-manifold. The picture is to think of a triangulation of a 4-manifold as a 5-valent graph with each edge of the graph refined into a bundle of 4 strands. Different matchings of the strands at an edge are different diagrams.

The vertices of the diagram correspond to the 4-simplices, the edges are the 3-simplices, and the strands are the faces of the 3-simplices. Mathematicians would describe this as the dual 2-skeleton of the triangulation, following strands around until they close into loops, and attaching disks along the loops.

A crucial observation is that the LCSS model in [1], unlike the topological models that preceded it [4], requires only the combinatorial data of the dual 2-skeleton to formulate it, since it has no terms on the edges or vertices of the triangulated manifold. The GFT picture actually goes beyond this and produces all possible strand diagrams as terms in an expansion of a field theory into Feynman graphs.

A standard argument from PL topology tells us when the complex we would build up from such a diagram by adding simplices of dimensions 1 and 0 is a manifold: the links of all simplices must be spheres of appropriate dimension. (The combinatorial picture described above in fact tells us how they should be added). In the situation we are considering, this will be automatically satisfied for the 4- 3- and 2- simplices, the link of a 1-simplex can be any 2-manifold, and the link of a vertex can be any 3-manifold with conical singularities on the surfaces corresponding to the links of the incident edges. The links of vertices can also be described as 3-manifolds with boundary components the links of the edges incident to the vertex, leaving the cones out for simplicity.

For the nonmathematical reader, we note that a cone over any space is the cross product of an interval with the space, with the copy at one end of the interval contracted to a point. A point in a manifold has neighborhoods which are homeomorphic to the cone over a sphere, which is just a ball. A point with a neighborhood homeomorphic to a cone over some other manifold is not a manifold point, and is referred to as a conic singularity. In the case of an isolated singularity, the submanifold over which the cone is constructed is called the link of the point. If instead we have a simplex crossed with a cone on a lower dimensional submanifold, it is the link of the simplex. In a triangulated manifold, all links of simplices are spheres of appropriate dimension.

The proposal we are making is to interpret the web of singularities in such a complex as a Feynman graph; that is to say, we want to interpret the low energy part of the geometry around the cones over surfaces as particles, and the 3-manifolds with boundary connecting them as interaction vertices.

We shall make an attempt at this below, using several approximate techniques. At this point, we wish to underscore the extreme parsimoniousness of this proposal. Nothing is added to the model for quantum GR, no extra dimensions, no larger group, We simply allow a natural larger set of configurations. Once we abandon smooth manifolds for PL ones, there is really no reason not to allow such configurations. In the GFT picture, where geometries appear as fluctuations of a nongeometric vacuum, they are on an equal footing.

4. Conical matter

Now we want to get some picture of what degrees of freedom would appear on a conical singularity in the model of [1]. Since we need to average over all triangulations this is not easy. Also, in order to obtain a model to compare with particle physics as we see it today, we need to describe a universe which has cooled enormously from the Planck temperature, i.e. we need a low energy limit of the model. At present we do not know how to abstract such a limit from the model directly, so we approach this problem by making use of the connection between the discrete models for TQFTs in [4] and the model in [1].

A TQFT is an automatic solution to the renormalization group, in the sense that if we make a refinement of the triangulation on which it is computed the result is unchanged. We want to suggest that the state space of low energy states which survive summing over refinements of triangulation of a cone over a surface is given by the space of states for an associated 2+1 dimensional TQFT on the surface with a puncture. (The puncture would allow information to flow out, thus imitating the conical singular point. The state space for a TQFT with puncture is larger than the one for a closed surface in a TQFT.)

This should be taken as a physical hypothesis at this moment. One reason for believing it is that the LCSS model in [1] is itself obtained by constraining a TQFT.

Then there is the question of what TQFT to expect. Since the state sum in [1] is from the unitary representations of $SL(2,C)$, which is a sort of double for $Su(2)$, the TQFT should be the one for $SU_q(2) \times SU_q(2)$, i.e. a left-right symmetric TQFT produced from a quantum group in the by now standard way. This is also the TQFT we constrain to produce the euclidean signature model for quantum general relativity in [5].

Approximating limits of theories by states of other theories is not an unknown technique in theoretical physics. At this point we do not know how to set q or what value it should take. We could introduce a q into our original model by passing to the Quantum Lorentz Algebra [7]. It may also be that a q emerges from the poorly understood limit of low energy in the model, as a cosmological constant. As we note below, certain choices for q have interesting implications for the particle content of the low energy theory.

We believe that in the future it may be possible to make a stronger argument for this. The reason has to do with the relationship between conformal structures on a surface and flat Lorentzian metrics on the cone over the surface. A Riemann surface can be obtained by quotienting the hyperboloid in three dimensional Minkowski space by a discrete subgroup of the 2+1 dimensional Lorentz group, which is isomorphic to $SL(2,R)$. Quotienting the entire forward timelike cone by the same group yields a flat Lorentzian metric on the entire cone over the surface, except that the conical singularity (the origin in Minkowski space), is not a manifold point, so naive definitions of metrics fail there. Thus, the approach to producing CSW theory by quantizing a bundle over Riemann moduli space could be interpreted as a quantization over the space of flat geometries around a conic singularity. States arising from effects around flat geometry should

be important in understanding the low energy behavior of the model. This argument will be difficult because it will be necessary to treat the effect of the singular point, so we do not attempt it here. We will make further use of the relationship between flat Lorentzian metrics on a cone and constant negative sectional curvature (hyperbolic) metrics on the boundary of the cone in what follows.

At this point, let us note that the space of states assigned to a once punctured torus by a TQFT is a very special object. As demonstrated in [6], it is always a Hopf algebra object in the category associated to the TQFT. In the case of the TQFT produced from $SU(2)_q$, also known as the CSW [8] model, it is a sum of matrix rings, one at each dimension, up to the cutoff determined by q . The unitary part of this has been suggested by Connes and Lott [9] as a natural origin for the gauge symmetry of the standard model.

Thus, according to our ansatz using TQFT states, we find a copy of the gauge bosons of the standard model in the states on a toroidal conical singularity. If we choose the q in such a way as to get exactly 3 matrix blocks in our space [10], we could get exactly the standard model, otherwise we could be led to the conjecture that the standard model is really part of a gauge theory with group $U(1)+SU(2)+SU(3)+SU(4)\dots$ where particles charged in the higher dimensional pieces acquire very large masses and are therefore unseen.

It is therefore interesting to ask what sort of interaction vertices toroidal and other conical singularities might admit. Are the toroidal singularities special, as compared to the higher genus ones?

5. Hyperbolic manifolds and interaction vertices for conic matter.

We remind the reader that we are interpreting regions which look like a conic singularity over a surface crossed with an interval as propagating particles. Now we want to think of the vertices where such topologies meet as interaction vertices. As we explained above, the regions around these vertices are cones over 3-manifolds with conic singularities over surfaces.

We now want to propose a second approximation. The low energy vertices corresponding to these cones over 3-manifolds should be dominated by the flat Lorentzian metrics on them. The physical argument justifying this is that topologies which did not admit flat geometries would become very high energy as we summed over refinements of the triangulation. In the related context of 3d manifolds discussed above we noted a possible connection between this approximation and the TQFT ansatz.

Now we discover an interesting connection. Flat Lorentzian geometries on the cone over a 3-manifold arise in a natural way from hyperbolic structures on it. This is because hyperbolic structures can be recovered as the quotient space of the forward timelike hyperboloid of Minkowski space by discrete subgroups of $SL(2,C)$ acting isometrically on it. Extending the action to the entire forward cone yields a flat Lorentzian 4-geometry on the cone. If we do a similar

construction to produce a 3-manifold with boundary, we obtain a conformal (=hyperbolic) structure on the two dimensional boundary of the 3-manifold at the same time. Thus we are led to a picture where we match the hyperbolic structures on the surfaces linking the edges to the hyperbolic structures assigned to the boundary components of the 3-manifolds linking the vertices to obtain flat geometries surrounding the entire singular part of a 4-D simplicial complex which could arise in our model.

An interesting theorem about hyperbolic structures on 3-manifolds with boundary, called Mostow rigidity [11], tells us that the degrees of freedom of a hyperbolic structure on the bulk are exactly the degrees of freedom of the conformal structure on the boundary components. This means that when we sum over flat geometries in our situation, we get a multiple integral over Teichmuller parameters. This produces a sort of mathematical convergence with the Polyakov approach to string theory. We do not yet know if when we go to quantizing over the space of flat structures any deeper connections to string theory will result.

Now we make another critical observation: the only complete hyperbolic 3-manifolds with finite volume are the ones whose boundary components are tori and Klein bottles [12]. We believe that infinite volume metrics would not make an important low energy contribution to the model, while incomplete hyperbolic metrics would not match flatly at the surfaces linking the edges.

This leads us to a picture in which the low energy interacting world would contain only toroidal and Klein bottle singularities, leaving the higher genus surfaces to decouple and form dark matter. Since our TQFT ansatz suggested that the states on tori could reproduce the gauge bosons for the standard model, while the Klein bottle, being nonorientable, would produce fermionic states, this yields a picture with many similarities to the standard model plus dark matter. We have not yet tried to find an argument for the state space on a Klein bottle.

We would also like to find an approximate argument for how TQFT states might propagate across a vertex described by some cobordism between the incoming and outgoing surfaces. The most obvious would be to simply take the linear map between the surface states given by the TQFT itself. It is interesting to note that for a particularly simple cobordism from two tori to a third this would just give the multiplication of the associative algebra we mentioned above, yielding the gauge algebra of the standard model.

Conclusions

It is clear that the arguments presented here to analyse the behavior of the LCSS model near singular points are very preliminary; it would be rash to jump to the conclusion that we had the unified field theory in hand. Nevertheless the trilogy of standard model bosons, fermions, and weakly interacting higher genus states, especially embedded in a plausible model for quantum general relativity, cannot be ignored.

It does seem safe to say that the simplicity of this model makes it an interesting problem for mathematical physics to study. The connection between particle interactions and hyperbolic structures on 3-manifolds has the advantage that it poses problems for a mathematical subject which has been deeply studied from several points of view and concerning which much is known [12]. It is interesting to note that the question of hyperbolic structure on 3-manifolds with toroidal boundaries is deeply connected to knot theory. We may find knotted vertex structures play a role in this theory; interestingly, they are chiral.

Given the finiteness proof in [2], the conjectures in this paper pertain to limits of families of finite integrals; at least in principle there is reason to hope they can be rigorously formulated and proven.

The fact that the families of flat structures which appear here are parametrized by the Teichmüller parameters on the boundary (Mostow rigidity) means that the subject takes on an unexpected mathematical resemblance to string theory, although it is “world sheets” rather than loops which propagate through spacetime.

We do think the moral can be drawn from this model that there are more possibilities for forming fundamental quantum theories of nature than contemporary theoretical physics seems to recognize.

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