Differential Geometry and its Applications 2 (1992) 385–397
North-Holland

Three-dimensional Einstein-like manifolds

Jürgen Berndt
Mathematisches Institut, Universität zu Köln, W–5000 Köln 41, Germany

Communicated by A. Gray
Received 20 October 1991

Berndt J., Three-dimensional Einstein-like manifolds, Diff. Geom. Appl. 2 (1992) 385–397.

Abstract: One derives a local classification of all three-dimensional Riemannian manifolds whose Ricci tensor satisfies the equation $\nabla(\text{ric} - \frac{1}{4}sg) = \frac{1}{20}ds \odot g$.

Keywords: Einstein manifolds, three-manifolds, conformally flat Stäckel systems, covariant derivative of the Ricci tensor.

MS classification: 53B20, 53C25.

Introduction

One of the nicest metrics a manifold might be equipped with is an Einstein metric. Alas, in dimension three the concept of Einstein metrics leads merely to the spaces of constant curvature. In 1978 A. Gray [5] suggested to study (in arbitrary dimensions) three generalizations of Einstein metrics. Background for his investigation was the fact that any Einstein manifold obviously has parallel Ricci tensor, and conversely, that any Riemannian manifold with parallel Ricci tensor is locally a Riemannian product of Einstein manifolds. Using representation theory of the orthogonal group Gray decomposed the covariant derivative ∇ ric of the Ricci tensor ric of a Riemannian manifold into its irreducible components and derived so in a natural way three classes of Riemannian manifolds, namely

1. the class $\mathfrak{B}$ of Riemannian manifolds whose Ricci tensor is a Codazzi tensor, that is, ∇ ric is symmetric in all its variables (this is precisely the class of Riemannian manifolds with harmonic curvature),
2. the class $\mathfrak{A}$ of Riemannian manifolds whose Ricci tensor is a Killing tensor, that is, the cyclic sum of $(\nabla_X \text{ric})(Y,Z)$ is zero for all vector fields X, Y, Z on the manifold,
3. the class $\mathfrak{Q}$ of Riemannian manifolds whose Ricci tensor satisfies

$$\nabla\Big(\text{ric} - \frac{1}{2n-2}sg\Big) = \frac{n-2}{2(n+2)(n-1)}ds \odot g, \tag{1}$$

where n is the dimension of the manifold M, g its Riemannian metric, ds the differential of the scalar curvature s of M, and $\odot$ denotes the symmetric product of symmetric

Correspondence to: J. Berndt, Mathematisches Institut, Universität zu Köln, Weyertal 86–90, W–5000 Köln 41, Germany

0926-2245/92/$05.00

tensors.

Clearly any Riemannian manifold with parallel Ricci tensor, in particular any locally symmetric space, belongs to each of these three classes. A detailed discussion of these classes of Riemannian manifolds can also be found in Chapter 16 of [2].

While a reasonable amount is known about the geometry of manifolds in $\mathfrak{A}$ or $\mathfrak{B}$, only few seems to be known about the class $\mathfrak{Q}$. In particular there are many examples for manifolds with non-parallel Ricci tensor in $\mathfrak{A}$ or $\mathfrak{B}$, but only few are known which belong to $\mathfrak{Q}$. An exception is dimension two, since any two-dimensional Riemannian manifold is a $\mathfrak{Q}$-space (for the sake of brevity we call manifolds belonging to $\mathfrak{Q}$ also $\mathfrak{Q}$-spaces). The two-dimensional Riemannian manifolds in $\mathfrak{A}$ or $\mathfrak{B}$ are precisely the spaces of constant curvature. Gray remarks in [5, p. 265] that he does not know any interesting manifolds in $\mathfrak{Q}$ which are of dimension greater than two. Later a class of examples for $\mathfrak{Q}$-spaces was presented in [2, p. 448]. These examples arise as certain bundles over one-dimensional manifolds and whose fiber is an Einstein manifold with negative scalar curvature. Moreover, Besse has investigated equation (1), but he could not obtain any classification for manifolds in $\mathfrak{Q}$. The purpose of this article is to classify locally all three-dimensional Riemannian manifolds belonging to $\mathfrak{Q}$. This classification provides also some new examples of $\mathfrak{Q}$-spaces which seem to be unknown up to now.

Theorem. *A three-dimensional Riemannian manifold M belongs to the class $\mathfrak{Q}$ if and only if M is almost everywhere (that is, on an open and dense subset of M) locally isometric to one of the following Riemannian manifolds:*

(I) *a Riemannian symmetric space;*

(II) *a warped product of the form $M_1 \times_f M_2$, where M_1 is a one-dimensional Riemannian manifold, M_2 is a two-dimensional flat Riemannian manifold, and $1/f$ is a positive solution of the second order equation $y'' + cy = 0$ with some $c \in \mathbb{R}$;*

(III) *a warped product of the form $M_2 \times_f M_1$, where M_1 is a one-dimensional Riemannian manifold, M_2 is a Liouville surface with Riemannian metric of (local) Liouville form*

$$ds^2 = (\varphi(x) + \psi(y))(dx^2 + dy^2),$$

and f is given by

$$f^2(x,y) = |\varphi(x)\psi(y)|.$$

Moreover, φ and ψ satisfy one of the following two conditions:

(IIIa) *φ and ψ are non-constant and satisfy the equations*

$$\varphi'^2 = \quad a\varphi^4 + b\varphi^3 + c\varphi^2 + d\varphi,$$

$$\psi'^2 = -a\psi^4 + b\psi^3 - c\psi^2 + d\psi$$

with some $a, b, c, d \in \mathbb{R}$, $a \neq 0$;

(IIIb) *ψ is constant and φ is non-constant and satisfies the equation*

$$\varphi'^2 = \varphi(\varphi + \psi)^2(a\varphi + b)$$

with some $a, b \in \mathbb{R}, a \neq 0$;

(IV) *a Riemannian manifold with Riemannian metric of the form*

$$ds^2 = \underset{1,2,3}{\mathfrak{S}} F_1(x_1)|x_1 - x_2||x_1 - x_3|dx_1^2,$$

where $\mathfrak{S}$ *denotes the cyclic sum and* $1/F_i(x_i) = P(x_i)$ $(i = 1,2,3)$ *with a polynomial* $P(x)$ *of degree five.*

Remarks. 1. The complete, simply connected three-dimensional Riemannian symmetric spaces are the standard models for the spaces of constant curvature, namely the sphere S^3, the Euclidean space E^3 and the real hyperbolic space H^3, and furthermore the Riemannian products $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$. These manifolds provide also the model spaces for all three-dimensional Riemannian manifolds with parallel Ricci tensor.

2. The general solution of $y'' + cy = 0$ is of the form

$$a\cos(\sqrt{c}x) + \frac{b}{\sqrt{c}}\sin(\sqrt{c}x), \qquad \text{if } c > 0,$$

$$a + bx, \qquad \text{if } c = 0,$$

$$a\cosh(\sqrt{-c}x) + \frac{b}{\sqrt{-c}}\sinh(\sqrt{-c}x), \qquad \text{if } c < 0,$$

with some $a, b \in \mathbb{R}$. If M is of type (II), then M has parallel Ricci tensor if and only if $c \leqslant 0, a > 0$ and $a^2c + b^2 = 0$ (in this case M is of constant curvature c).

3. With regard to (III) we put $Q(x) = ax^4 + bx^3 + cx^2 + dx$ and $R(x) = -ax^4 + bx^3 - cx^2 + dx$. Then α is a zero of Q if and only if $-\alpha$ is a zero of R. If the zeros of Q are all distinct, then φ and ψ are (real) elliptic functions. The close relation between Q and R suggests that there is also a close relation between φ and ψ. Indeed, consider φ as a (complex) elliptic function in a complex variable z given by the equation $\varphi'^2 = Q \circ \varphi$. We define a new (complex) elliptic function ψ by the Jacobi transformation of φ, that is, $\psi(z) := -\varphi(iz)$. Then ψ satisfies $\psi'^2 = R \circ \psi$.

If the zeros of Q are not distinct, then φ and ψ can be expressed in terms of elementary functions. This is always possible in case (IIIb). We omit the explicit expression of φ and ψ in these cases, since it is lengthy and not helpful in order to understand the geometry of spaces of type (III).

4. We want to point out that spaces of type (III) and (IV) have always nonparallel Ricci tensor. (We could also consider $a = 0$ in case (III) or a polynomial $P(x)$ of degree less than five in case (IV), but this would lead us merely to spaces of constant curvature.)

The structure of the proof can be described as follows. At first we show that in any three-dimensional $\mathfrak{Q}$-space the Jacobi operator and its covariant derivative commute. Manifolds whose Jacobi operator has this property have recently been classified locally by the author and L. Vanhecke [1] in another context. By means of this classification we get four classes of Riemannian manifolds which contain necessarily all three-dimensional $\mathfrak{Q}$-spaces. These classes are the spaces of constant curvature (which obviously belong to $\mathfrak{Q}$), warped products of the form $M_1 \times_f M_2$ (where M_k is a k-dimensional Riemannian manifold), and two more classes consisting of certain Stäckel manifolds. On the warped products we impose condition (1), which leads to the spaces of type (I) and (II) of the Theorem. For manifolds belonging to the remaining two classes we prove that they are in $\mathfrak{Q}$ if and only if they are conformally flat. This enables us to use Eisenhart's [4] classification of all three-dimensional Stäckel systems in conformal Euclidean space in order to complete the proof of the Theorem.

The Jacobi operator

Let M be a three-dimensional connected C^∞ Riemannian manifold. We denote by TM the tangent bundle of M, by g the Riemannian metric of M, by ∇ the Levi Civita connection of M, by R the Riemannian curvature tensor of M with the convention $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, by Ric and ric the Ricci tensor of M in its $(1,1)$ and $(0,2)$ version respectively, and by s the scalar curvature of M.

Evaluating explicitly the right-hand term of (1) in this situation we get that M is a $\mathfrak{Q}$-space if and only if

$$(\nabla_X \operatorname{ric})(Y,Z) = \tfrac{3}{10}ds(X)g(Y,Z) + \tfrac{1}{20}ds(Y)g(X,Z) + \tfrac{1}{20}ds(Z)g(X,Y), \quad (2)$$

or equivalently,

$$(\nabla_X \operatorname{Ric})Y = \tfrac{3}{10}ds(X)Y + \tfrac{1}{20}ds(Y)X + \tfrac{1}{20}g(X,Y)S, \quad (3)$$

where S denotes the gradient of s. It is known that the curvature tensor R of M can be expressed in terms of the Ricci tensor and the scalar curvature, namely

$$\begin{aligned} R(X,Y)Z = {} & \operatorname{ric}(Y,Z)X - \operatorname{ric}(X,Z)Y + g(Y,Z)\operatorname{Ric}X - g(X,Z)\operatorname{Ric}Y \\ & - \tfrac{1}{2}s(g(Y,Z)X - g(X,Z)Y). \end{aligned}$$

This immediately implies

$$\begin{aligned} (\nabla_W R)(X,Y)Z = {} & (\nabla_W \operatorname{ric})(Y,Z)X - (\nabla_W \operatorname{ric})(X,Z)Y \\ & + g(Y,Z)(\nabla_W \operatorname{Ric})X - g(X,Z)(\nabla_W \operatorname{Ric})Y \\ & - \tfrac{1}{2}ds(W)(g(Y,Z)X - g(X,Z)Y). \end{aligned} \quad (4)$$

We recall that for each tangent vector $v \in T_pM$ $(p \in M)$ the associated Jacobi operator R_v and its "derivative" R'_v (with respect to v) are the selfadjoint endomorphisms of T_pM defined by $R_v = R(\cdot, v)v$ and $R'_v = (\nabla_v R)(\cdot, v)v$. Note that $R_v v = 0$ and $R'_v v = 0$.

Lemma 1. *If $M \in \mathfrak{Q}$, then R_v and R'_v commute for each $v \in TM$.*

Proof. For $v, w \in T_pM$ we get from (2), (3) and (4)

$$R'_v w = \tfrac{1}{5}ds(v)(g(v,v)w - g(v,w)v).$$

Thus any vector $w \in T_pM \setminus \{0\}$ orthogonal to v is an eigenvector of R'_v. On the other hand, from $R_v v = 0$ we see that R_v maps the orthogonal complement $(\mathbb{R}v)^\perp$ of $\mathbb{R}v$ in T_pM into itself. From this we readily get that R_v and R'_v are simultaneously diagonalizable and hence commute with each other. □

The three-dimensional Riemannian manifolds for which R_v and R'_v commute have been classified by the author and L. Vanhecke [1, Theorem 7]. (In order to apply Theorem 7 one should also read the first three lines of its proof.) Combining this classification and Lemma 1 we get

Proposition 1. *If $M \in \mathfrak{Q}$, then M is almost everywhere (that is, on an open and dense subset of M) locally isometric to one of the following spaces:*

(A) *a space of constant Riemannian sectional curvature,*

(B) *a warped product of the form $M_1 \times_f M_2$, where M_1 is a one-dimensional Riemannian manifold, M_2 is a two-dimensional Riemannian manifold, and f is a positive function on M_1,*

(C) *a warped product of the form $M_2 \times_f M_1$, where M_1 is a one-dimensional Riemannian manifold, M_2 is a Liouville surface, and f is given by*

$$f^2(x,y) = |\varphi(x)\psi(y)|,$$

where the functions φ and ψ come from a (local) Liouville form

$$(\varphi(x) + \psi(y))(dx^2 + dy^2)$$

of the Riemannian metric of M_2,

(D) *a three-dimensional Riemannian manifold with Riemannian metric of the form*

$$\mathfrak{S}_{1,2,3} F_1(x_1)|x_1 - x_2||x_1 - x_3|dx_1^2,$$

where $\mathfrak{S}$ denotes the cyclic sum and F_1, F_2, F_3 are positive functions.

As any space of constant curvature is a locally symmetric space and thus locally isometric to a Riemannian symmetric space, it remains to select the $\mathfrak{Q}$-spaces from the spaces of Type (B), (C) or (D).

Manifolds of Type (B)

Let M be a warped product of the form $M_1 \times_f M_2$. We choose a local orthonormal frame field E_1, E_2, E_3 of TM such that E_1 is tangent to M_1 and E_2, E_3 are tangent to M_2 (here, and henceforth, we identify in the usual way functions and vector fields, which are defined on M_1 or M_2, with the corresponding objects defined on M). These vector fields are eigenvectors of the Ricci tensor everywhere, namely (see for example [6, p. 211])

$$\operatorname{Ric} E_1 = \lambda E_1 \quad \text{and} \quad \operatorname{Ric} E_j = \mu E_j \ (j = 2, 3) \tag{5}$$

with

$$\lambda = -2\frac{f''}{f} \quad \text{and} \quad \mu = K - \frac{f''}{f} - \frac{f'^2}{f^2}\,, \tag{6}$$

where K is the Gaussian curvature of M_2 and $'$ denotes differentiation with respect to E_1. The scalar curvature s of M is given by

$$s = 2\Big(K - 2\frac{f''}{f} - \frac{f'^2}{f^2}\Big)\,. \tag{7}$$

The submanifolds $M_1 \times \{q\}$ are totally geodesic in M for each $q \in M_2$. Hence we have

$$\nabla_{E_1} E_1 = 0\,. \tag{8}$$

The submanifolds $\{p\} \times M_2$ are spherical in M for each $p \in M_1$, that is, they are totally umbilical with parallel mean curvature vector field. Since they are totally umbilical, we get

$$g(\nabla_{E_2} E_3, E_1) = 0 = g(\nabla_{E_3} E_2, E_1)\,. \tag{9}$$

Moreover, their mean curvature α is given by

$$\alpha = g(\nabla_{E_2} E_2, E_1) = g(\nabla_{E_3} E_3, E_1) = -\frac{f'}{f}\,, \tag{10}$$

where the last equation follows from the well-known formula for the Levi-Civita connection of warped products (see for example [6, p. 206]).

Proposition 2. *M is in $\mathfrak{Q}$ if and only if*
(i) *M_2 is flat and $1/f$ is a solution of $y'' + cy = 0$ for some $c \in \mathbb{R}$, or*
(ii) *M_2 is of constant curvature and f is constant.*

Proof. In view of (3) we define a $(1,2)$ tensor field H on M by

$$H(X,Y) = \tfrac{3}{10}ds(X)Y + \tfrac{1}{20}ds(Y)X + \tfrac{1}{20}g(X,Y)S.$$

Our goal is to find necessary and sufficient conditions in order that the equations $(\nabla_{E_i}\operatorname{Ric})E_j = H(E_i,E_j)$ are valid for all $i,j \in \{1,2,3\}$. By means of (5), (6), (8), (9) and (10) we calculate

$$\begin{aligned}
(\nabla_{E_1}\operatorname{Ric})E_1 - H(E_1,E_1) &= d(\lambda - \tfrac{2}{5}s)(E_1)E_1 - \tfrac{1}{20}ds(E_2)E_2 - \tfrac{1}{20}ds(E_3)E_3,\\
(\nabla_{E_2}\operatorname{Ric})E_2 - H(E_2,E_2) &= (\alpha(\mu-\lambda) - \tfrac{1}{20}ds(E_1))E_1 + d(\mu - \tfrac{2}{5}s)(E_2)E_2\\
&\quad - \tfrac{1}{20}ds(E_3)E_3,\\
(\nabla_{E_1}\operatorname{Ric})E_2 - H(E_1,E_2) &= -\tfrac{1}{20}ds(E_2)E_1 + d(\mu - \tfrac{3}{10}s)(E_1)E_2,\\
(\nabla_{E_2}\operatorname{Ric})E_1 - H(E_2,E_1) &= -\tfrac{3}{10}ds(E_2)E_1 + (\alpha(\mu-\lambda) - \tfrac{1}{20}ds(E_1))E_2,\\
(\nabla_{E_2}\operatorname{Ric})E_3 - H(E_2,E_3) &= -\tfrac{1}{20}ds(E_3)E_2 + d(\mu - \tfrac{3}{10}s)(E_2)E_3,
\end{aligned}$$

and the corresponding equations obtained by changing E_2 and E_3. These equations yield that M is in $\mathfrak{Q}$ if and only if

$$ds(E_j) = 0 \quad (j = 2,3), \tag{11}$$

$$d(\mu - \tfrac{2}{5}s)(E_j) = 0 = d(\mu - \tfrac{3}{10}s)(E_j) \quad (j = 2,3), \tag{12}$$

$$d(\lambda - \tfrac{2}{5}s)(E_1) = 0 = d(\mu - \tfrac{3}{10}s)(E_1), \tag{13}$$

$$\alpha(\mu - \lambda) - \tfrac{1}{20}ds(E_1) = 0. \tag{14}$$

By means of (7) we see that (11) is valid if and only if K is constant. This and (6) shows that (12) is already a consequence of (11). Next, using (6) and (7), we calculate

$$\lambda - \frac{2}{5}s = \frac{2}{5}\Big(2\frac{f'^2}{f^2} - \frac{f''}{f}\Big) - \frac{4}{5}K = \frac{2}{5}f\Big(\frac{1}{f}\Big)'' - \frac{4}{5}K, \tag{15}$$

$$\mu - \frac{3}{10}s = \frac{2}{5}K - \frac{1}{5}\Big(2\frac{f'^2}{f^2} - \frac{f''}{f}\Big) = \frac{2}{5}K - \frac{1}{5}f\Big(\frac{1}{f}\Big)'',$$

by which we see that (13) is valid if and only if $1/f$ satisfies $y'' + cy = 0$ for some $c \in \mathbb{R}$. By differentiation of (15) and using (13) we conclude

$$\Big(\frac{f''}{f}\Big)' = 2\Big(\frac{f'^2}{f^2}\Big)' = 4\Big(\frac{f'f''}{f^2} - \frac{f'^3}{f^3}\Big). \tag{16}$$

Finally, (6), (7), (10) and (16) yield

$$\begin{aligned}\alpha(\mu - \lambda) - \frac{1}{20}ds(E_1) &= -\frac{f'}{f}\Big(K + \frac{f''}{f} - \frac{f'^2}{f^2}\Big) + \frac{1}{10}\Big(2\frac{f''}{f} + \frac{f'^2}{f^2}\Big)' \\ &= -\frac{f'}{f}\Big(K + \frac{f''}{f} - \frac{f'^2}{f^2}\Big) + \frac{1}{2}\Big(\frac{f'^2}{f^2}\Big)' = -K\frac{f'}{f}.\end{aligned}$$

Hence (20) is valid if and only if $K = 0$ or f is constant. Summing up we see that M is in $\mathfrak{Q}$ if and only if $K = 0$ and $1/f$ is a solution of $y'' + cy = 0$ for some $c \in \mathbb{R}$, or if K and f are constant. □

We now determine those of the warped products of the form $M_1 \times_f M_2$ in $\mathfrak{Q}$, whose Ricci tensor is parallel (and thus are locally isometric to a Riemannian symmetric space). Clearly, if M_2 is of constant curvature and if f is constant, then M is locally isometric to a Riemannian product $S^2 \times \mathbb{R}$, $\mathbb{R}^2 \times \mathbb{R}$ or $H^2 \times \mathbb{R}$—hence the Ricci tensor of M is parallel.

From now on we assume that M_2 is flat and $1/f$ is a solution of $y'' + cy = 0$ for some $c \in \mathbb{R}$. Then M is of constant curvature if and only if $\lambda = \mu$ everywhere. By means of (6) this holds precisely if $(\ln f)'' = 0$, that is, if $f(x) = d\exp(rx)$ for some $d, r \in \mathbb{R}, d > 0$. For such f we get

$$\Big(\frac{1}{f}\Big)'' + c\frac{1}{f} = (r^2 + c)\frac{1}{f}.$$

Thus $1/f$ a solution of $y'' + cy = 0$ if and only if $r^2 + c = 0$. Obviously $c > 0$ is impossible. If $c = 0$, then $r = 0$ and f is constant, which means that $b = 0$, $a > 0$ and M is flat (for a and b see Remark 2). Finally, if $c < 0$, we calculate that

$$d \exp(rx) = a \cosh(\sqrt{-c}x) + \frac{b}{\sqrt{-c}} \sinh(\sqrt{-c}x)$$

holds if and only if either $r = \sqrt{-c}$ and $d = a = b/\sqrt{-c}$ or $r = -\sqrt{-c}$ and $d = a = -b/\sqrt{-c}$. This proves that M is of constant curvature precisely if $a > 0$ and $a^2c + b^2 = 0$. The value K of the curvature can be determined from (6), namely $\lambda = -2f''/f = -2r^2 = 2c$ implies $K = \lambda/2 = c$. Summing up we have seen so far that M is of constant curvature (of value c) if and only if $c \leqslant 0, a > 0$ and $a^2c + b^2 = 0$. Now, if M has parallel Ricci tensor, then in particular we have $0 = (\nabla_{E_2} \operatorname{Ric})E_2 = \alpha(\mu - \lambda)E_1$. Regarding to (10) $\alpha = 0$ implies that f is constant and hence M is flat. Thus we may assume $\alpha \neq 0$ almost everywhere. But this leads to $\lambda = \mu$ almost everywhere and therefore to the spaces of constant curvature. Thus we have proved Remark 2.

Manifolds of Type (C) or (D)

At first we shall derive some formulae for the eigenvalues of the Ricci tensor of manifolds of Type (C) or (D). To begin with we consider a three-dimensional Riemannian manifold M whose Riemannian metric is of the form

$$ds^2 = \sum_{i=1}^{3} \mu_i^2(x_1, x_2, x_3) dx_i^2$$

with some positive functions μ_1, μ_2, μ_3 depending on coordinates x_1, x_2, x_3 on M. Such a Riemannian manifold is often called a triply orthogonal system of surfaces. We put

$$X_i := \frac{\partial}{\partial x_i}, \quad \nu_i := \ln(\mu_i), \quad \nu_{i,j} := \frac{\partial \nu_i}{\partial x_j}, \quad \nu_{i,jk} := \frac{\partial^2 \nu_i}{\partial x_j \partial x_k},$$

and define functions R_{kij}^l and S_j^k on M by

$$R(X_i, X_j)X_k = \sum_{l=1}^{3} R_{kij}^l X_l \quad \text{and} \quad \operatorname{Ric} X_j = \sum_{k=1}^{3} S_j^k X_k.$$

Explicitly we have in terms of the functions μ_1, μ_2, μ_3 for distinct i, j, k (see for instance [3, p.17-22])

$$R_{kij}^i = \nu_{i,j}\nu_{j,k} + \nu_{i,k}\nu_{k,j} - \nu_{i,j}\nu_{i,k} - \nu_{i,jk},$$

$$R_{jij}^i = \nu_{i,j}\nu_{j,j} - \nu_{i,j}\nu_{i,j} - \nu_{i,jj} + \left(\frac{\mu_j^2}{\mu_i^2}\right)(\nu_{j,i}\nu_{i,i} - \nu_{j,i}\nu_{j,i} - \nu_{j,ii}) - \left(\frac{\mu_j^2}{\mu_k^2}\right)\nu_{i,k}\nu_{j,k},$$

$$S_j^k = \left(\frac{1}{\mu_k^2}\right) R_{kij}^i,$$

$$S_i^i = \left(\frac{1}{\mu_j^2}\right) R_{jij}^i + \left(\frac{1}{\mu_k^2}\right) R_{kik}^i.$$

From now on we assume that M is of Type (C) or (D). The Riemannian metric of M is locally of the form

$$ds^2 = (\varphi(x_1) + \psi(x_2))(dx_1^2 + dx_2^2) + |\varphi(x_1)\psi(x_2)|dx_3^2$$

and

$$ds^2 = \mathop{\mathfrak{S}}_{1,2,3} F_1(x_1)|x_1 - x_2||x_1 - x_3|dx_1^2,$$

respectively. Hence M is locally a triply orthogonal system of surfaces. Using the above formulae we obtain by a lengthy, but straightforward computation, that X_1, X_2 and X_3 are eigenvectors of the Ricci tensor everywhere, say $\operatorname{Ric} X_i = \lambda_i X_i$, and the eigenvalue functions λ_1, λ_2 and λ_3 are subject to the relations
– *in case of Type* (C):

$$\lambda_2 - \lambda_3 = \psi T, \quad \lambda_3 - \lambda_1 = \varphi T, \quad \lambda_1 - \lambda_2 = -(\varphi + \psi)T, \tag{17}$$

where

$$\begin{aligned} T = \frac{1}{4(\varphi+\psi)^3}\Big(&\frac{1}{\varphi}(2\varphi''(\varphi+\psi) - \frac{\varphi'^2}{\varphi}(3\varphi+\psi)) \\ &-\frac{1}{\psi}(2\psi''(\varphi+\psi) - \frac{\psi'^2}{\psi}(3\psi+\varphi))\Big), \end{aligned} \tag{18}$$

– *in case of type* (D):

$$\lambda_i - \lambda_j = (x_i - x_j)T \quad (i, j = 1, 2, 3)\ , \tag{19}$$

where

$$T = \frac{\mathfrak{S}_{1,2,3}\left(\frac{2}{F_1}\left(\frac{1}{x_1 - x_3} + \frac{1}{x_1 - x_2}\right) - \left(\frac{1}{F_1}\right)'\right)(x_2 - x_3)^2}{4(x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2}. \tag{20}$$

We now normalize X_1, X_2 and X_3 to unit length and denote the resulting vector fields by E_1, E_2 and E_3, respectively. Note that $\operatorname{Ric} E_i = \lambda_i E_i$ and $X_i = \mu_i E_i$. For the following it will be useful to introduce three notations, namely

$$\omega_{ij}^k := g(\nabla_{E_i} E_j, E_k)\ ,$$

$$s_i := \lambda_{i+1} - \lambda_{i+2} \ \text{(index modulo 3)}\ ,$$

$$\Lambda := s_1 d\lambda_1 + s_2 d\lambda_2 + s_3 d\lambda_3\ .$$

From the classification we have used to obtain Proposition 1 we know already that the Jacobi operator of M and its covariant derivative commute, that is, $L_v := R'_v \circ R_v - R_v \circ R'_v = 0$ for all $v \in TM$. As we are in dimension three, we can express each endomorphism L_v in terms of the Ricci tensor and its covariant derivative. Investigating the equation $L_v = 0$ for various linear combinations v of $E_1(p), E_2(p)$ and $E_3(p)$ (at any fixed point $p \in M$) leads to some relations between the entities $\lambda_i, \omega_{ij}^k, s_i$ and Λ. Since this investigation has been carried out rigorously in [1] we omit it here and just state the conclusion: For all distinct $i, j, k \in \{1,2,3\}$ we have

$$(\lambda_j - \lambda_i)\omega_{jj}^i = (\lambda_k - \lambda_i)\omega_{kk}^i \ , \tag{21}$$

$$\Lambda(E_k) = 2s_i s_k \omega_{jj}^k \quad (k = (j+1) \ \text{modulo} \ 3) \ , \tag{22}$$

and moreover, on the set $\{p \in M \mid T(p) \neq 0\}$ we have

$$\omega_{ij}^k = 0. \tag{23}$$

Note that (23) means $(\nabla_{E_i} \mathrm{ric})(E_j, E_k) = 0$ and that (21) means $(\nabla_{E_j} \mathrm{ric})(E_j, E_i) = (\nabla_{E_k} \mathrm{ric})(E_k, E_i)$.

Equations (17) and (19) show that $T = 0$ holds precisely if M is a space of constant curvature. We will now see that the constancy of T plays an important role with regard to our problem.

Proposition 3. *Let M be of Type* (C) *or* (D). *Then the following statements are equivalent:*

(i) $M \in \mathfrak{Q}$,

(ii) *M is conformally flat,*

(iii) *T is constant.*

Proof. "(i) $\Rightarrow$ (ii)". If $M \in \mathfrak{Q}$, then we see immediately from (1) that $\mathrm{ric} - \frac{1}{4}sg$ is a Codazzi tensor. A well-known theorem due to Weyl and Schouten then says that M is conformally flat. (Note that this argument is true for any three-dimensional $\mathfrak{Q}$-space.)

"(ii) $\Rightarrow$ (iii)". We assume that M is conformally flat. Then $L := \mathrm{ric} - \frac{1}{4}sg$ is a Codazzi tensor and we get for distinct i, j

$$\begin{aligned} d(\lambda_j - \tfrac{1}{4}s)(E_i) &= (\nabla_{E_i} L)(E_j, E_j) = (\nabla_{E_j} L)(E_i, E_j) \\ &= (\lambda_i - \lambda_j)\omega_{ji}^j = (\lambda_j - \lambda_i)\omega_{jj}^i. \end{aligned}$$

This and (21) implies $d\lambda_j(E_i) = d\lambda_k(E_i)$ for all distinct i, j, k by which eventually the constancy of T follows from (17) and (19).

"(iii) $\Rightarrow$ (i)". In the following we always assume that $i, j, k \in \{1,2,3\}$ are distinct. From the second Bianchi identity for the curvature tensor we derive

$$\begin{aligned} 0 &= \sum_{\nu=1}^{3} \big(2(\nabla_{E_\nu} \mathrm{ric})(E_\nu, E_i) - (\nabla_{E_i} \mathrm{ric})(E_\nu, E_\nu)\big) \\ &= 2(\lambda_j - \lambda_i)\omega_{jj}^i + 2(\lambda_k - \lambda_i)\omega_{kk}^i - 2d\lambda_j(E_i) - 2d\lambda_k(E_i) + ds(E_i). \end{aligned}$$

By the assumption T is constant. Hence (17) and (19) imply

$$d\lambda_j(E_i) = d\lambda_k(E_i). \tag{24}$$

Then (21) and the preceding two equations yield

$$(\lambda_j - \lambda_i)\omega^i_{jj} = d(\lambda_j - \tfrac{1}{4}s)(E_i). \tag{25}$$

Using (24), (22), (25), and taking the index modulo three we calculate

$$\begin{aligned} s_i d(\lambda_i - \lambda_{i+2})(E_i) &= s_i d\lambda_i(E_i) + (s_{i+1} + s_{i+2})d\lambda_{i+2}(E_i) = \Lambda(E_i) \\ &= 2s_i s_{i+1}\omega^i_{i+2,i+2} = 2s_i(\lambda_{i+2} - \lambda_i)\omega^i_{i+2,i+2} = 2s_i d(\lambda_{i+2} - \tfrac{1}{4}s)(E_i), \end{aligned}$$

and therefore, again using (24),

$$s_i d\lambda_i(E_i) = \tfrac{4}{3}s_i d\lambda_{i+1}(E_i) = \tfrac{4}{3}s_i d\lambda_{i+2}(E_i). \tag{26}$$

$T = 0$ in a point implies $\lambda_1 = \lambda_2 = \lambda_3$ in that point. Thus on the open kernel U of $\{p \in M | T(p) = 0\}$ the manifold M is of constant curvature and hence in $\mathfrak{Q}$. We define $V = \{p \in M | T(p) \neq 0\}$, which is an open subset of M. We shall prove that $V \in \mathfrak{Q}$. Since $U \cup V$ is an open and dense subset of M we then conclude that (1) holds everywhere and hence the whole of M is in $\mathfrak{Q}$. The following computations and arguments are all restricted to V, which is henceforth to be assumed non-empty. In view of (17) and (19) the functions s_i are non-zero everywhere, hence (26) implies

$$\tfrac{2}{5}ds(E_i) = d\lambda_i(E_i), \tag{27}$$

$$\tfrac{3}{10}ds(E_i) = d\lambda_j(E_i), \tag{28}$$

$$\tfrac{1}{20}ds(E_i) = d(\lambda_j - \tfrac{1}{4}s)(E_i). \tag{29}$$

Again, let H be the $(1,2)$ tensor field

$$H(X,Y) = \tfrac{3}{10}ds(X)Y + \tfrac{1}{20}ds(Y)X + \tfrac{1}{20}g(X,Y)S.$$

Then we calculate with (25), (27) and (29)

$$\begin{aligned} (\nabla_{E_i}\operatorname{Ric})E_i &= d\lambda_i(E_i)E_i + (\lambda_i - \lambda_j)\omega^j_{ii}E_j + (\lambda_i - \lambda_k)\omega^k_{ii}E_k \\ &= d\lambda_i(E_i)E_i + d(\lambda_i - \tfrac{1}{4}s)(E_j)E_j + d(\lambda_i - \tfrac{1}{4}s)(E_k)E_k \\ &= \tfrac{2}{5}ds(E_i)E_i + \tfrac{1}{20}ds(E_j)E_j + \tfrac{1}{20}ds(E_k)E_k = H(E_i)E_i, \end{aligned}$$

and with (25), (23), (28) and (29)

$$\begin{aligned} (\nabla_{E_i}\operatorname{Ric})E_j &= (\lambda_i - \lambda_j)\omega^j_{ii}E_i + d\lambda_j(E_i)E_j + (\lambda_j - \lambda_k)\omega^k_{ij}E_k \\ &= d(\lambda_i - \tfrac{1}{4}s)(E_j)E_i + d\lambda_j(E_i)E_j \\ &= \tfrac{1}{20}ds(E_j)E_i + \tfrac{3}{10}ds(E_i)E_j = H(E_i, E_j). \end{aligned}$$

Thus (3) is valid on V and hence $V \in \mathfrak{Q}$. □

In view of Proposition 3 it remains to find the solutions of $T \equiv \text{const}$.

Case 1. *M of Type* (C). If φ and ψ are both constant on open subsets, then M is flat and hence in $\mathfrak{Q}$ over the corresponding open subset of M. We now consider the case where ψ is constant and φ is non-constant everywhere (that is, φ is non-constant on any open subset). The equation $T = a/4$ for some $a \in \mathbb{R}$ then becomes

$$2\varphi'' - \frac{3\varphi + \psi}{\varphi(\varphi + \psi)}\varphi'^2 = a\varphi(\varphi + \psi)^2.$$

In the regular points of φ we can substitute $p(\varphi) = \varphi'$ and get the Bernoulli equation

$$2pp' - \frac{3\varphi + \psi}{\varphi(\varphi + \psi)}p^2 = a\varphi(\varphi + \psi)^2.$$

The substitution $u = p^2$ then leads to the linear equation

$$u' - \frac{3\varphi + \psi}{\varphi(\varphi + \psi)}u = a\varphi(\varphi + \psi)^2.$$

The general solution of this equation is of the form

$$u(\varphi) = \varphi(\varphi + \psi)^2(a\varphi + b) \text{ with some } b \in \mathbb{R}.$$

Thus we see that φ satisfies

$$\varphi'^2 = \varphi(\varphi + \psi)^2(a\varphi + b)$$

on the set of its regular points. Since by our assumption this set is open and dense, φ satisfies the above equation everywhere. Conversely, if φ satisfies this equation, a straightforward computation proves that $T = a/4$ holds.

Eventually we consider the case where φ and ψ are non-constant everywhere. At first we solve the equation $T = a/4$ for fixed y in an analogous way by using the same substitutions as above. Here we deduce that φ satisfies necessarily the equation

$$\varphi'^2 = a\varphi^4 + b(y)\varphi^3 + c(y)\varphi^2 + d(y)\varphi$$

with some $b(y), c(y), d(y) \in \mathbb{R}$. Then we fix x and solve $T = a/4$ for ψ analogously, which leads to

$$\psi'^2 = -a\psi^4 + \beta(x)\psi^3 + \gamma(x)\psi^2 + \delta(x)\psi$$

with some $\beta(x), \gamma(x), \delta(x) \in \mathbb{R}$. Finally, inserting these expressions for φ and ψ in the equation $T = a/4$ shows that $\beta(x) = b(y), \gamma(x) = -c(y)$ and $\delta(x) = d(y)$. Thus we have $T = a/4$ for some $a \in \mathbb{R}$ if and only if

$$\begin{aligned}\varphi'^2 &= \quad a\varphi^4 + b\varphi^3 + c\varphi^2 + d\varphi \quad \text{and} \\ \psi'^2 &= -a\psi^4 + b\psi^3 - c\psi^2 + d\psi\end{aligned}$$

with some $b, c, d \in \mathbb{R}$. A different proof for the last statement can be found in [4, p. 64/65].

Case 2. *M of Type* (D). We consider the equation $T = -a/4$ for some $a \in \mathbb{R}$ and substitute $G_\nu = 1/F_\nu$ ($\nu = 1,2,3$). Let $i, j, k \in \{1,2,3\}$ be distinct. For fixed x_j and

x_k the equation $T = -a/4$ is linear and can be solved explicitly in the usual way. Its solution is of the general form

$$G_i(x_i) = ax_i^5 + \sum_{\nu=0}^{4} b_{\nu i}(x_j, x_k) x_i^\nu$$

with some $b_{\nu i}(x_j, x_k) \in \mathbb{R}$ $(\nu = 0, \dots, 4)$. Now, inserting these expressions for G_1, G_2 and G_3 into the equation $T = -a/4$ leads via a straightforward and lengthy computation to the necessary and sufficient conditions

$$b_{\nu 1}(x_2, x_3) = b_{\nu 2}(x_1, x_3) = b_{\nu 3}(x_1, x_2) \quad (v = 0, \dots, 4).$$

Thus we have $T = -a/4$ if and only if $1/F_i(x_i) = P(x_i)$ $(i = 1, 2, 3)$ with a real polynomial $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. Also in this case one can find a different proof in [4, p. 65/66].

We remark that in both cases and in view of (17) and (19) the vanishing of T (or equivalently, of the coefficient a) corresponds to the spaces of constant curvature. In case of $T \neq 0$ the number d of distinct eigenvalues of the Ricci tensor is three, whereas for a three-dimensional Riemannian manifold with parallel Ricci tensor always $d \leqslant 2$ (see Remark 1). Thus M has non-parallel Ricci tensor in case $T \neq 0$.

In view of Propositions 1, 2 and 3 and the preceding computations regarding the constancy of T we have finished the proof of the Theorem.

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