Gauge Theories and Fiber Bundles: Definitions, Pictures, and Results

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Abstract

A pedagogical but concise overview of fiber bundles and their connections is provided, in the context of gauge theories in physics. The emphasis is on defining and visualizing concepts and relationships between them, as well as listing common confusions, alternative notations and jargon, and relevant facts and theorems. Special attention is given to detailed figures and geometric viewpoints, some of which would seem to be novel to the literature. Topics are avoided which are well covered in textbooks, such as historical motivations, proofs and derivations, and tools for practical calculations. The present paper is best read in conjunction with the similar paper on Riemannian geometry cited herein.

Contents

1 Introduction

A manifold includes a tangent space associated with each point. A frame defines a basis for the tangent space at each point, and a connection allows us to compare vectors at different points, leading to concepts including the covariant derivative and curvature. All of these concepts, covered in a similar style in [6], can be applied to an arbitrary vector space associated with each point in place of the tangent space. This is the idea behind gauge theories. Both manifolds with connection and gauge theories can then be described using the mathematical language of fiber bundles.

Throughout the paper, warnings concerning a common confusion or easily misunderstood concept are separated from the core material by boxes, as are intuitive interpretations or heuristic views that help in understanding a particular concept. Quantities are written in **bold** when first mentioned or defined.

2 Gauge theory

2.1 Matter fields and gauges

Gauge theories associate each point x on the spacetime manifold M with a (usually complex) vector space $V_x \cong \mathbb{C}^n$, called the **internal space**. A V-valued 0-form $\vec{\Phi}$ on M is called a **matter field**. A matter field lets us define analogs of the quantities associated with a change of frame (see [6]) as follows.

A basis for each V_x is called a **gauge**, and is the analog of a frame; choosing a gauge is sometimes called **gauge fixing**. Like the frame, a gauge is generally considered on a region $U \subseteq M$. The analog of a change of frame is then a (local) **gauge transformation** (AKA gauge transformation of the second kind), a change of basis for each V_x at each point $x \in U$. This is viewed as a representation of a **gauge group** (AKA symmetry group, structure group) G

acting on V at each point $x \in U$, so that we have

$$\begin{aligned} \gamma^{-1} &\colon U \to G \\ \rho &\colon G \to GL(V) \\ \Rightarrow \check{\gamma}^{-1} \equiv \rho\gamma^{-1} &\colon U \to GL(V), \end{aligned} \tag{2.1}$$

and if we choose a gauge it can thus be associated with a matrix-valued 0-form or tensor field

$$(\gamma^{-1})^{\beta}{}_{\alpha} \colon U \to GL(n, \mathbb{C}), \tag{2.2}$$

so that the components of the matter field Φ^{α} transform according to

$$\Phi'^{\beta} = \gamma^{\beta}{}_{\alpha}\Phi^{\alpha}. \tag{2.3}$$

Since all reps of a compact G are similar to a unitary rep, for compact G we can then choose a **unitary gauge**, which is defined to make gauge transformations unitary, so

$$\check{\gamma}^{-1} \colon U \to U(n). \tag{2.4}$$

This is the analog of choosing an orthonormal frame, where a change of orthonormal frame then consists of a rotation at each point. A **global gauge transformation** (AKA gauge transformation of the first kind) is a gauge transformation that is the same at every point. If the gauge group is non-abelian (i.e. most groups considered beyond $U(1)$), the matter field is called a **Yang-Mills field** (AKA YM field).

△ The term "gauge group" can refer to the abstract group G, the matrix rep of this group on $GL(V)$, the matrix rep on $U(n)$ under a unitary gauge, or the infinite-dimensional group of maps γ^{-1} under composition.

△ As with vector fields, the matter field $\vec{\Phi}$ is considered to be an intrinsic object, with only the components Φ^{α} changing under gauge transformations.

△ Unlike with the frame, whose global existence is determined by the topology of M, there can be a choice as to whether a global gauge exists or not. This is the essence of fiber bundles, as we will see in Section 3.1.

2.2 The gauge potential and field strength

We can then define the parallel transporter for matter fields to be a linear map

$$\|_{C} \colon V_p \to V_q, \tag{2.5}$$

where C is a curve in M from p to q. Choosing a gauge, the parallel transporter can be viewed as a (gauge-dependent) map

$$\|^{\beta}{}_{\alpha} \colon \{C\} \to GL(n, \mathbb{C}). \tag{2.6}$$

This determines the (gauge-dependent) matter field connection 1-form

$$\Gamma^{\beta}{}_{\alpha}(v) : T_x M \to gl(n, \mathbb{C}), \tag{2.7}$$

which can also be written when acting on a $\mathbb{C}^n$-valued 0-form as $\check{\Gamma}(v)\vec{\Phi}$. The values of the parallel transporter are again viewed as a rep of the gauge group G, so that the values of the connection are a rep of the Lie algebra $\mathfrak{g}$, and if G is compact we can choose a unitary gauge so that $\mathfrak{g}$ is represented by anti-hermitian matrices. We then define the **gauge potential** (AKA gauge field, vector potential) $\check{A}$ by

$$\check{\Gamma} \equiv -iq\check{A}, \tag{2.8}$$

where q is called the **coupling constant** (AKA charge, interaction constant, gauge coupling parameter). Note that $A^{\beta}{}_{\alpha}$ are then hermitian matrices in a unitary gauge. The covariant derivative is then

$$\nabla_v \Phi = \mathrm{d}\vec{\Phi}(v) - iq\check{A}(v)\vec{\Phi}, \tag{2.9}$$

which can be generalized to $\mathbb{C}^n$-valued k-forms in terms of the exterior covariant derivative as

$$D\vec{\Phi} = \mathrm{d}\vec{\Phi} - iq\check{A} \wedge \vec{\Phi}. \tag{2.10}$$

For a matter field (0-form), this is often written after being applied to e_μ as

$$D_\mu \vec{\Phi} = \partial_\mu \vec{\Phi} - iq\check{A}_\mu \vec{\Phi}, \tag{2.11}$$

where μ is then a spacetime index and

$$\check{A}_\mu \equiv \check{A}(e_\mu) \tag{2.12}$$

are $gl(n, \mathbb{C})$-valued components.

This connection defines a curvature

$$\check{R} \equiv \mathrm{d}\check{\Gamma} + \check{\Gamma} \wedge \check{\Gamma}, \tag{2.13}$$

which lets us define the **field strength** (AKA gauge field) $\check{F}$ by

$$\begin{aligned} \check{R} &\equiv -iq\check{F} \\ \Rightarrow \check{F} &= d\check{A} - iq\check{A} \wedge \check{A}. \end{aligned} \tag{2.14}$$

2.3 Spinor fields

A matter field can also transform as a spinor, in which case it is called a **spinor matter field** (AKA spinor field), and is a 0-form on M which e.g. for Dirac spinors takes values in $V \otimes \mathbb{C}^4$. The gauge component then responds to gauge transformations, while the spinor component responds to changes of frame. Similarly, a matter field on M^{r+s} taking values in $V \otimes \mathbb{R}^{r+s}$ is called a **vector matter field** (AKA vector field), where the vector component responds to changes of frame. Finally, a matter field without any frame-dependent component is called a **scalar matter field** (AKA scalar field), and a matter field taking values in $\mathbb{C}$ (which can be viewed as either vectors or scalars) is called a **complex scalar matter field** (AKA complex scalar field, scalar field). A spinor matter field with gauge group $U(1)$ is called a **charged spinor field**.

$\triangle$ It is important remember that spinor and vector matter fields use the tensor product, not the direct sum, and therefore cannot be treated as two independent fields. In particular, the field value $\phi \otimes \psi \in V \otimes \mathbb{C}^4$ is identical to the value $-\phi \otimes -\psi$, which has consequences regarding the existence of global spinor fields, as we will see in Section 5.7.

In order to directly map changes of frame to spinor field transformations, one must use an orthonormal frame so all changes of frame are rotations. The connection associated with an orthonormal frame is therefore called a **spin connection**, and takes values in $so(3,1) \cong \mathrm{spin}(3,1)$. Thus the spin connection and gauge potential together provide the overall transformation of a spinor field under parallel transport. All of the above can be generalized to arbitrary dimension and signature.

Tangent space $T_pM = \mathbb{R}^{(r+s)}$	Spinor space $S_p = \mathbb{K}^m$	Internal space $V_p = \mathbb{C}^n$
Frame	Standard basis of $\mathbb{K}^m$ identified with an initial orthonormal frame on M	Gauge
Change of frame	$U_p \in \mathrm{Spin}(r,s)$ associated with change of orthonormal frame $\check{\gamma}_p$	Gauge transformation
Vector field $p \mapsto w \in T_pM$	Spinor field $p \mapsto \psi \in S_p$	Complex / YM field $p \mapsto \phi \in V_p$
Connection $v \mapsto \check{\Gamma}(v) \in gl(r,s)$	Spin connection $v \mapsto \check{\omega}(v) \in so(r,s)$, the bivectors	Gauge potential $v \mapsto \check{A}(v) \in gl(n,\mathbb{C})$
Curvature $\check{R} = \mathrm{d}\check{\Gamma} + \check{\Gamma} \wedge \check{\Gamma}$	Curvature $\check{R} = \mathrm{d}\check{\omega} + \check{\omega} \wedge \check{\omega}$	Field strength $\check{F} = d\check{A} - iq\check{A} \wedge \check{A}$

Table 2.1: Constructs as applied to the various spaces associated with a point $p \in M$ in spacetime and a vector v at p.

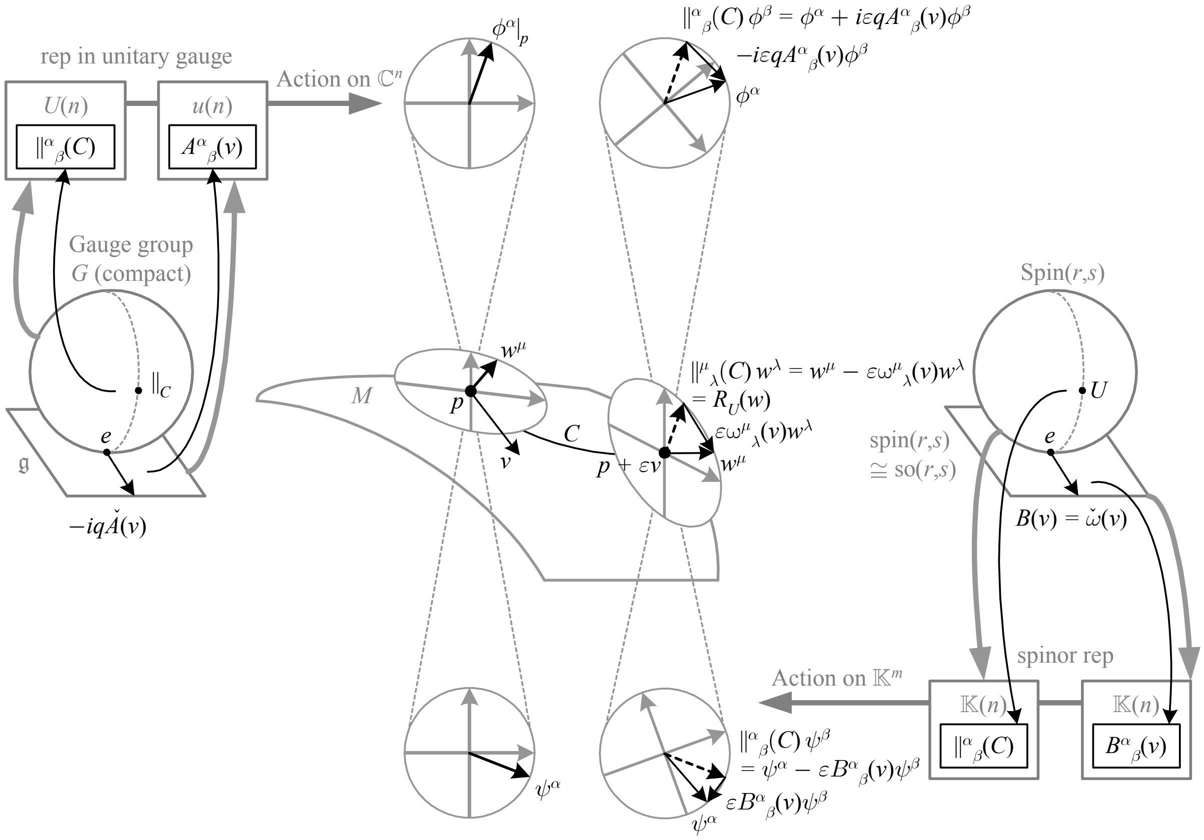

FIGURE 2.1: A matter field can be the tensor product of a complex scalar or Yang-Mills field ϕ and a spinor field ψ. YM fields use a connection and gauge (frame) which are independent of the spacetime manifold frame, while spinor fields mirror the connection and changes in frame of the spacetime manifold. YM fields are acted on by reps of the gauge group and its Lie algebra, while spinor fields are acted on by reps of the Spin group and its Lie algebra. In the figure we assume an infinitesimal curve C with tangent v, an orthonormal frame, a spin connection, and a unitary gauge.

△ Note that a Lorentz transformation on all of flat Minkowski space, which is the setting for many treatments of this material, induces a change of coordinate frame that is the same Lorentz transformation on every tangent space, thus simplifying the above picture by eliminating the need to consider parallel transport on the curved spacetime manifold.

△ The spinor space is an internal space, but its changes of frame are driven by those of the spacetime manifold. The question of whether a global change of orthonormal frame can be mapped to globally defined elements in Spin(r, s) across coordinate charts in a consistent way is resolved in Section 5.7 in terms of fiber bundles.

3 Defining bundles

When introducing tangent spaces on a manifold M^n, the tangent bundle is usually defined to be the set of tangent spaces at every point within the region of a coordinate chart $U \to \mathbb{R}^n$, i.e. it is defined as the cartesian product $U \times \mathbb{R}^n$. Globally, one uses an atlas of charts covering M, with coordinate transformations $\mathbb{R}^n \to \mathbb{R}^n$ defining how to consider a vector field across charts. Here we want to take the same approach to define the global version of the tangent bundle, with analogs for frames and internal spaces.

3.1 Fiber bundles

In defining fiber bundles we first consider a **base space** M and a **bundle space** (AKA total space, entire space) E, which includes a surjective **bundle projection** (AKA bundle submersion, projection map)

$$\pi\colon E \to M. \tag{3.1}$$

In the special case that M and F are manifolds, we require the bundle projections π to be (infinitely) differentiable, and E without any further structure is called a **fibered manifold**.

The space E becomes a **fiber bundle** (AKA fibre bundle) if each **fiber over *x*** $\pi^{-1}(x)$, where $x \in M$, is homeomorphic to an **abstract fiber** (AKA standard fiber, typical fiber, fiber space, fiber) F; specifically, we must have the analog of an atlas, a collection of open **trivializing neighborhoods** $\{U_i\}$ that cover M, each with a **local trivialization**, a homeomorphism

$$\begin{aligned} \phi_i\colon \pi^{-1}(U_i) &\to U_i \times F \\ p &\mapsto (\pi(p), f_i(p))\,, \end{aligned} \tag{3.2}$$

which in a given $\pi^{-1}(x)$ allows us to ignore the first component and consider the last as a homeomorphism

$$f_i\colon \pi^{-1}(x) \to F. \tag{3.3}$$

This property of a bundle is described by calling it **locally trivial** (AKA a **local product space**), and if all of M can be made a trivializing neighborhood, then E is a **trivial bundle**, i.e. $E \cong M \times F$. The topology of a non-trivial bundle can be defined via E itself, or imputed by the local trivializations. Note that if F is discrete, then E is a covering space of M, and if M is contractible, then E is trivial. If F is given additional structure, f_i must remain an isomorphism with respect to this structure.

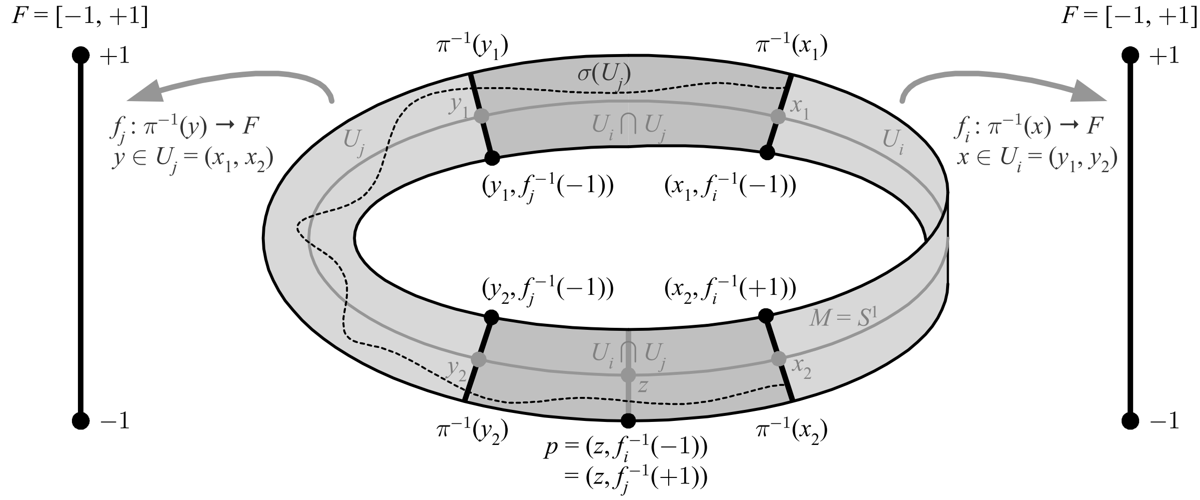

FIGURE 3.1: The **Mï¿½bius strip** (AKA Mï¿½bius band) has a base space which is a circle $M = S^1$, a fiber which is a line segment $F = [-1, +1]$, and is non-trivial, since it requires at least two trivializing neighborhoods. In the figure, the fiber over z has two different descriptions under the two local trivializations, and a local section σ (defined below) is depicted.

△ Fiber bundles are denoted by various combination of components and maps in various orders, frequently (E, M, F), (E, M, π), or (E, M, π, F). Other notations include $\pi\colon E \to M$ and $F \longrightarrow E \xrightarrow{\pi} M$ or just $F \longrightarrow E \longrightarrow M$.

△ The distinction between the fiber and the fiber over x is sometimes not made clear; it is important to remember that the abstract fiber F is not part of the bundle space E.

A **bundle map** (AKA bundle morphism) is a pair of maps

$$\Phi_E\colon E \to E' \tag{3.4}$$

$$\Phi_M\colon M \to M' \tag{3.5}$$

between bundles that map fibers to fibers, i.e.

$$\pi'(\Phi_E(p)) = \Phi_M(\pi(p)). \tag{3.6}$$

Note that if Φ_M is the identity map, the bundles are over the same base space M and this reduces to a single map satisfying

$$\pi'(\Phi(p)) = \pi(p). \tag{3.7}$$

A **section** (AKA cross section) of a fiber bundle is a continuous map

$$\sigma\colon M \to E \tag{3.8}$$

that satisfies

$$\pi(\sigma(x)) = x. \tag{3.9}$$

At a point $x \in M$ a **local section** always exists, being only defined in a neighborhood of x; however global sections may not exist.

△ It is important to remember that the base space M is not part of the bundle space E. In particular, since a global section may not exist, the base space cannot in general be viewed as being embedded in the bundle space, and even when it can be, such an embedding is in general arbitrary. An exception is when there is a canonical global section, for example the zero section as depicted in the Mï¿½bius strip above (and in a vector bundle in general, see Section 4.2).

3.2 G-bundles

In the fiber over a point $\pi^{-1}(x)$ in the intersection of two trivializing neighborhoods on a bundle (E, M, F), we have a homeomorphism

$$f_i f_j^{-1} \colon F \to F. \tag{3.10}$$

If each of these homeomorphisms is the (left) action of an element $g_{ij}(x) \in G$, then G is called the **structure group** of E. This action is usually required to be faithful, so that each $g \in G$ corresponds to a distinct homeomorphism of F. The map

$$g_{ij} \colon U_i \cap U_j \to G \tag{3.11}$$

is called a **transition function**; the existence of transition functions for all overlapping charts makes $\{U_i\}$ a ***G*-atlas** and turns the bundle into a ***G*-bundle**. Applying the action of g_{ij} to an arbitrary $f_j(p)$ yields

$$f_i(p) = g_{ij}\left(f_j(p)\right). \tag{3.12}$$

For example, the Mï¿½bius strip in the previous figure has a structure group $G = \mathbb{Z}_2$, where the action of $0 \in G$ is multiplication by $+1$, and the action of $1 \in G$ is multiplication by -1. In the top intersection $U_i \cap U_j$, $g_{ij} = 0$, so that f_i and f_j are identical, while in the lower intersection $g_{ij} = 1$, so that

$$f_i(p) = g_{ij}\left(f_j(p)\right) = 1\left(f_j(p)\right) = -f_j(p). \tag{3.13}$$

At a point in a triple intersection $U_i \cap U_j \cap U_k$, the **cocycle condition**

$$g_{ij} g_{jk} = g_{ik} \tag{3.14}$$

can be shown to hold, which implies

$$g_{ii} = e \tag{3.15}$$

and

$$g_{ji} = g_{ij}^{-1}. \tag{3.16}$$

Going the other direction, if we start with transition functions from M to G acting on F that obey the cocycle condition, then they determine a unique G-bundle E.

△ It is important to remember that the left action of G is on the abstract fiber F, which is not part of the entire space E, and whose mappings to E are dependent upon local trivializations. A left action on E itself based on these mappings cannot in general be consistently defined, since for non-abelian G it will not commute with the transition functions.

A given G-atlas may not need all the possible homeomorphisms of F between trivializing neighborhoods, and therefore will not "use up" all the possible values in G. If there exists trivializing neighborhoods on a G-bundle whose transition functions take values only in a subgroup H of G, then we say the structure group G is **reducible** to H. For example, a trivial bundle's structure group is always reducible to the trivial group consisting only of the identity element.

3.3 Principal bundles

A **principal bundle** (AKA principal G-bundle) (P, M, π, G) has a topological group G as both abstract fiber and structure group, where G acts on itself via left translation as a transition function across trivializing neighborhoods, i.e.

$$f_i(p) = g_{ij} f_j(p), \tag{3.17}$$

where the operation of g_{ij} is the group operation. Note that the fiber over a point $\pi^{-1}(x)$ is only homeomorphic as a space to G in a given trivializing neighborhood, and so is missing a unique identity element and is a G-torsor, not a group.

A principal bundle lets us introduce a consistent right action of G on $\pi^{-1}(x)$ (as opposed to the left action on the abstract fiber). This right action is defined by

$$\begin{aligned} g(p) &\equiv f_i^{-1}\left(f_i(p) g\right) \\ \Rightarrow f_i\left(g(p)\right) &= f_i(p) g \end{aligned} \tag{3.18}$$

for $p \in \pi^{-1}(U_i)$, where in an intersection of trivializing neighborhoods $U_i \cap U_j$ we see that

$$\begin{aligned} g(p) &= f_j^{-1}\left(f_j(p) g\right) \\ &= f_i^{-1} f_i f_j^{-1}\left(f_j(p) g\right) = f_i^{-1}\left(g_{ij} f_j(p) g\right) \\ &= f_i^{-1}\left(f_i(p) g\right) = g(p), \end{aligned} \tag{3.19}$$

i.e. $g(p)$ is consistently defined across trivializing neighborhoods. Via this fiber-wise action, G then has a right action on the bundle P.

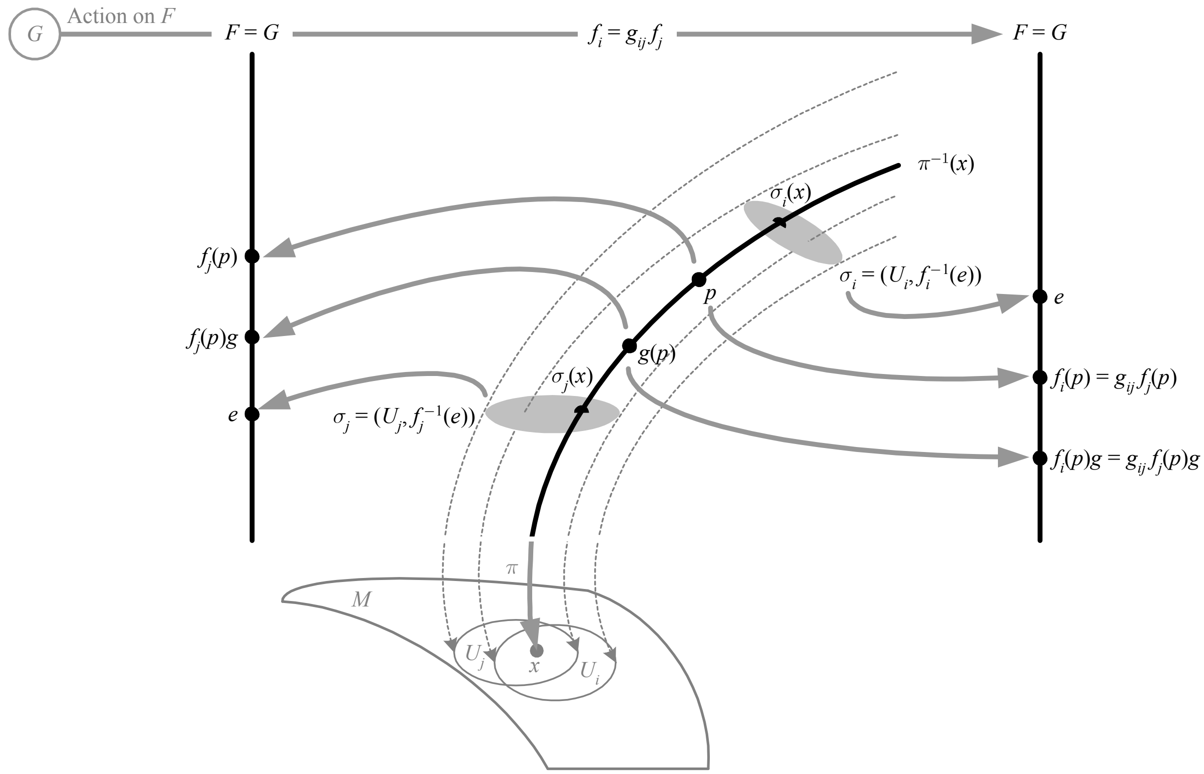

FIGURE 3.2: A principal bundle has the same group G as both abstract fiber and structure group, where G acts on itself via left translation. G also has a right action on the bundle itself, which is consistent across trivializing neighborhoods. The identity sections (defined below) are also depicted.

△ It is important to remember that M is not part of E, and that the depiction of each fiber in the bundle $\pi^{-1}(x) \in E$ as "hovering over" the point $x \in M$ is only valid locally.

△ Note that from its definition and basic group properties, the right action of G on $\pi^{-1}(x)$ is automatically free and transitive (making $\pi^{-1}(x)$ a "right G-torsor"). An equivalent definition of a principal bundle excludes G as a structure group but includes this free and transitive right action of G. Also note that the definition of the right action is equivalent to saying that $f_i \colon \pi^{-1}(x) \to G$ is equivariant with respect to the right action of G on $\pi^{-1}(x)$ and the right action of G on itself.

△ A principal bundle is sometimes defined so that the structure group acts on itself by right translation instead of left. In this case the action of G on the bundle must be a left action.

△ A principal bundle can also be denoted $P(M, G)$ or $G \hookrightarrow P \xrightarrow{\pi} M$.

Since the right action is an intrinsic operation, a **principal bundle map** between principal G-bundles (e.g. a principal bundle automorphism) is required to be equivariant with regard to it, i.e. we require

$$\Phi_E(g(p)) = g(\Phi_E(p)), \tag{3.20}$$

or in juxtaposition notation,

$$\Phi_E(pg) = \Phi_E(p)g. \tag{3.21}$$

In fact, any such equivariant map is automatically a principal bundle map, and if the base spaces are identical and unchanged by Φ_E, then Φ_E is an isomorphism. For a principal bundle map

$$\Phi_E\colon (P', M', G') \to (P, M, G) \tag{3.22}$$

between bundles with different structure groups, we must include a homomorphism

$$\Phi_G\colon G' \to G \tag{3.23}$$

between structure groups so that the equivariance condition becomes

$$\Phi_E(g(p)) = \Phi_G(g)(\Phi_E(p)), \tag{3.24}$$

or in juxtaposition notation,

$$\Phi_E(pg) = \Phi_E(p)\Phi_G(g). \tag{3.25}$$

△ Note that the right action of a fixed $g \in G$ is thus not a principal bundle automorphism, since for non-abelian G it will not commute with another right action.

A principal bundle has a global section iff it is trivial. However, within each trivializing neighborhood on a principal bundle we can define a local **identity section**

$$\sigma_i(x) \equiv f_i^{-1}(e), \tag{3.26}$$

where e is the identity element in G. In $U_i \cap U_j$, we can then use $f_i(\sigma_i) = e$ to see that the identity sections are related by the right action of the transition function:

$$\begin{aligned} g_{ij}(\sigma_i) &= f_i^{-1}\left(f_i(\sigma_i)g_{ij}\right) \\ &= f_i^{-1}\left(g_{ij}\right) \\ &= f_i^{-1}\left(g_{ij}f_j(\sigma_j)\right) \\ &= f_i^{-1}\left(f_i(\sigma_j)\right) \\ &= \sigma_j, \end{aligned} \tag{3.27}$$

or in juxtaposition notation,

$$\sigma_j = \sigma_i g_{ij}. \tag{3.28}$$

△ The different actions of G are a potential source of confusion. g_{ij} has a left action on the abstract fiber of a G-bundle, which on a principal bundle becomes left group multiplication, and also has a right action on the bundle itself that relates the elements in the identity section.

If G is a closed subgroup of a Lie group P (and thus also a Lie group by Cartan's theorem), then $(P, P/G, G)$ is a principal G-bundle with base space the (left) coset space P/G. The right action of G on the entire space P is just right translation.

4 Generalizing tangent spaces

In this section we use matrix notation to reduce clutter, remembering that bases are row vectors and are acted on by matrices from the right. We retain index notation when acting on vector components to avoid confusion with operations on intrinsic vectors.

4.1 Associated bundles

If two G-bundles (E, M, F) and (E', M, F'), with the same base space and structure group, also share the same trivializing neighborhoods and transition functions, then they are each called an **associated bundle** with regard to the other. It is possible to construct (up to isomorphism) a unique principal G-bundle associated to a given G-bundle; going in the other direction, given a principal G-bundle and a left action of G on a fiber F, we can construct a unique associated G-bundle with fiber F. In particular, given a principal bundle (P, M, G), the rep of G on itself by inner automorphisms defines an associated bundle $(\mathrm{Inn}P, M, G)$, and the adjoint rep of G on $\mathfrak{g}$ defines an associated bundle $(\mathrm{Ad}P, M, \mathfrak{g})$. If G has a linear rep on a vector space $\mathbb{K}^n$, this rep defines an associated bundle $(E, M, \mathbb{K}^n)$, which we explore next.

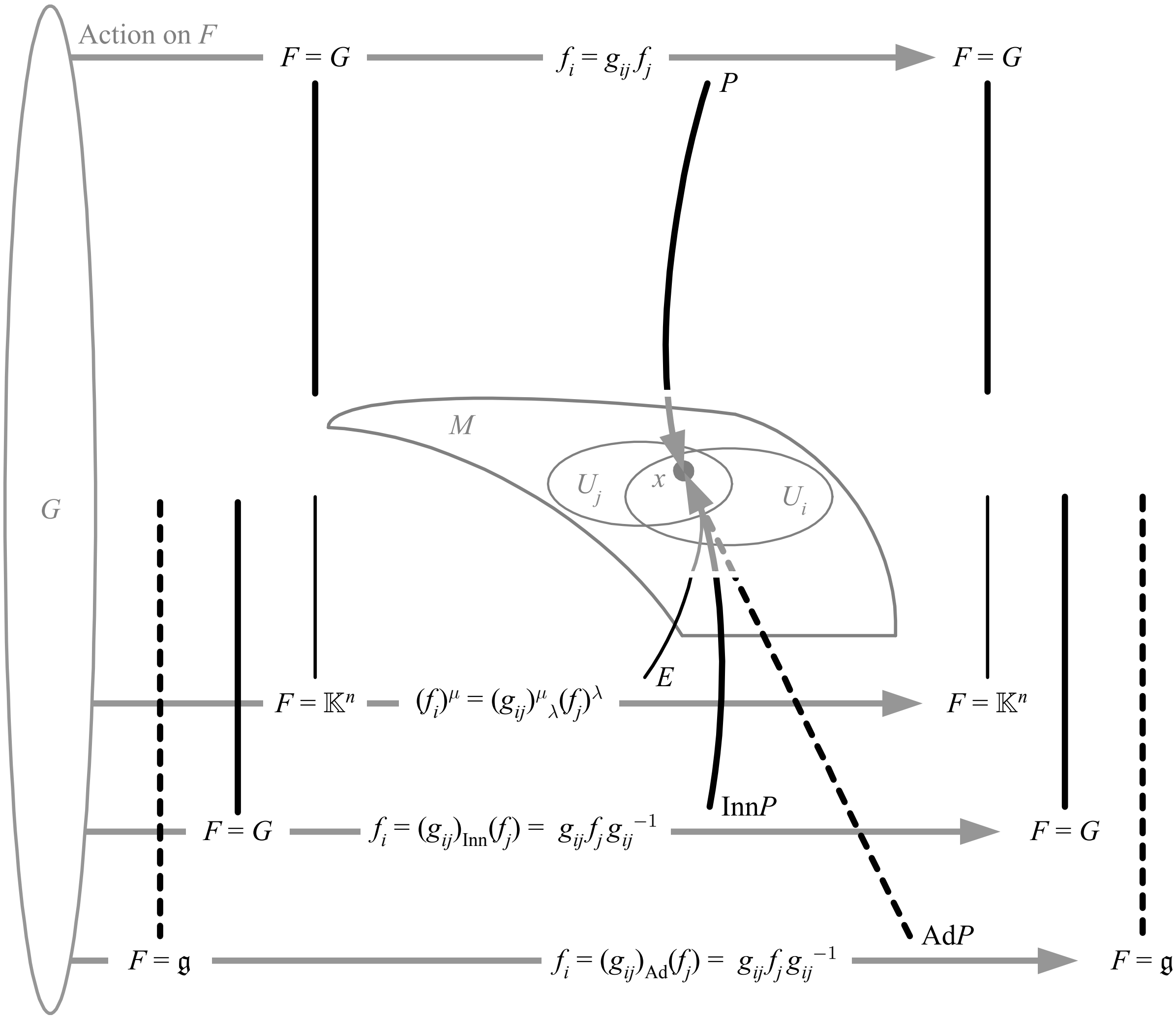

FIGURE 4.1: Given a principal bundle, we can construct an associated bundle for the action of G on a vector space $\mathbb{K}^n$ by a linear rep, on itself by inner automorphisms, and on its Lie algebra $\mathfrak{g}$ by the adjoint rep. The action of the structure group is shown in general and for the case in which G is a matrix group, with matrix multiplication denoted as juxtaposition. Although denoted identically, the f_i are those corresponding to each bundle.

△ The G-bundle E with fiber F associated to a principal bundle P is sometimes written

$$E = P \times_G F \equiv (P \times F)/G, \tag{4.1}$$

where the quotient space collapses all points in the product space which are related by the right action of some $g \in G$ on P and the right action of g^{-1} on F.

4.2 Vector bundles

A **vector bundle** $(E, M, \pi, \mathbb{K}^n)$ has a vector space fiber $\mathbb{K}^n$ (assumed here to be $\mathbb{R}^n$ or $\mathbb{C}^n$) and a structure group that is linear ($G \subseteq GL(n, \mathbb{K})$) and therefore acts as a matrix across trivializing neighborhoods, i.e.

$$f_i(p) = g_{ij} f_j(p), \tag{4.2}$$

where the operation of g_{ij} is now matrix multiplication on the vector components $f_j(p) \in \mathbb{K}^n$. If we view $V_x \equiv \pi^{-1}(x)$ as an internal space on M with intrinsic vector elements v, the linear map $f_i \colon \pi^{-1}(x) \to \mathbb{K}^n$ is equivalent to choosing a basis $e_{i\mu}$ to get vector components, i.e.

$$f_i(v) = v_i^\mu, \tag{4.3}$$

where

$$v_i^\mu e_{i\mu} = v \tag{4.4}$$

and latin letters are labels while greek letters are the usual indices for vectors and labels for bases. The action of the structure group can then be written

$$v_i^\mu = (g_{ij})^\mu{}_\lambda v_j^\lambda, \tag{4.5}$$

which is equivalent to a change of basis

$$e_{i\mu} = (g_{ij}^{-1})^\lambda{}_\mu e_{j\lambda}, \tag{4.6}$$

or as matrix multiplication on basis row vectors

$$e_j = e_i g_{ij}, \tag{4.7}$$

so that the action of $g_{ij}(x)$ in $U_i \cap U_j$ is equivalent to a change of frame or gauge transformation from e_i to e_j, which is equivalent to a transformation of internal space vector components in the opposite direction.

> △ The frame is not a part of the vector bundle, it is a way of viewing the local trivializations; therefore the view of $g_{ij}(x)$ as effecting a change of basis should not be confused with a group action on either $\pi^{-1}(x)$ or E. As the structure group of E, the action of G is on the fiber $\mathbb{K}^n$, which is not part of E.

If the structure group of a vector bundle is reducible to $GL(n, \mathbb{K})^e$, then it is called an **orientable bundle**; all complex vector bundles are orientable, so orientability usually refers to real vector bundles. The tangent bundle of M (formally defined in Section 5.5) is then orientable iff M is orientable. On a pseudo-Riemannian manifold M, the structure group is reducible to $O(r, s)$, and if M is orientable then it is reducible to $SO(r, s)$; if the structure group can be further reduced to $SO(r, s)^e$, then M and its tangent bundle are called **time and space orientable**. Note that this additional distinction is dependent only upon the metric, and two metrics on the same manifold can have different time and space orientabilities.

△ The orientability of a vector bundle as a bundle is different than its orientability as a manifold itself; therefore it is important to understand which version of orientability is being referred to. In particular, the tangent bundle of M is always orientable as a manifold, but it is orientable as a bundle only if M is.

A gauge transformation on a vector bundle is a smoothly defined linear transformation of the basis inferred by the components due to local trivializations at each point, i.e.

$$e'_{i\mu} = (\gamma_i^{-1})^{\lambda}{}_{\mu} e_{i\lambda}, \tag{4.8}$$

which is equivalent to new local trivializations where

$$v_i'^{\mu} = (\gamma_i)^{\mu}{}_{\lambda} v_i^{\lambda}, \tag{4.9}$$

giving us new transition functions

$$g'_{ij} = \gamma_i g_{ij} \gamma_j^{-1}, \tag{4.10}$$

where we have suppressed indices for pure matrix relationships. Thus the gauge group is the same as the structure group, and a gauge transformation γ_i^{-1} is equivalent to the transition function $g_{i'i}$ from U_i to U'_i, the same neighborhood with a different local trivialization.

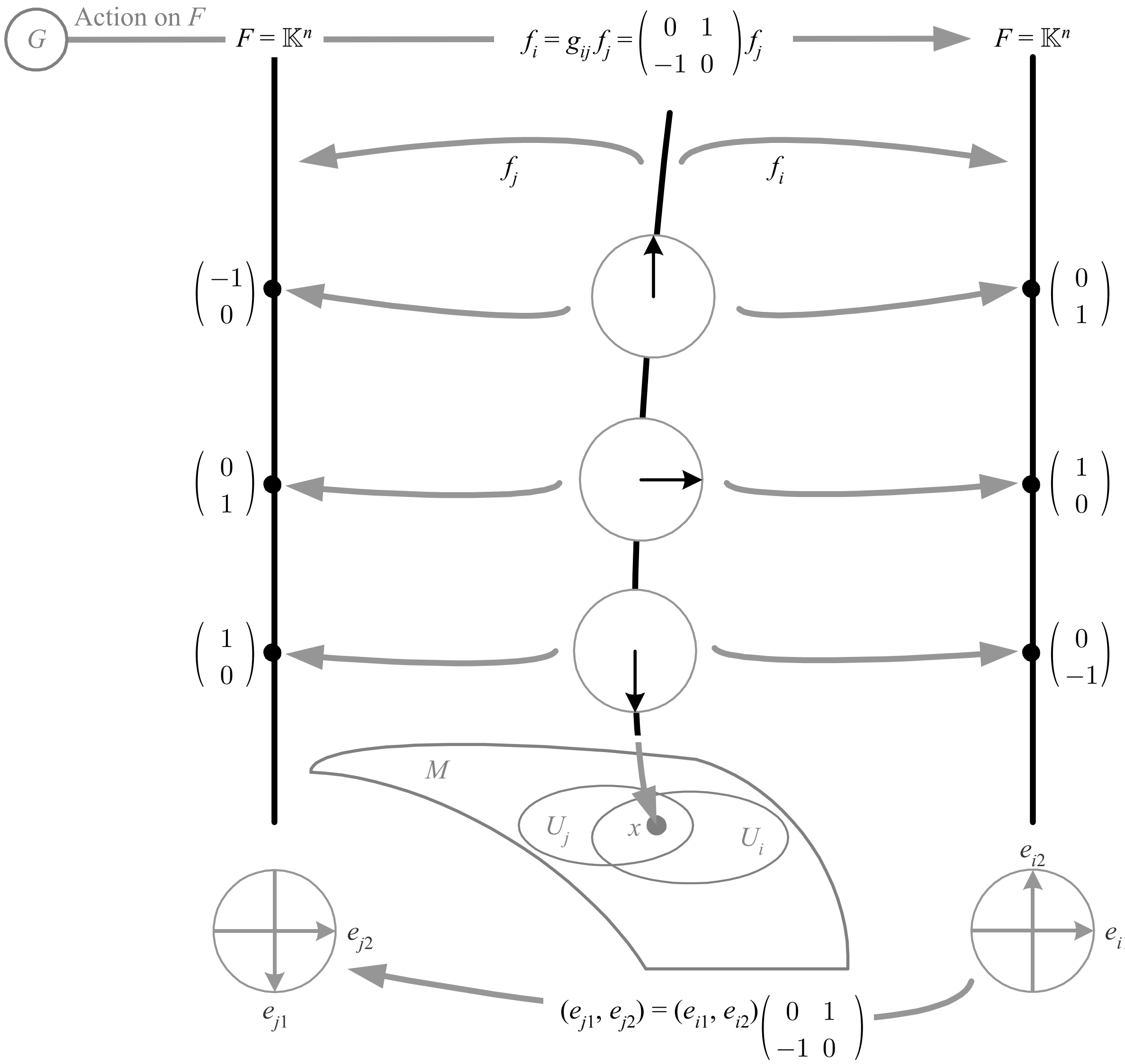

FIGURE 4.2: The elements of the fiber over x in a vector bundle can be viewed as abstract vectors in an internal space, with the local trivialization acting as a choice of basis from which the components of these vectors can be calculated. The structure group then acts as a matrix transformation between vector components, and between bases in the opposite direction. A gauge transformation is also a new choice of basis, and so can be handled similarly.

A vector bundle always has global sections (e.g. the zero vector in the fiber over each point). A vector bundle with fiber $\mathbb{R}$ is called a **line bundle**.

4.3 Frame bundles

Given a vector bundle $(E, M, \mathbb{K}^n)$, the **frame bundle** of E is the principal $GL(n, \mathbb{K})$-bundle associated to E, and is denoted

$$F(E) \equiv (F(E), M, \pi, GL(n, \mathbb{K})). \tag{4.11}$$

The elements $p \in \pi^{-1}(x)$ are viewed as ordered bases of the internal space $V_x \cong \mathbb{K}^n$, which we denote

$$p \equiv e_p, \tag{4.12}$$

or $e_{p\mu}$ if operated on by a matrix in index notation. Each trivializing neighborhood U_i is associated with a fixed frame e_i, which we take from the local trivializations in the vector bundle E, letting us define

$$f_i \colon \pi^{-1}(x) \to GL(n, \mathbb{K}) \tag{4.13}$$

by the matrix relation

$$e_p = e_i f_i(p). \tag{4.14}$$

In other words $f_i(p)$ is the matrix that transforms (as row vectors) the fixed basis e_i into the basis element e_p of $F(E)$; in particular, the identity section is

$$\sigma_i = f_i^{-1}(I) = e_i, \tag{4.15}$$

where I is the identity matrix. If we again write vector components in these bases as $v_i^\mu e_{i\mu} = v_p^\mu e_{p\mu} = v$, then we have

$$v_i^\mu = f_i(p)^\mu{}_\lambda v_p^\lambda. \tag{4.16}$$

The left action of g_{ij} is defined by $f_i(p) = g_{ij} f_j(p)$, and applying both sides to vector components v_p^μ we get

$$v_i^\mu = (g_{ij})^\mu{}_\lambda v_j^\lambda, \tag{4.17}$$

the same transition functions as in E. The transition functions can be viewed as changes of frame $e_j = e_i g_{ij}$, or gauge transformations, between the identity sections of $F(E)$ in $U_i \cap U_j$, i.e. this can be written as a matrix relation

$$\sigma_j = \sigma_i g_{ij}, \tag{4.18}$$

which as we see next is the usual right action of the transition functions on identity sections.

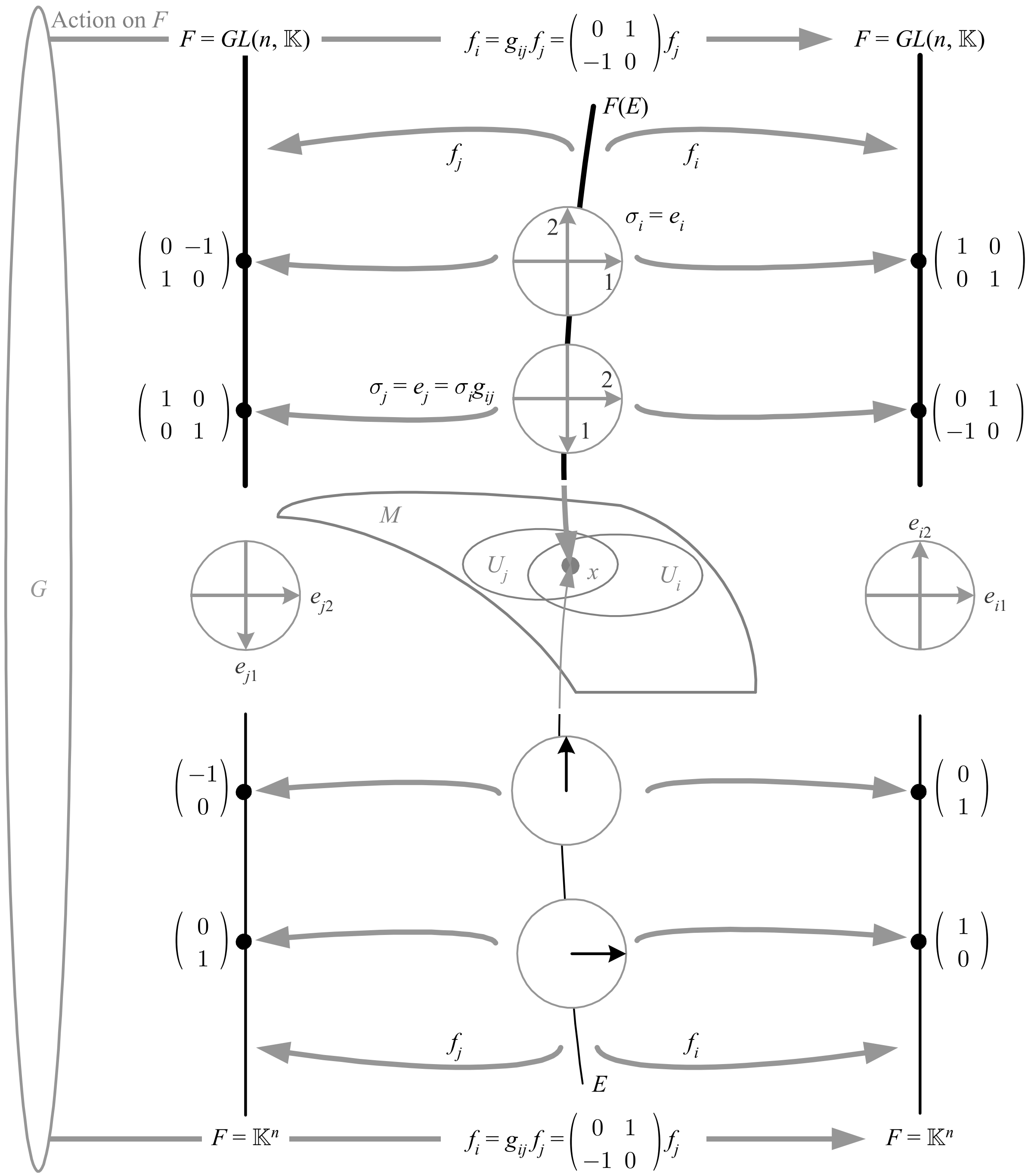

FIGURE 4.3: Given a vector bundle E, we can construct an associated frame bundle $F(E)$. The elements of the fiber over x in the frame bundle can be viewed as bases for the internal space, with the local trivialization acting as a choice of a fixed basis against which linear transformations generate these bases. These fixed bases are the same as those chosen in the corresponding local trivialization on the vector bundle, and are acted on by the same transition functions. Although denoted identically, the f_i are those corresponding to each bundle.

△ Unlike with E, the frame is in fact part of the bundle $F(E)$, but vectors and vector components are not. The left action of g_{ij} on the abstract fiber $GL(n,\mathbb{K})$ is equivalent to a transformation in the opposite direction from the fixed frame in U_i to the fixed frame in U_j, which is a right action on the identity sections from $\sigma_i = e_i$ to $\sigma_j = e_j$.

△ It is important to remember that the elements of $\pi^{-1}(x)$ in $F(E)$ are bases of the vector space V_x, and in a given trivializing neighborhood it is only the basis in the identity section that is identified with the basis underlying the vector components in the same trivializing neighborhood of E.

The right action of $g \in GL(n,\mathbb{K})$ on $\pi^{-1}(x)$ is defined by $f_i(g(p)) = f_i(p)g$, and applying both sides to e_i from the right and using $e_p = e_i f_i(p)$ we immediately obtain

$$e_{g(p)} = e_p g, \tag{4.19}$$

so that the right action of the matrix g is literally matrix multiplication from the right on the basis row vector $p = e_{p\mu}$. Alternative ways of writing this relation include

$$\begin{aligned} e_{g(p)\mu} &= e_{p\mu} g^{\mu}{}_{\lambda}, \\ g(p) &= pg. \end{aligned} \tag{4.20}$$

In particular, if $f_i(p) = g$ then we have

$$p = e_p = e_i g = g(e_i) = e_{g(e_i)}. \tag{4.21}$$

☼ Note that since the right action on $\pi^{-1}(x)$ is by a fixed matrix, it acts as a transformation relative to each e_p, not as a transformation on the internal space V_x in which all of the bases in $\pi^{-1}(x)$ live. As a concrete example, if $g^0{}_0 = 1$ and $g^{\lambda\neq 0}{}_0 = 0$, then $e_{g(p)0} = e_{p0}$, meaning that the transformation $p \mapsto g(p)$ leaves first vector of all bases in $\pi^{-1}(x)$ unaffected, regardless of that vector's direction. This behavior contrasts with that of a transformation on V_x itself, which as we will see in the next section is a gauge transformation.

4.4 Gauge transformations on frame bundles

Recall that a gauge transformation on a vector bundle E is an active transformation of the bases underlying the components defining a local trivialization, which is equivalent to a new set of local trivializations and transition functions (and is not a transformation on the space E itself). On the frame bundle $F(E)$, we perform the same basis change for the fixed frames associated with each trivializing neighborhood

$$e'_i = e_i \gamma_i^{-1}, \tag{4.22}$$

which also defines the new identity sections, and is equivalent to new local trivializations where

$$f'_i(p) = \gamma_i f_i(p), \tag{4.23}$$

giving us new transition functions

$$g'_{ij} = \gamma_i g_{ij} \gamma_j^{-1}, \tag{4.24}$$

which are the same as those in the associated vector bundle E. We will call this transformation a **neighborhood-wise gauge transformation**.

An alternative (and more common) way to view gauge transformations on $F(E)$ is to transform the actual bases in $\pi^{-1}(x)$ via a bundle automorphism

$$p' \equiv \gamma^{-1}(p), \tag{4.25}$$

and then change the fixed bases in each trivializing neighborhood to

$$\begin{aligned} e'_i &= \gamma^{-1}(e_i) \\ &\equiv e_i \gamma_i^{-1} \end{aligned} \tag{4.26}$$

in order to leave the maps $f_i(p)$ the same (which also leaves the identity sections and transition functions the same). This immediately implies a constraint on the basis changes in $U_i \cap U_j$: since $g'_{ij} = \gamma_i g_{ij} \gamma_j^{-1}$, requiring constant g_{ij} means we must have

$$\gamma_i^{-1} = g_{ij} \gamma_j^{-1} g_{ij}^{-1}. \tag{4.27}$$

We will call this transformation an **automorphism gauge transformation**.

△ Note that this constraint means that automorphism gauge transformations are a subset of neighborhood-wise gauge transformations, which allow arbitrary changes of frame in every trivializing neighborhood. Also note that for automorphism gauge transformations, the matrices γ_i^{-1} (and therefore the new identity section elements e'_i) are determined by the automorphism γ^{-1}, while neighborhood-wise gauge transformations are defined by arbitrary matrices γ_i^{-1} in each neighborhood which are not necessarily consistent in $U_i \cap U_j$.

☼ As with the associated vector bundle, for either type of gauge transformation the gauge group is the same as the structure group, and a gauge transformation γ_i^{-1} is equivalent to the transition function $g_{i'i}$ from U_i to U'_i, the same neighborhood with a different local trivialization.

We now define the matrices γ_p^{-1} to be those which result from the transformation $\gamma^{-1}(p)$ on the rest of $\pi^{-1}(x)$, i.e.

$$e'_p \equiv e_p \gamma_p^{-1}. \tag{4.28}$$

Note that γ_p^{-1} is determined by γ_i^{-1}: since we require that $f'_i = f_i$, we have

$$\begin{aligned} e'_i f_i(p) &= e'_p \\ \Rightarrow e_i \gamma_i^{-1} f_i(p) &= e_p \gamma_p^{-1} \\ &= e_i f_i(p) \gamma_p^{-1} \\ \Rightarrow \gamma_p^{-1} &= f_i(p)^{-1} \gamma_i^{-1} f_i(p), \end{aligned} \tag{4.29}$$

or more generally, using the definition of a right action $f_i(g(p)) = f_i(p)g$ we get

$$\gamma_{g(p)}^{-1} = g^{-1}\gamma_p^{-1}g. \tag{4.30}$$

△ It is important to remember that the matrices γ_i^{-1} are dependent upon the local trivialization (since they are defined as the matrix acting on the element $e_i \in \pi^{-1}(x)$ for $x \in U_i$), but the matrices γ_p^{-1} are independent of the local trivialization, and are the action of the automorphism γ^{-1} on the basis e_p.

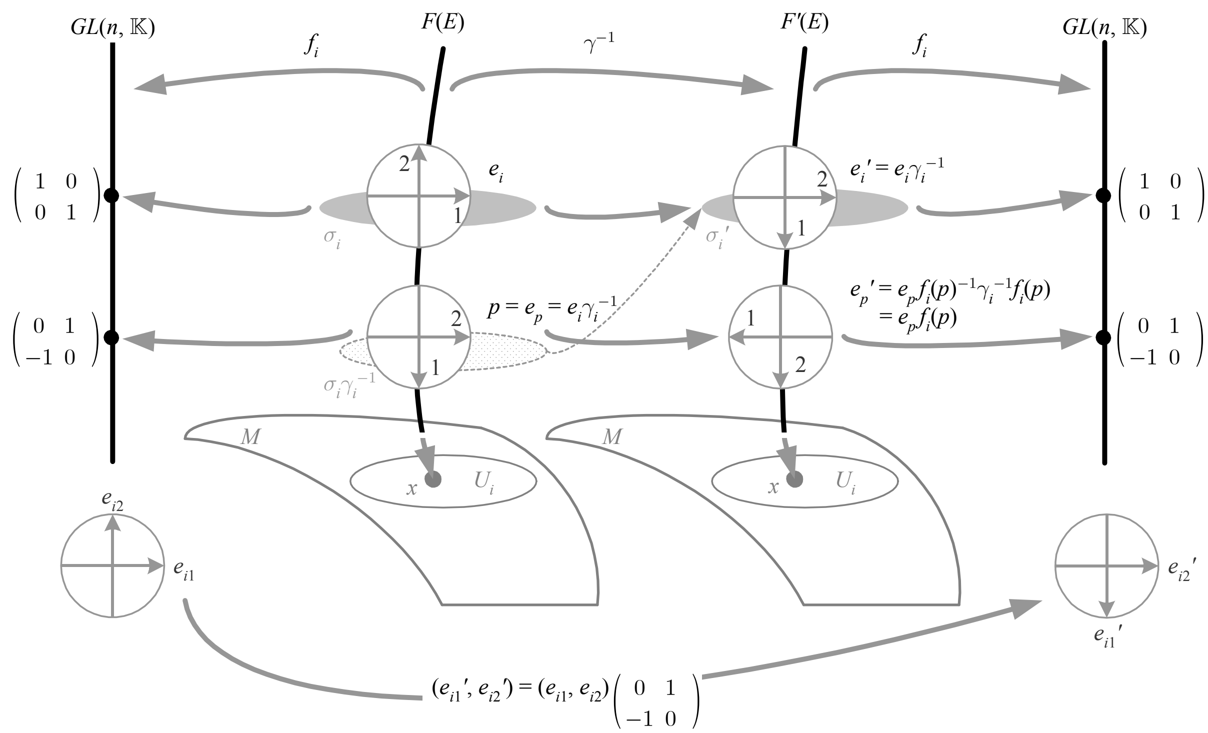

FIGURE 4.4: An automorphism gauge transformation on $F(E)$ transforms the actual elements of the fiber over x, including the identity section elements corresponding to the fixed bases in each local trivialization, thus leaving the local trivializations unchanged.

☼ This result can be understood as γ^{-1} being a transformation on the internal space V_x itself, applied to all the elements of $\pi^{-1}(x)$, each of which is a basis of V_x. For example, in the figure above, γ^{-1} rotates all bases clockwise by $\pi/2$. To see why this is so, note that the matrix in the transformation $v_i'^{\mu} = (\gamma_i)^{\mu}{}_{\lambda} v_i^{\lambda}$ has components which are those of $\gamma_i \in GL(V_x)$ in the basis $e_{i\mu}$. Therefore in a different basis $e_{p\mu} \in \pi^{-1}(x)$ we must apply a different matrix $v_p'^{\mu} = (\gamma_p)^{\mu}{}_{\lambda} v_p^{\lambda}$ which reflects the change of basis $e_{p\mu} = f_i(p)^{\lambda}{}_{\mu} e_{i\lambda}$ via a similarity transformation

$$\begin{aligned} \gamma_p &= f_i(p)^{-1}\gamma_i f_i(p) \\ \Rightarrow \gamma_p^{-1} &= f_i(p)^{-1}\gamma_i^{-1} f_i(p). \end{aligned} \tag{4.31}$$

Viewed as a transformation on V_x, γ^{-1} will then commute with any fixed matrix applied to the bases, which as we saw is the right action; as we see next, this corresponds to the equivariance of γ^{-1} required by it being a bundle automorphism.

We now check that γ^{-1} is a bundle automorphism with respect to the right action of G, i.e. that $\gamma^{-1}(g(p)) = g\left(\gamma^{-1}(p)\right)$:

$$\begin{aligned}\gamma^{-1}(g(p)) &= e_{g(p)}\gamma_{g(p)}^{-1} \\ &= e_{g(p)}g^{-1}\gamma_p^{-1}g \\ &= e_p\gamma_p^{-1}g \\ &= e_{\gamma^{-1}(p)}g \\ &= g\left(\gamma^{-1}(p)\right)\end{aligned} \tag{4.32}$$

△ A possible source of confusion is that a local gauge transformation (different at different points) can be defined globally on $F(E)$; meanwhile, a global gauge transformation (the same matrix γ_i^{-1} at every point) can only be defined locally (unless $F(E)$ is trivial).

Consider the associated bundle to $F(E)$ with fiber $GL(\mathbb{K}^n)$, where the local trivialization of the fiber over x is defined to be the possible automorphism gauge transformations γ_i^{-1} on the identity section element over x in the trivializing neighborhood U_i. Then recalling that $\gamma_i^{-1} = g_{ij}\gamma_j^{-1}g_{ij}^{-1}$, we see that the action of the structure group on the fiber is by inner automorphism. Since the values of γ^{-1} on $F(E)$ are determined by those in the identity section, we can thus view automorphism gauge transformations as sections of the associated bundle $(\mathrm{Inn}F(E), M, GL(\mathbb{K}^n))$.

4.5 Smooth bundles and jets

Nothing we have done so far has required the spaces of a fiber bundle to be manifolds; if they are, then we require the bundle projections π to be (infinitely) differentiable and $\pi^{-1}(x)$ to be diffeomorphic to F, resulting in a **smooth bundle**. A **smooth *G*-bundle** then has a structure group G which is a Lie group, and whose elements correspond to diffeomorphisms of F.

If we consider a local section σ of a smooth fiber bundle (E, M, π, F) with $\sigma(x) = p$, the equivalence class of all local sections that have both $\sigma(x) = p$ and also the same tangent space $T_p\sigma$ is called the **jet** $j_p\sigma$ with **representative** σ. We can also require that further derivatives of the section match the representative, in which case the order of matching derivatives defines the **order** of the jet, which is also called a ***k*-jet** so that the above definition would be that of a 1-jet. x is called the **source** of the jet and p is called its **target**. With some work to transition between local sections, one can then form a **jet manifold** by considering jets with all sources and representative sections, which becomes a **jet bundle** by considering jets to be fibers over their source.

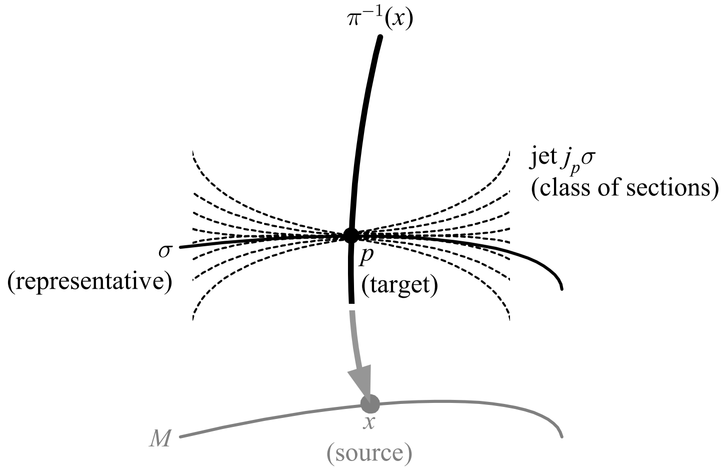

FIGURE 4.5: A jet with representative σ, source x, and target p.

4.6 Vertical tangents and horizontal equivariant forms

A smooth bundle (E, M, π) is a manifold itself, and thus has tangent vectors. A tangent vector v at $p \in E$ is called a **vertical tangent** if

$$\mathrm{d}\pi(v) = 0, \tag{4.33}$$

i.e. if it is tangent to the fiber over x where $\pi(p) = x$, so the projection down to the base space vanishes. The **vertical tangent space** V_p is then the subspace of the tangent space T_p at p consisting of vertical tangents, and viewing the vertical tangent spaces as fibers over E we can form the **vertical bundle** (VE, E, π_V), which is a subbundle of TE. We can also consider differential forms on a smooth bundle, which take arguments that are tangent vectors on E. A form is called a **horizontal form** if it vanishes whenever any of its arguments are vertical.

On a smooth principal bundle (P, M, G), we have a consistent right action

$$\rho\colon G \to \mathrm{Diff}(P), \tag{4.34}$$

and the corresponding Lie algebra action

$$\mathrm{d}\rho\colon \mathfrak{g} \to \mathrm{vect}(P) \tag{4.35}$$

is then a Lie algebra homomorphism. The fundamental vector fields corresponding to elements of $\mathfrak{g}$ are vertical tangent fields; in fact, at a point p, $\mathrm{d}\rho\,|_p$ is a vector space isomorphism from $\mathfrak{g}$ to V_p:

$$\mathrm{d}\rho\,|_p : \mathfrak{g} \xrightarrow{\cong} V_p \tag{4.36}$$

In addition, the right action

$$g\colon P \to P \tag{4.37}$$

of a given element g corresponds to a right action

$$\mathrm{d}g\colon TP \to TP, \tag{4.38}$$

which maps tangent vectors on P via

$$\mathrm{d}g(v)\colon T_pP \to T_{g(p)}P.$$

This map is an automorphism of TP restricted to $\pi_P^{-1}(x)$, which we denote $T_{\pi^{-1}(x)}P$, and it preserves vertical tangent vectors. We can then consider the pullback

$$g^*\varphi(v_1,\ldots,v_k) = \varphi(\mathrm{d}g(v_1),\ldots,\mathrm{d}g(v_k)) \tag{4.39}$$

as a right action on the space $\Lambda^k P$ of k-forms on P.

If we have a bundle (E, M, π_E, F) associated to (P, M, π_P, G), we can define an F-valued form φ_P, which can be viewed on each $\pi_P^{-1}(x)$ as a mapping

$$\varphi_P\colon T_{\pi^{-1}(x)}P \otimes \cdots \otimes T_{\pi^{-1}(x)}P \to F \times \pi_P^{-1}(x), \tag{4.40}$$

where $g \in G$ has a right action $\mathrm{d}g$ on $T_{\pi^{-1}(x)}P$ and a left action g on the abstract fiber F of E. The form φ_P is called an **equivariant form** if this mapping is equivariant with respect to these actions, i.e. if

$$g^*\varphi_P = g^{-1}\left(\varphi_P\right). \tag{4.41}$$

If φ_P is also horizontal, then it is called a **horizontal equivariant form** (AKA basic form, tensorial form). If we pull back a horizontal equivariant form to the base space M using the identity sections, we get forms

$$\varphi_i \equiv \sigma_i^*\varphi_P \tag{4.42}$$

on each $U_i \subset M$. Using the identity section relation $\sigma_i = g_{ij}^{-1}(\sigma_j)$ and the pullback composition property $\left(g(h)\right)^*\varphi = h^*\left(g^*\varphi\right)$, we see that the values of these forms satisfy

$$\begin{aligned}
\varphi_i &= \left(g_{ij}^{-1}(\sigma_j)\right)^*\varphi_P \\
&= \sigma_j^*\left(\left(g_{ij}^{-1}\right)^*\varphi_P\right) \\
&= \sigma_j^*\left(g_{ij}\left(\varphi_P\right)\right) \\
&= g_{ij}\left(\varphi_j\right),
\end{aligned} \tag{4.43}$$

where in the third line g_{ij} is acting on the value of φ_P. This means that at a point x in $U_i \cap U_j$, the values of φ_i and φ_j in the abstract fiber F correspond to a single point in $\pi_E^{-1}(x) \in E$, so that the union $\bigcup \varphi_i$ can be viewed as comprising a single E-valued form φ on M. Such a form is sometimes called a **section-valued form**, since for fixed argument vector fields its value on M is a section of E. It can be shown that the correspondence between the E-valued forms φ on M and the horizontal equivariant F-valued forms on P is one-to-one. Equivariant F-valued 0-forms on P are automatically horizontal (since one cannot pass in a vertical argument), and are thus one-to-one with sections on E.

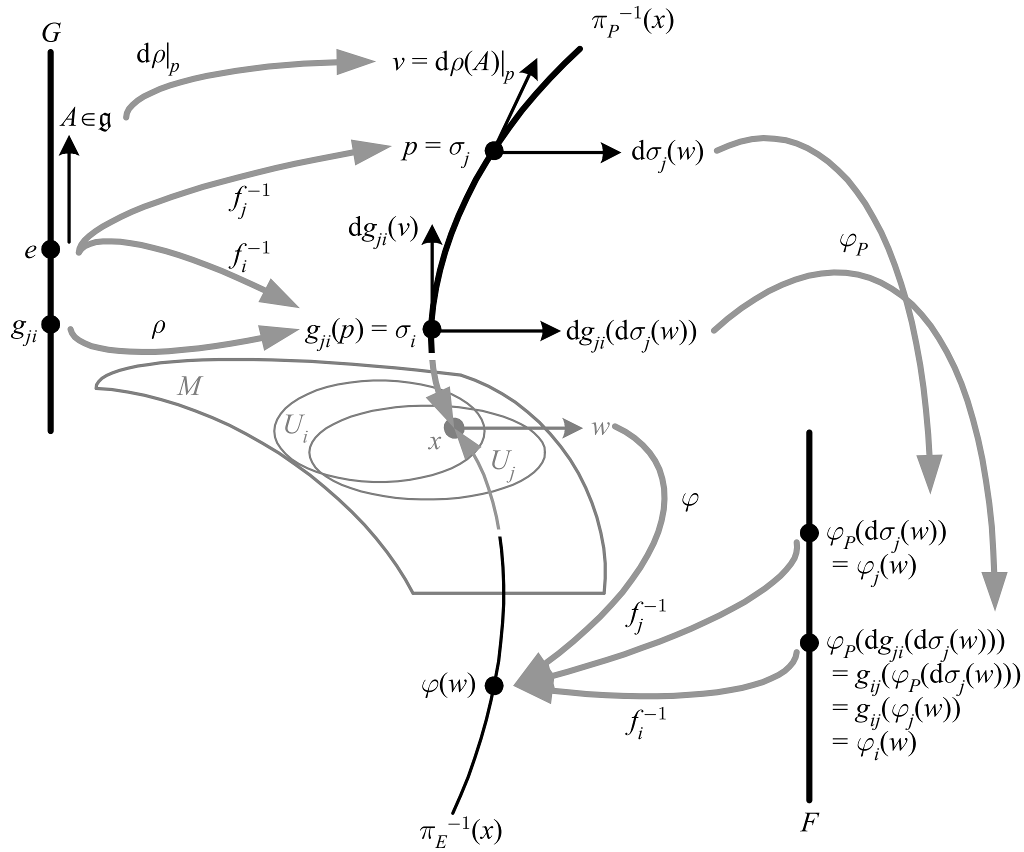

FIGURE 4.6: The differential of the right action of G on $\pi_P^{-1}(x) \in P$ creates an isomorphism to the vertical tangent space $\mathfrak{g} \cong V_p$. A horizontal equivariant form φ_P on P maps non-vertical vectors to the abstract fiber F of an associated bundle, and pulling back by the identity sections yields an E-valued form φ on M. Although denoted identically, the f_i are those corresponding to each bundle.

On the frame bundle $(P, M, \pi_P, GL(n, \mathbb{K}))$ associated with a vector bundle $(E, M, \pi_E, \mathbb{K}^n)$, a $\mathbb{K}^n$-valued form $\vec{\varphi}_P$ is then equivariant if

$$g^* \vec{\varphi}_P = \check{g}^{-1} \vec{\varphi}_P, \tag{4.44}$$

where $\check{g}^{-1}$ is a matrix-valued 0-form on P operating on the $\mathbb{K}^n$-valued form $\vec{\varphi}_P$. The pullback of a horizontal equivariant form on P to the base space M using the identity sections satisfies

$$\vec{\varphi}_i = \check{g}_{ij} \vec{\varphi}_j, \tag{4.45}$$

where $\check{g}_{ij}$ is now a matrix-valued 0-form on M. At a point x in $U_i \cap U_j$, the values of $\vec{\varphi}_i$ and $\vec{\varphi}_j$ in the abstract fiber $\mathbb{K}^n$ correspond to a single abstract vector in $V_x = \pi_E^{-1}(x) \in E$, so that the union $\bigcup \vec{\varphi}_i$ can be viewed as comprising a single V-valued form $\vec{\varphi}$ on M. Thus an equivariant $\mathbb{K}^n$-valued 0-form on P is a matter field on M.

☼ This correspondence can be viewed as follows. The right action of g on P is a transformation on bases, so that the equivalent transformation of vector components is g^{-1}. The left action of g^{-1} on the abstract fiber of E is also a transformation of vector components. Thus the equivariant property can be viewed as "keeping the same value when changing basis on both bundles," so that the values of $\vec{\varphi}_P$ on $\pi_P^{-1}(x) \in P$ correspond to a single point in $\pi_E^{-1}(x) \in E$, i.e a single abstract vector over M. In other words, $\vec{\varphi} \in T_x M$ is determined by the value of $\vec{\varphi}_P$ at a single point in $\pi_P^{-1}(x) \in P$. The horizontal requirement means we do not consider forms which take non-zero values given argument vectors which project down to a zero vector on M.

Under an automorphism gauge transformation, the transformation of a horizontal equivariant form on the frame bundle P is defined by the pullback of the automorphism

$$\vec{\varphi}'_P \equiv \left(\gamma^{-1}\right)^* \vec{\varphi}_P. \tag{4.46}$$

The automorphism does not give us a right action on $T_{\pi^{-1}(x)}P$ by a fixed element, but it does give a right action when acting on the element in the identity section, so since the identity sections remain constant we have

$$\begin{aligned}
\vec{\varphi}'_i &= \sigma_i^* \left(\gamma^{-1}\right)^* \vec{\varphi}_P \\
&= \left(\gamma^{-1}\sigma_i\right)^* \vec{\varphi}_P \\
&= \left(\gamma_i^{-1}\sigma_i\right)^* \vec{\varphi}_P \\
&= \sigma_i^* \left(\gamma_i^{-1}\right)^* \vec{\varphi}_P \\
&= \sigma_i^* \check{\gamma}_i \vec{\varphi}_P \\
&= \check{\gamma}_i \vec{\varphi}_i,
\end{aligned} \tag{4.47}$$

where we have used $(g(h))^* \varphi = h^* (g^* \varphi)$ twice, and in the penultimate line we used the equivariance of $\vec{\varphi}_P$. Under neighborhood-wise gauge transformations, there is no change in $\vec{\varphi}_P$ but we have new identity sections $\sigma'_i(x) = \gamma_i^{-1}(\sigma_i(x))$, so that we get

$$\begin{aligned}
\vec{\varphi}'_i &= \sigma_i'^* \vec{\varphi}_P \\
&= \left(\gamma_i^{-1}\left(\sigma_i\right)\right)^* \vec{\varphi}_P \\
&= \sigma_i^* \left(\gamma_i^{-1}\right)^* \vec{\varphi}_P \\
&= \check{\gamma}_i \vec{\varphi}_i,
\end{aligned} \tag{4.48}$$

matching the behavior for both automorphism gauge transformations and for gauge transformations as previously defined directly on M in Section 2.1.

Note that if a horizontal equivariant form takes values in the abstract fiber F of another bundle associated to the frame bundle, the same reasoning applies, but with $\check{\gamma}_i$ applied using the left action of G on F. In particular, recalling from Section 4.1 that the adjoint rep $\rho = \mathrm{Ad}$ of G on $\mathfrak{g}$ defines an associated bundle $(\mathrm{Ad}P, M, \mathfrak{g})$ to P, we can consider a $\mathfrak{g}$-valued horizontal equivariant form $\check{\Theta}_P$ on P, whose pullback by the identity section under a gauge transformation satisfies

$$\check{\Theta}'_i = \check{\gamma}_i \check{\Theta}_i \check{\gamma}_i^{-1}, \tag{4.49}$$

and which similarly across trivializing neighborhoods also undergoes a gauge transformation

$$\check{\Theta}_i = \check{g}_{ij} \check{\Theta}_j \check{g}_{ij}^{-1}. \tag{4.50}$$

5 Generalizing connections

5.1 Connections on bundles

The fibers of a smooth bundle (E, M, π) let us define vertical tangents, but we have no structure that would allow us to canonically define a horizontal tangent. This structure is introduced via the **Ehresmann connection 1-form** (AKA bundle connection 1-form), a vector-valued 1-form on E that defines the vertical component of its argument v, which we denote v^{Φ}, and therefore also defines the horizontal component, which we denote $v^{\ominus}$:

$$\begin{aligned} \vec{\Gamma}(v) &\equiv v^{\Phi}, \\ H_p &\equiv \left\{ v \in T_p E \mid \vec{\Gamma}(v) = 0 \right\} \\ \Rightarrow v &= v^{\Phi} + v^{\ominus}, \end{aligned} \tag{5.1}$$

where $v^{\Phi} \in V_p$, $v^{\ominus} \in H_p$, and H_p is called the **horizontal tangent space**. Viewing the H_p as fibers over E then yields the **horizontal bundle** (HE, E, π_H), and a **vertical form** is defined to vanish whenever any of its arguments are horizontal. Alternatively, one can start by defining the horizontal tangent spaces as smooth sections of the jet bundle of order 1 over E, which uniquely determines a Ehresmann connection 1-form.

> △ "Ehresmann connection" can refer to the horizontal tangent spaces, the horizontal bundle, the connection 1-form, or the complementary 1-form that maps to the horizontal component of its argument.

Recall that on a smooth principal bundle (P, M, π, G), the right action $\rho\colon G \to \mathrm{Diff}(P)$ has a corresponding Lie algebra action $\mathrm{d}\rho\colon \mathfrak{g} \to \mathrm{vect}(P)$ where $\mathrm{d}\rho|_p$ is a vector space isomorphism from $\mathfrak{g}$ to V_p. The **principal connection 1-form** (AKA principal G-connection, G-connection 1-form) is a $\mathfrak{g}$-valued vertical 1-form $\check{\Gamma}_P$ on P that defines the vertical part of its argument v at p via this isomorphism, i.e. the right action of the structure group transforms it into the Ehresmann connection 1-form:

$$\begin{aligned} \mathrm{d}\rho\left(\check{\Gamma}_P(v)\right)|_p &\equiv v^{\Phi} \\ &= \vec{\Gamma}(v) \end{aligned} \tag{5.2}$$

For $g \in G$, $\mathrm{d}g(v)\colon T_p P \to T_{g(p)} P$ preserves horizontal tangent vectors as well as vertical.

> △ As with the Ehresmann connection, a "connection" on a principal bundle can refer to the principal connection 1-form, the horizontal tangent spaces, or other related quantities.

5.2 Parallel transport on the frame bundle

On a frame bundle $(P = F(E), M, \pi, GL(n, \mathbb{K}))$ with connection, we consider the horizontal tangent space to define the direction of parallel transport. More precisely, we define a **horizontal lift** of a curve C from x to y on M to be a curve C_P that projects down to C and whose tangents are horizontal:

$$\begin{aligned} \pi\left(C_P\right) &= C \\ \dot{C}_P|_p &\in H_p \end{aligned} \tag{5.3}$$

There is a unique horizontal lift of C that starts at any $p \in \pi^{-1}(x)$, whose endpoint lets us define the parallel transporter on $F(E)$

$$\|_C \colon \pi^{-1}(x) \to \pi^{-1}(y). \tag{5.4}$$

The parallel transporter is a diffeomorphism between fibers, and it commutes with the right action:

$$\|_C\ (g\,(p)) = g\,(\|_C\ (p)) \tag{5.5}$$

We can then recover the parallel transporter on M by choosing a frame (i.e. a local trivialization), using the horizontal lift that starts at the element $\sigma_i = e_i$ in the identity section, and recalling the relation $e_p = e_i f_i(p)$:

$$\begin{aligned} \|_C\ (e_i\,|_x) &= e_i\,|_y\ f_i\,(\|_C\ (e_i\,|_x)) \\ \Rightarrow (\|_C\ (v))_i^\mu\,|_y &= f_i\,(\|_C\ (e_i\,|_x))^\mu{}_\lambda v_i^\lambda\,|_x \\ \Rightarrow \|^\mu{}_\lambda\,(C) &= f_i\,(\|_C\ (e_i\,|_x))^\mu{}_\lambda \end{aligned} \tag{5.6}$$

The second line transforms vector components using the change of basis matrix in the opposite direction.

Similarly, on the frame bundle we can recover the connection 1-form on $v \in T_x M$ within a trivializing neighborhood by using the pullback of the identity section:

$$\begin{aligned} \check{\Gamma}_i(v) &= \sigma_i^* \check{\Gamma}_P(v) \\ &= \check{\Gamma}_P\,(\mathrm{d}\sigma_i(v)) \end{aligned} \tag{5.7}$$

On $F(E)$, $\sigma_i = e_i$ is the frame used to define the components of vectors in the internal space on U_i, and $\check{\Gamma}_i(v)$ then is the element of $gl(n, \mathbb{K})$ corresponding to the vertical component of v after being mapped to a tangent of the identity section. Thus since we consider the horizontal tangent space to define the direction of parallel transport, $\check{\Gamma}_i(v)$ is the infinitesimal linear transformation that takes the parallel transported frame to the frame in the direction v, the same interpretation found for manifolds in [6].

> △ It is important to remember that $\check{\Gamma}_i$ takes values that are dependent upon the local trivialization that defines the identity section (i.e. it is frame-dependent), while the values of $\check{\Gamma}_P$ are intrinsic to the frame bundle. This reflects the fact that the connection is a choice of horizontal correspondences between frames, and so cannot have any value intrinsic to E.

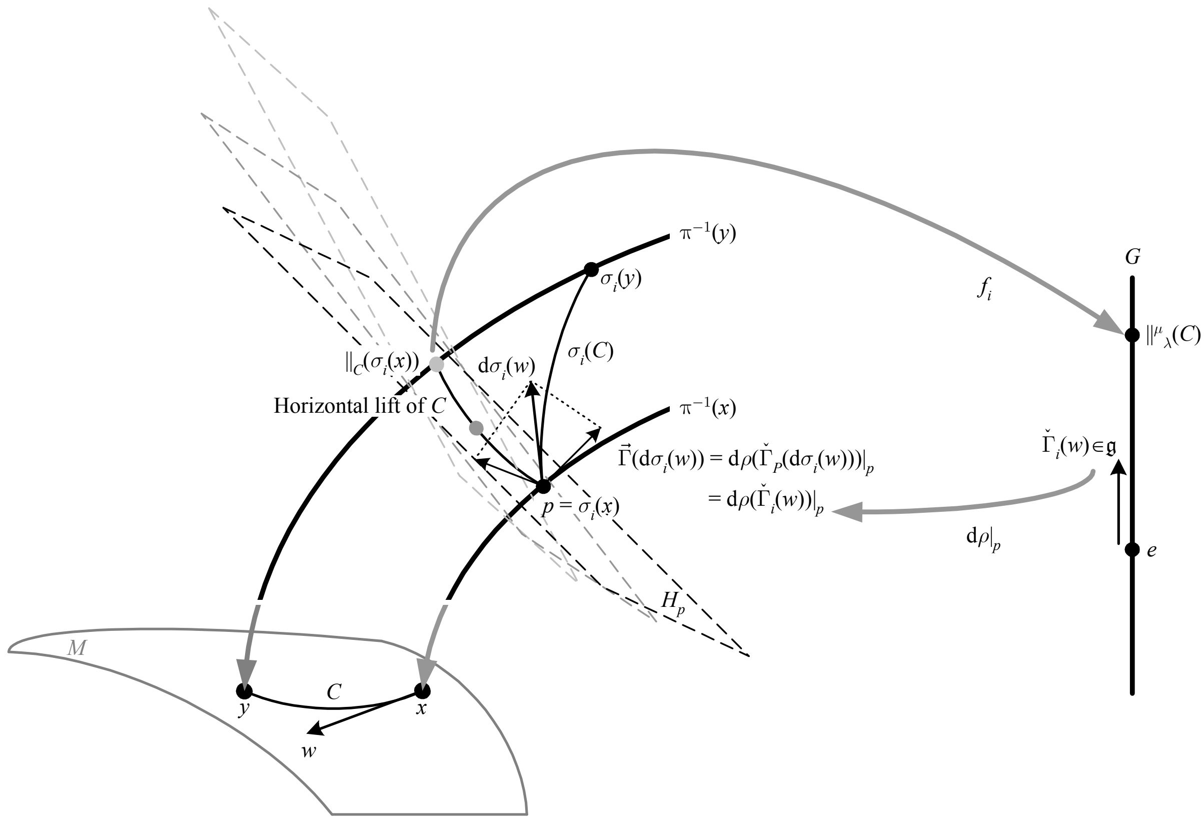

FIGURE 5.1: A principal connection 1-form on (P, M, G) defines the vertical component of its argument as a value in the Lie algebra $\mathfrak{g}$ via the isomorphism defined by the differential of the right action $\mathrm{d}\rho$. A horizontal lift of a curve C yields the parallel transporter, and the pullback by the identity section recovers the connection 1-form on M.

The transition functions on the frame bundle can be viewed as $GL(n, \mathbb{K})$-valued 0-forms $\check{g}_{ij}$ on $U_i \cap U_j$, and it can be shown that

$$\check{\Gamma}_i(v) = \check{g}_{ij}\check{\Gamma}_j(v)\check{g}_{ij}^{-1} + \check{g}_{ij}\mathrm{d}\check{g}_{ij}^{-1}(v), \tag{5.8}$$

which is the transformation of the connection 1-form under a change of frame $\check{g}_{ij}^{-1}$ on manifolds. This is consistent with the interpretation of the action of g_{ij} as a change of frame g_{ij}^{-1} in Section 4.2, and it can be shown that a unique connection on $F(E)$ is determined by locally defined connection 1-forms on M and sections that are related by the same transition functions.

☼ The inhomogeneous transformation of the connection 1-form can be viewed as reflecting the fact that both the location and "shape" of the identity section is different across local trivializations (although we have depicted the identity sections as "flat," the values of each $\sigma_i(x)$ are smooth but arbitrary).

☼ This demonstrates the advantage of the principal bundle formulation, in that the connection 1-form on M is frame-dependent, and therefore cannot in general be defined on all of M, while in contrast the principal connection 1-form is defined on all of $F(E)$, and can be used to determine a consistent connection 1-form on M within each trivializing neighborhood.

Under either type of gauge transformation, it can also be shown that as expected we have

$$\check{\Gamma}'_i(v) = \check{\gamma}_i \check{\Gamma}_i(v) \check{\gamma}_i^{-1} + \check{\gamma}_i \mathrm{d}\check{\gamma}_i^{-1}(v). \tag{5.9}$$

5.3 The exterior covariant derivative on bundles

The exterior covariant derivative of a form on a smooth bundle with connection is the horizontal form that results from taking the exterior derivative on the horizontal components of all its arguments, i.e. for a k-form φ we define

$$\mathrm{D}\varphi(v_0, \ldots, v_k) \equiv \mathrm{d}\varphi(v_0^{\ominus}, \ldots, v_k^{\ominus}). \tag{5.10}$$

On a smooth bundle, $\mathrm{D}\varphi$ can then be viewed as the "sum of φ on the boundary of the horizontal hypersurface defined by its arguments." Note that these boundaries are all defined by horizontal vectors except those including a Lie bracket, which may have a vertical component. So for example, if φ is a vertical 1-form we have

$$\mathrm{D}\varphi(v, w) = -\varphi([v^{\ominus}, w^{\ominus}]), \tag{5.11}$$

the other terms all vanishing.

For a vector bundle $(E, M, \mathbb{K}^n)$ associated to a smooth principal bundle with connection $(P, M, GL(n, \mathbb{K}))$, it can be shown that an $\mathbb{K}^n$-valued horizontal equivariant form $\vec{\varphi}_P$ on P satisfies the familiar equation

$$\mathrm{D}\vec{\varphi}_P = \mathrm{d}\vec{\varphi}_P + \check{\Gamma}_P \wedge \vec{\varphi}_P, \tag{5.12}$$

where the derivatives are taken on the components of $\vec{\varphi}_P$, and the action of $gl(n, \mathbb{K})$-valued $\check{\Gamma}_P$ on the values of $\vec{\varphi}_P$ in is the differential of the left action of $GL(n, \mathbb{K})$. $\mathrm{D}\vec{\varphi}_P$ is then also a horizontal equivariant form. Applying the pullback by the identity section to the exterior covariant derivative, we obtain the expected

$$\mathrm{D}\vec{\varphi}_i = \mathrm{d}\vec{\varphi}_i + \check{\Gamma}_i \wedge \vec{\varphi}_i. \tag{5.13}$$

△ As with the connection 1-form, it is important to remember that the values of $\vec{\varphi}_i$ on M are components operated on by the matrix $\check{\Gamma}_i$, both of which are defined by a local trivialization.

The immediate application of the above is to a $\mathbb{K}^n$-valued form on the frame bundle. However, we can also apply it to other associated bundles to P. In particular, recalling Section 4.6, in the associated bundle $(\mathrm{Ad}P, M, gl(n, \mathbb{K}))$ we can apply it to a $gl(n, \mathbb{K})$-valued horizontal equivariant form $\check{\Theta}_P$ on P, where the left action of $GL(n, \mathbb{K})$ is $\rho = \text{Ad}$, and the left action of $gl(n, \mathbb{K})$ on itself is therefore $\text{d}\rho = \text{ad}$, i.e. the Lie bracket. For such a form we then have

$$\text{D}\check{\Theta}_P = \text{d}\check{\Theta}_P + \check{\Gamma}_P[\wedge]\check{\Theta}_P, \tag{5.14}$$

where again the exterior derivative is taken on the matrix components of $\check{\Theta}_P$, and the action of $gl(n, \mathbb{K})$-valued $\check{\Gamma}_P$ on the values of $\check{\Theta}_P$ is the Lie bracket, the differential of the left action of $GL(n, \mathbb{K})$. Applying the pullback by the identity section recovers the same formula for algebra-valued forms on M (see [6] for an explanation of the notation $[\wedge]$).

5.4 Curvature on principal bundles

On a smooth principal bundle with connection (P, M, G), the exterior covariant derivative gives us a definition for the curvature of the principal connection, the horizontal $\mathfrak{g}$-valued 2-form on P

$$\check{R}_P \equiv D\check{\Gamma}_P. \tag{5.15}$$

Note that the analog of the above equation on M itself does not hold. Since $\check{\Gamma}_P$ is vertical, this can be written

$$\begin{aligned} \check{R}_P\left(v, w\right) &= -\check{\Gamma}_P\left(\left[v^{\ominus}, w^{\ominus}\right]\right) \\ \Rightarrow \text{d}\rho\left(\check{R}_P\left(v, w\right)\right)|_p &= -\left[v^{\ominus}, w^{\ominus}\right]^{\phi}, \end{aligned} \tag{5.16}$$

so that the curvature of the principal connection is the element of $\mathfrak{g}$ corresponding to the vertical component of the Lie bracket of the horizontal components of its arguments.

On a frame bundle, we associate the horizontal tangent space with parallel transport, and the curvature is the "infinitesimal linear transformation between parallel transport in opposite directions around the boundary of the horizontal hypersurface defined by its arguments," or equivalently the "infinitesimal linear transformation associated with the vertical component of the negative Lie bracket of the horizontal components of its arguments." The curvature on M can be recovered using identity sections σ_i as with the connection:

$$\check{R}_i \equiv \sigma_i^* \check{R}_P \tag{5.17}$$

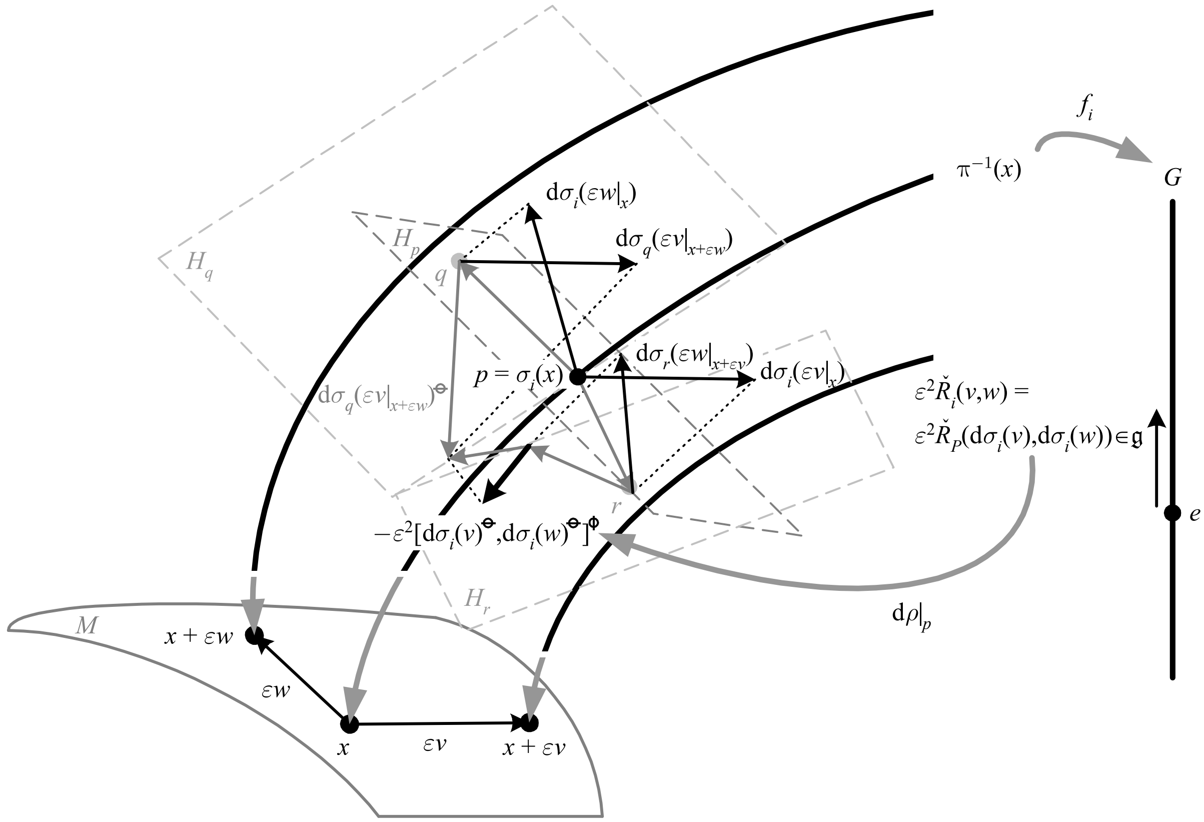

FIGURE 5.2: The curvature of the principal connection is the element of $\mathfrak{g}$ corresponding to the vertical component of the negative Lie bracket of the horizontal components of its arguments. The sections used at q and r are arbitrary, since they don't affect the vertical component of the loop remainder. If the arguments are pulled back using the identity section, we recover the curvature on the base space M.

When G is a matrix group, we find analogs of equations for curvature on M using the relations from the previous section:

$$\begin{aligned} \check{R}_P &= \mathrm{d}\check{\Gamma}_P + \frac{1}{2}\check{\Gamma}_P[\wedge]\check{\Gamma}_P \\ \check{R}_i &= \mathrm{d}\check{\Gamma}_i + \frac{1}{2}\check{\Gamma}_i[\wedge]\check{\Gamma}_i \\ \mathrm{D}\check{R}_P &= 0 \end{aligned} \tag{5.18}$$

Note that $\check{R}_P$ is a map from 2-forms on P to $\mathfrak{g}$, where G has a left action via the adjoint rep of G on $\mathfrak{g}$. One can then show that $\check{R}_P$ is equivariant with respect to this action and that of G on 2-forms, i.e. we have

$$g^* \check{R}_P = g_{\mathrm{Ad}}^{-1}\left(\check{R}_P\right). \tag{5.19}$$

Thus $\check{R}_P$ is a horizontal equivariant form, and recalling Section 4.6 we have the expected transformations

$$\begin{aligned} \check{R}_i &= \check{g}_{ij}\check{R}_j\check{g}_{ij}^{-1}, \\ \check{R}'_i &= \check{\gamma}_i\check{R}_i\check{\gamma}_i^{-1}. \end{aligned} \tag{5.20}$$

If a flat connection (zero curvature) can be defined on a principal bundle, then the structure group is discrete. If in addition the base space is simply connected, then the bundle is trivial.

5.5 The tangent bundle and solder form

Returning to our motivating example, the **tangent bundle** on a manifold M^n, denoted TM, is a smooth vector bundle $(E, M^n, \mathbb{R}^n)$ with a (possibly reducible) structure group $GL(n, \mathbb{R})$ that acts as an inverse change of local frame across trivializing neighborhoods. These trivializing neighborhoods can be obtained from an atlas on M, with fiber homeomorphisms $f_i \colon T_x M \to \mathbb{R}^n$ defined by components in the coordinate frame $e_{i\mu} = \partial/\partial x_i^\mu$, so that the transition functions are Jacobian matrices

$$\begin{aligned} v_i^\mu &= (g_{ij})^\mu{}_\lambda v_j^\lambda \\ &= \frac{\partial x_i^\mu}{\partial x_j^\lambda} v_j^\lambda \end{aligned} \tag{5.21}$$

associated with the transformation of vector components. M is orientable iff these Jacobians all have positive determinant, i.e. iff the structure group is reducible to $GL(n, \mathbb{R})^e$ (the definition of TM being orientable). A section of the tangent bundle is a vector field on M. A change of coordinates within each coordinate patch then generates a change of frame

$$\frac{\partial}{\partial x_i'^\mu} = \frac{\partial x_i^\lambda}{\partial x_i'^\mu} \frac{\partial}{\partial x_i^\lambda}, \tag{5.22}$$

which is equivalent to new local trivializations where

$$v_i'^\mu = \frac{\partial x_i'^\mu}{\partial x_i^\lambda} v_i^\lambda, \tag{5.23}$$

giving us new transition functions

$$\frac{\partial x_i'^\mu}{\partial x_j'^\lambda} = \frac{\partial x_i'^\mu}{\partial x_i^\sigma} \frac{\partial x_i^\sigma}{\partial x_j^\nu} \frac{\partial x_j^\nu}{\partial x_j'^\lambda}. \tag{5.24}$$

The **tangent frame bundle** (AKA frame bundle), denoted FM, is the smooth frame bundle of TM, i.e. $(FM, M^n, GL(n, \mathbb{R}))$, where the fixed bases in each trivializing neighborhood are again obtained from the atlas on M, giving the same transition functions as in the tangent bundle. The bases in $\pi^{-1}(x)$ are thus defined by

$$e_{p\mu} = f_i(p)^\lambda{}_\mu \frac{\partial}{\partial x_i^\lambda}. \tag{5.25}$$

A section of the frame bundle is a frame on M, and a global section is a global frame, so that M is parallelizable iff FM is trivial. The right action of a matrix $g^\mu{}_\lambda \in GL(n, \mathbb{R})$ operates on bases as row vectors, and an automorphism of FM along with a redefinition of fixed bases to preserve identity sections generates changes of frame in each trivializing neighborhood that preserve the transition functions.

△ The tangent frame bundle is also denoted $F(M)$, but rarely $F(TM)$, which is what would be consistent with general frame bundle notation.

The tangent frame bundle is special in that we can relate its tangent vectors to the elements of the bundle as bases. Specifically, we define the **solder form** (AKA soldering form, tautological 1-form, fundamental 1-form), as a $\mathbb{R}^n$-valued 1-form $\vec{\theta}_P$ on $P = FM^n$ which at a point $p = e_p$ projects its argument $v \in T_pFM$ down to M and then takes the resulting vector's components in the basis e_p, i.e.

$$\vec{\theta}_P(v) \equiv \mathrm{d}\pi(v)^{\mu}_{p}. \tag{5.26}$$

The projection makes the solder form horizontal, and it is also not hard to show it is equivariant, since both actions essentially effect a change of basis:

$$g^* \vec{\theta}_P(v) = \check{g}^{-1} \vec{\theta}_P(v). \tag{5.27}$$

The pullback by the identity section

$$\vec{\theta}_i \equiv \sigma_i^* \vec{\theta}_P \tag{5.28}$$

simply returns the components of the argument in the local basis, and thus is identical to the dual frame $\vec{\beta}$ viewed as a frame-dependent $\mathbb{R}^n$-valued 1-form. Thus recalling Section 4.6, the values of $\vec{\theta}_P$ in the fiber over x correspond to a single point in the associated bundle TM, so that the union of the pullbacks $\vec{\theta}_i$ can be viewed as a single TM-valued 1-form on M

$$\vec{\theta} \colon TM \to TM \tag{5.29}$$

which identifies, or "solders," the tangent vectors on M to elements in the bundle TM associated to FM (explaining the alternative name "tautological 1-form").

△ The TM-valued 1-form $\vec{\theta}$ is also sometimes called the solder form, and can be generalized to bundles E with more general fibers as

$$\theta_E(v) \colon TM \to E \tag{5.30}$$

or

$$\theta_{\sigma_0}(v) \colon TM \to V_{\sigma_0}E, \tag{5.31}$$

where in the second case σ_0 is a distinguished section (e.g. the zero section in a vector bundle). This is called a **soldering** of E to M; for example a Riemannian metric provides a soldering of the cotangent bundle to M. In classical dynamics, if M is a configuration space then the solder form to the cotangent bundle is called the Liouville 1-form, Poincarï¿½ 1-form, canonical 1-form, or symplectic potential.

△ The solder form can also be used to identify the tangent space with a subspace of a vector bundle over M with higher dimension than M.

5.6 Torsion on the tangent frame bundle

The covariant derivative of the solder form defines the torsion on P

$$\begin{aligned} \vec{T}_P &\equiv \mathrm{D}\vec{\theta}_P \\ &= \mathrm{d}\vec{\theta}_P + \check{\Gamma}_P \wedge \vec{\theta}_P. \end{aligned} \tag{5.32}$$

$\vec{T}_P$ is a horizontal equivariant form since $\vec{\theta}_P$ is. Examining the first few components, we have:

$$\begin{aligned}\vec{T}_P\left(v,w\right) &= \mathrm{d}\vec{\theta}_P\left(v^{\ominus},w^{\ominus}\right)\\ \Rightarrow \varepsilon^2\vec{T}_P\left(v,w\right) &= \vec{\theta}_P\left(\varepsilon w^{\ominus}\big|_{p+\varepsilon v^{\ominus}}\right) - \vec{\theta}_P\left(\varepsilon w^{\ominus}\big|_p\right) - \dots\\ &= \mathrm{d}\pi\left(\varepsilon w^{\ominus}\big|_{p+\varepsilon v^{\ominus}}\right)^{\mu}_{p+\varepsilon v^{\ominus}} - \mathrm{d}\pi\left(\varepsilon w^{\ominus}\big|_p\right)^{\mu}_p - \dots\end{aligned} \tag{5.33}$$

The first term projects the horizontal component of w at $p+\varepsilon v^{\ominus}$ down to M, which is the same as projecting w itself down to M since the projection of the vertical part vanishes. Then we take its components in the basis at $p+\varepsilon v^{\ominus}$, which is the parallel transport of the basis at p in the direction v. These are the same components as that of the projection of w at $p+\varepsilon v^{\ominus}$ parallel transported back to p in the basis at p. Thus the torsion on P is the "sum of the boundary vectors of the surface defined by the projection of its arguments down to M after being parallel transported back to p."

This analysis makes it clear that the pullback of the torsion on P by the identity section

$$\vec{T}_i \equiv \sigma_i^* \vec{T}_P, \tag{5.34}$$

which by our previous pullback results recovers the torsion on M, just bounces the argument vectors to the identity section and back.

It can also be shown that the analog of the first Bianchi identity on M holds on P, with the original being recovered upon pulling back by the identity section:

$$\begin{aligned}\mathrm{D}\vec{T}_P &= \check{R}_P \wedge \vec{\theta}_P\\ \mathrm{D}\vec{T}_i &= \check{R}_i \wedge \vec{\theta}_i\end{aligned} \tag{5.35}$$

5.7 Spinor bundles

A **spin structure** on an orientable Riemannian manifold M is a principal bundle map

$$\Phi_P\colon (P, M^n, \mathrm{Spin}(n)) \to (F_{SO}, M^n, SO(n)) \tag{5.36}$$

from the **spin frame bundle** (AKA bundle of spin frames) P to the orthonormal frame bundle F_{SO} with respect to the double covering map

$$\Phi_G\colon \mathrm{Spin}(n) \to SO(n). \tag{5.37}$$

The equivariance condition on the bundle map is then

$$\Phi_P(U(p)) = \Phi_G(U)(\Phi_P(p)), \tag{5.38}$$

so that the right action of a spinor transformation $U \in \mathrm{Spin}(n)$ on a spin basis corresponds to the right action of a rotation $\Phi_G(U)$ on the corresponding orthonormal basis $\Phi_P(p)$. On a time and space orientable pseudo-Riemannian manifold, a spin structure is a principal bundle map with respect to the double covering map $\Phi_G\colon \mathrm{Spin}(r,s)^e \to SO(r,s)^e$ (except in the case $r=s=1$, which is not a double cover).

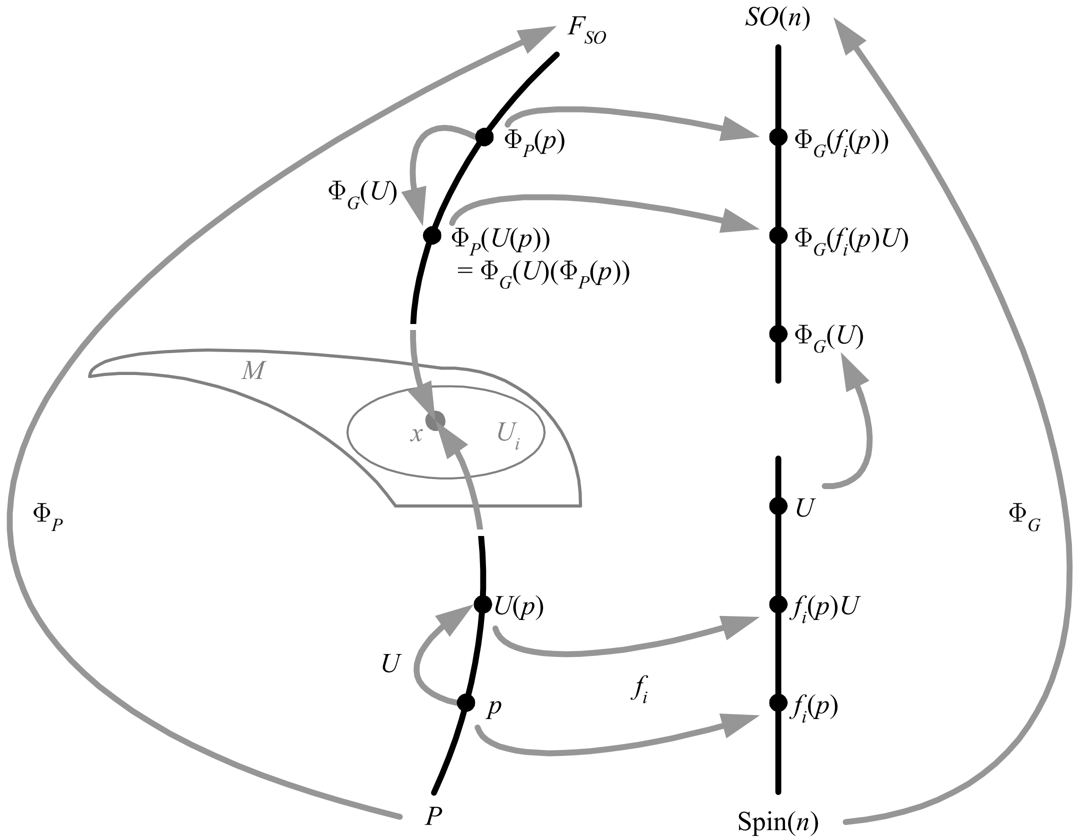

Figure 5.3: A spin structure is a principal bundle map that gives a global 2-1 mapping from the fibers of the spin frame bundle to the fibers of the orthonormal frame bundle. The existence of a spin structure means that a change of frame can be smoothly and consistently mapped to changes of spin frame, permitting the existence of spinor fields.

If a spin structure exists for M, then M is called a **spin manifold** (one also says M is spin; sometimes a spin manifold is defined to include a specific spin structure). Any manifold that can be defined with no more than two coordinate charts is then spin, and therefore any parallelizable manifold and any n-sphere is spin. As we will see in Section 6.2, the existence of spin structures can be related to characteristic classes. It also can be shown that any non-compact spacetime manifold with signature $(3, 1)$ is spin iff it is parallelizable. Finally, a vector bundle $(E, M^n, \mathbb{C}^m)$ associated to the spin frame bundle $(P, M, \text{Spin}(r, s)^e)$ under a rep of $\text{Spin}(r, s)^e$ on $\mathbb{C}^m$ is called a **spinor bundle**, and a section of this bundle is a spinor field on M.

For a charged spinor field taking values in $U(1) \otimes \mathbb{C}^m$, where $\mathbb{C}^m$ is acted on by a rep of $\text{Spin}(r, s)^e$, the action of $(e^{i\theta}, U) \in U(1) \times \text{Spin}(r, s)^e$ and $(-e^{i\theta}, -U)$ are identical, so that the structure group is reducible to

$$\begin{aligned} \text{Spin}^c(r, s)^e &\equiv U(1) \times_{\mathbb{Z}_2} \text{Spin}(r, s)^e \\ &\equiv \left(U(1) \times \text{Spin}(r, s)^e\right) / \mathbb{Z}_2, \end{aligned} \tag{5.39}$$

where the quotient space collapses all points in the product space which are related by changing the sign of both components. The superscript refers to the circle $U(1)$. A **spin^c structure** on an orientable pseudo-Riemannian manifold M is then a principal bundle map

$$\Phi_P\colon (P, M^n, \mathrm{Spin}^c(r,s)^e) \to (F_{SO}, M^n, SO(r,s)^e) \tag{5.40}$$

with respect to the double covering map

$$\Phi_G\colon \mathrm{Spin}^c(r,s)^e \to SO(r,s)^e \tag{5.41}$$

For spinor matter fields that take values in $V \otimes \mathbb{C}^m$ for some internal space V with structure (gauge) group G with $\mathbb{Z}_2$ in its center (e.g. a matrix group where the negative of every element remains in the group), we can analogously define a **spin^G structure**. It can be shown (see [1]) that spin^G structures exist on any four dimensional M if such a G is a compact simple simply connected Lie group, e.g. $SU(2i)$; therefore the spacetime manifold has no constraints due to spin structure in the standard model, or in any extension that includes $SU(2)$ gauged spinors.

6 Characterizing bundles

6.1 Universal bundles

Given a fiber bundle (E, M, π, F) and a continuous map to the base space

$$f\colon N \to M, \tag{6.1}$$

the **pullback bundle** (AKA induced bundle, pullback of E by f) is defined as

$$f^*(E) \equiv \{(n,p) \in N \times E \mid f(n) = \pi(p)\}, \tag{6.2}$$

and is a fiber bundle $(f^*(E), N, \pi_f, F)$ with the same fiber but base space N. Projection of $q = (n,p) \in f^*(E)$ onto n is just the bundle projection $\pi_f\colon f^*(E) \to N$, while projection onto p defines a bundle map

$$\Phi\colon f^*(E) \to E \tag{6.3}$$

such that

$$\pi\left(\Phi(q)\right) = f\left(\pi_f(q)\right) = x \in M. \tag{6.4}$$

For any topological group G, there exists a **universal principal bundle** (AKA universal bundle) (EG, BG, G) such that every principal G-bundle (P, M, G) (with M at least a CW-complex) is the pullback of EG by some $f\colon M \to BG$. The base space BG is called the **classifying space** for G. The pullbacks of a principal bundle by two homotopic maps are isomorphic, and thus for a given M the homotopy classes of the maps f are one-to-one with the isomorphism classes of principal G-bundles over M.

Every vector bundle $(E, M, \mathbb{K}^n)$ is therefore the pullback of the **universal vector bundle** $E_n(\mathbb{K}^\infty)$ (AKA tautological bundle, universal bundle), the vector bundle associated to the universal principal bundle for its structure group. It can be shown that any vector bundle admits an inner product, so we need only consider the structure groups $O(n)$ and $U(n)$, whose classifying spaces are each a **Grassmann manifold** (AKA Grassmannian) $G_n(\mathbb{K}^\infty)$. This is a limit of the finite-dimensional Grassmann manifold $G_n(\mathbb{K}^k)$, which is all n-planes in $\mathbb{K}^k$ through the origin. Each point $x \in G_n(\mathbb{K}^k)$ thus corresponds to a copy of $\mathbb{K}^n$, as does the fiber over x in the universal vector bundle, explaining the alternate name “tautological bundle.” The total space of the associated universal principal bundle is the **Stiefel manifold** $V_n(\mathbb{K}^\infty)$, a limit of the finite-dimensional $V_n(\mathbb{K}^k)$, defined as all ordered orthonormal n-tuples in $\mathbb{K}^k$; the bundle projection simply sends each n-tuple to the n-plane containing it.

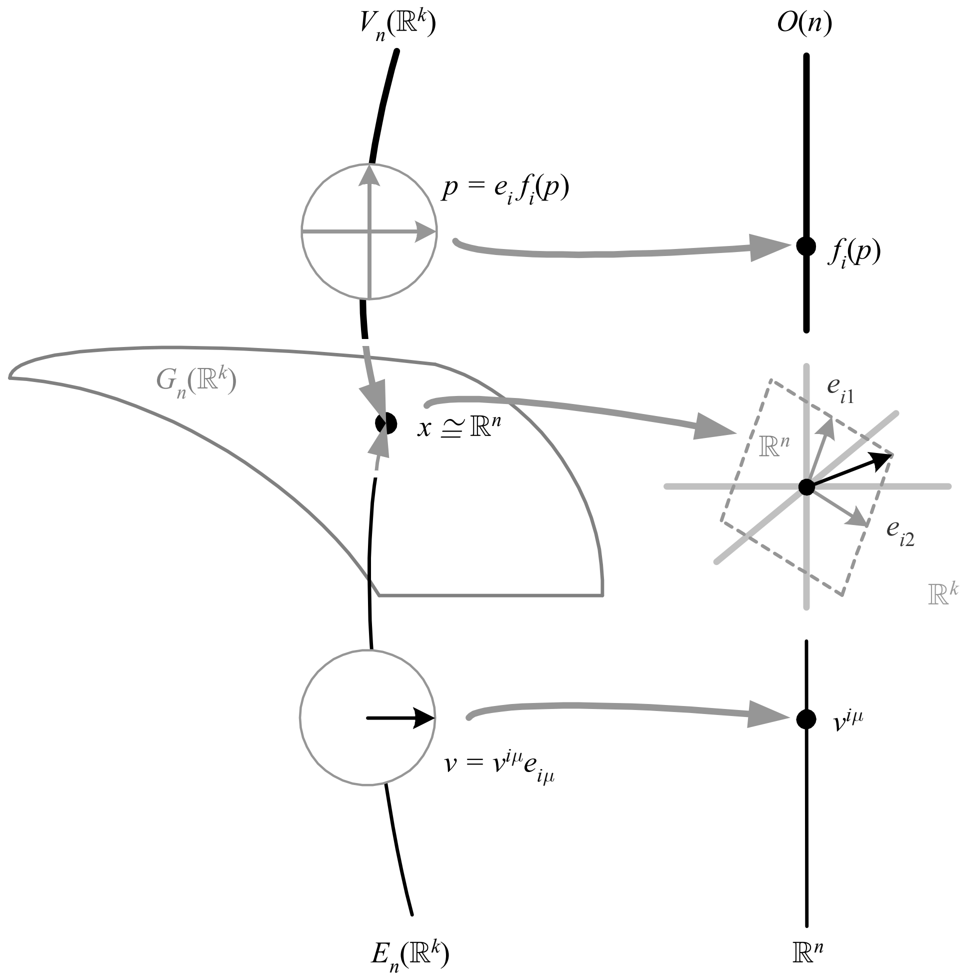

FIGURE 6.1: The Grassmann manifold $G_n(\mathbb{R}^k)$ is all n-planes in $\mathbb{R}^k$ through the origin, and is the base space of the Stiefel manifold $V_n(\mathbb{R}^k)$, defined as all ordered orthonormal n-tuples in $\mathbb{R}^k$ where the fiber is $O(n)$ and the bundle projection simply sends each n-tuple to the n-plane containing it. The tautological bundle is the associated vector bundle $E_n(\mathbb{R}^k)$ with fiber $\mathbb{R}^n$, and the universal principal bundle for $O(n)$ is the limit $V_n(\mathbb{K}^\infty)$.

△ Grassmann manifolds can also be denoted $Gr(n, \mathbb{K}^k)$, $Gr(n, k)$, $G_{n,k}$ or $g_{n,k}$ and the order of the parameters are sometimes reversed. Stiefel manifolds have similar alternative notations.

6.2 Characteristic classes

Vector bundles (and thus their associated principal bundles) can be examined using **characteristic classes**. For a given vector bundle $(E, M, \mathbb{K}^n)$ these are elements in the cohomology groups of the base space

$$c(E) \in H^*(M;R), \tag{6.5}$$

for some commutative unital ring R, which commute with the pullback of any $f\colon N \to M$:

$$c\left(f^*\left(E\right)\right) = f^*\left(c\left(E\right)\right) \tag{6.6}$$

In the second term, the pullback by f means that $f^*\left(c\left(E\right)\right) \in H^*(N;R)$. Since a trivial vector bundle $M \times \mathbb{K}^n$ is the pullback of $(E, 0, \mathbb{K}^n)$ by $f\colon M \to 0$ (where 0 is the space with a single point), we have

$$c\left(M \times \mathbb{K}^n\right) = c\left(f^*\left(E\right)\right) = f^*\left(c\left(E\right)\right) = 0, \tag{6.7}$$

(where 0 is the ring zero). Therefore the characteristic classes of a trivial bundle vanish, or in other words a characteristic class acts as an **obstruction** to a bundle being trivial. However there exist non-trivial bundles whose characteristic classes also all vanish. Similarly, if two vector bundles with the same base space are isomorphic, then they are related by the identity pullback; thus a necessary (but not sufficient) condition for isomorphism is identical characteristic classes. All characteristic classes can be determined via the cohomology classes of the classifying spaces $BO(n)$ and $BU(n)$, since e.g. for real vector bundles any $(E, M, \mathbb{R}^n)$ is the pullback of $BO(n)$ by some f, so that we have

$$c\left(E\right) = c\left(f^*\left(BO(n)\right)\right) = f^*\left(c\left(BO(n)\right)\right). \tag{6.8}$$

For a real vector bundle $(E, M, \mathbb{R}^n)$ there are three characteristic classes (none of which we will define here): the **Stiefel-Whitney classes**

$$w_i(E) \in H^i(M;\mathbb{Z}_2), \tag{6.9}$$

the **Pontryagin classes**

$$p_i(E) \in H^{4i}(M;\mathbb{Z}), \tag{6.10}$$

and if the bundle is oriented the **Euler class**

$$e(E) \in H^n(M;\mathbb{Z}). \tag{6.11}$$

For complex vector bundles, there are the **Chern classes**

$$c_i(E) \in H^{2i}(M;\mathbb{Z}). \tag{6.12}$$

The characteristic class of a manifold M is defined to be that of its tangent bundle, e.g.

$$w_i(M) \equiv w_i(TM). \tag{6.13}$$

If M is a compact orientable four-dimensional manifold, then it is parallelizable iff $w_2(M) = p_1(M) = e(M) = 0$.

A non-zero Stiefel-Whitney class $w_i(E)$ acts as an obstruction to the existence of $(n-i+1)$ everywhere linearly independent sections of E. Therefore, if such section do exist, then $w_j(E)$ vanishes for $j \geq i$; in particular, a non-zero $w_n(E)$ means there are no non-vanishing global sections. It can be shown that $w_1(E) = 0$ iff E is orientable, so that M is orientable iff $w_1(M) = 0$.

Spin structures exist on an oriented M iff $w_2(M) = 0$; if spin structures do exist, then their equivalency classes have a one-to-one correspondence with the elements of $H^1(M, \mathbb{Z}_2)$. Inequivalent spin structures have either inequivalent spin frame bundles or inequivalent bundle maps; in four dimensions, there is only one spin frame bundle up to isomorphism, so that different spin structures correspond to different bundle maps (i.e. different spin connections).

Spin^c structures exist on an oriented M if spin structures exist, but also in some cases where they do not; for example if M is simply connected and compact. If spin^c structures do exist, then their equivalency classes have a one-to-one correspondence with the elements of $H^2(M, \mathbb{Z})$, and in four dimensions, unlike the case for spin structures, inequivalent spin^c structures can have inequivalent spin frame bundles.

6.3 Related constructions and facts

The direct product of two vector bundles $(E, M, \mathbb{K}^m)$ and $(E', M', \mathbb{K}^n)$ is another vector bundle

$$(E \times E', M \times M', \mathbb{K}^{m+n}). \tag{6.14}$$

If we form the direct product of two vector bundles with the same base space, we can then restrict the base space to the diagonal via the pullback by

$$f : M \times M \to M \tag{6.15}$$

defined by

$$(x, x) \mapsto x. \tag{6.16}$$

The resulting vector bundle is called the **Whitney sum** (AKA direct sum bundle), and is denoted

$$(E \oplus E', M, \mathbb{K}^{m+n}). \tag{6.17}$$

The **total Whitney clas**s of a real vector bundle $(E, M, \mathbb{R}^n)$ is defined as

$$w(E) \equiv 1 + w_1(E) + w_2(E) + \cdots + w_n(E). \tag{6.18}$$

The series is finite since $w_i(E)$ vanishes for $i > n$, and is thus an element of $H^*(M, \mathbb{Z}_2)$. The total Whitney class is multiplicative over the Whitney sum, i.e.

$$w(E \oplus E') = w(E)w(E'). \tag{6.19}$$

The **total Chern class** is defined similarly, and has the same multiplicative property.

The **flag manifold** $F_n(\mathbb{K}^\infty)$ is a limit of the finite-dimensional flag manifold $F_n(\mathbb{K}^k)$, which is all ordered n-tuples of orthogonal lines in $\mathbb{K}^k$ through the origin. The name is due to the fact that an ordered n-tuple of orthogonal lines in $\mathbb{K}^k$ is equivalent to an ***n*-flag**, a sequence of subspaces $V_1 \subset \cdots \subset V_n$ in $\mathbb{K}^k$ where each V_i has dimension i.

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