

YANG-MILLS THEORY

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ABSTRACT. We give a short introduction to classical Yang-Mills theory. Starting from Abelian symmetries we motivate the transformation laws, the covariant derivative and the gauge field from the requirement of local invariance. The results are then generalized to non-Abelian symmetry groups. Finally we give some examples of Yang-Mills theories which have been applied in physics.

1. INTRODUCTION

In 1954, Yang and Mills published a paper [1] on the isotopic $SU(2)$ invariance of the proton-neutron system. At the time the idea was not fully recognized since it had some unsatisfying properties. In the late 1960s these problems were solved when the full quantized field theory was developed and today quantum Yang-Mills theory is one of the cornerstones of theoretical physics. However, the mathematical foundations remain unclear. For this reason, the Clay Mathematics institute formulated the Millennium Prize problem [2]. A solution of this problem would require a rigorous mathematical formulation of quantum Yang-Mills theory. In this report, we will have a more modest approach and mainly discuss Yang-Mills theory at the classical level. In section 2, we describe the transition from global to local symmetry in the case of an Abelian phase transformation. In section 3, we motivate the preceding arguments in the modern way, starting from fundamental principles. In section 4, we extend the discussion to include non-Abelian symmetry groups and discuss some special cases, some of which have been shown to be in excellent agreement with experimental data ¹. In the final section, we conclude the report and discuss some possible generalizations.

2. ABELIAN SYMMETRIES

In this section, we introduce the physics of Abelian global symmetries and show how the extension to a local symmetry enforce a certain structure on the Lagrangian. Suppose we have a theory (like QED) which is given by the following Lagrangian²

$$(1) \quad \mathcal{L} = \bar{\psi}(x)i\gamma^\mu\partial_\mu\psi - m\bar{\psi}(x)\psi(x).$$

This theory will be invariant under the following $U(1)$ phase transformations

$$(2) \quad \psi(x) \rightarrow e^{i\alpha}\psi(x),$$

$$(3) \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x)e^{-i\alpha}.$$

¹For example, the prediction from QED for the anomalous magnetic moment of the electron agrees with the experimental results to an accuracy larger than one part in a million.

²Strictly speaking, this is a Lagrangian *density*.

Now, we extend this symmetry to a local symmetry by letting $\alpha \rightarrow \alpha(x)$. Then, the Lagrangian in Eq. (1) is no longer invariant, since we pick up a term $\sim \partial_\mu \alpha(x)$. This can be taken care of by adding to the theory a gauge field A_μ as follows

$$(4) \quad \mathcal{L} = \bar{\psi} i \gamma^\mu (\partial_\mu - ieA_\mu) \psi - m \bar{\psi} \psi,$$

where the gauge field transforms as

$$(5) \quad A_\mu \rightarrow A_\mu + \frac{1}{e} \partial_\mu \alpha(x).$$

We denote the combination $D_\mu = \partial_\mu - ieA_\mu$ as the *covariant derivative*.

So what is the most general Lagrangian we can construct which is invariant under the local $U(1)$ transformations? First, we must rule out a mass term for the gauge field, since that would violate gauge invariance. On the other hand, we can introduce a kinetic energy term for A_μ , given by $-\frac{1}{4}F^2$, where the field strength tensor $F_{\mu\nu}$ is

$$(6) \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

So far so good, but now one could ask the following question: If we want to construct the most general Lagrangian consistent with the symmetries, why are there not infinitely many possibilities? After all there are terms of arbitrary mass dimension, which still respect the symmetries.

However, these terms will not give renormalizable Lagrangians since operators of dimension larger than four enforce coupling constants with negative mass dimension.³ Any theory with coupling constants of negative mass dimension is non-renormalizable (for a proof of this statement, see for example chapter 10 of [6]). Hence, if we require renormalizability, then we must rule out operators of higher dimension. We could nevertheless still add to \mathcal{L} the term $\epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}$, which is invariant under the $U(1)$ symmetry and is renormalizable. However, this term, however, is not invariant under parity so if we postulate parity invariance this term must also be neglected. Thus, we have seen that the most general renormalizable Lagrangian invariant under Eq. (2), Eq. (3) and parity is

$$(7) \quad \mathcal{L} = \bar{\psi} i \gamma^\mu D_\mu \psi - m \bar{\psi} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

3. GAUGE INVARIANCE AND GEOMETRY

In the previous section, the transition from global to local symmetries was introduced in a rather ad hoc fashion. The covariant derivative and the gauge field were not motivated in any depth. In this section, we will see that these objects follow from geometrical arguments. Once again we try to find the most general renormalizable Lagrangian consistent with the $U(1)$ symmetry and possibly some other symmetry, like parity. We saw in the previous section that the difficulties originated from the term containing derivatives of the fields. Maybe then there is

³Since the action is dimensionless it follows that \mathcal{L} in d dimensions must have mass dimension $(\text{mass})^d$. Thus, if we have a term gA , where g is a coupling constant and A is some operator, it follows that if A has mass dimension larger than d , then the coupling constant is enforced to have negative mass dimension. The mass dimensions of the various fields in d dimensions can be deduced from the corresponding free Lagrangians. For example, in two dimensions this means that we can have operators of the form ϕ^n for any integer n , since the scalar field ϕ for $d = 2$ has mass dimension 1.

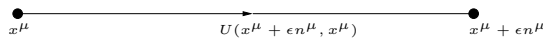


FIGURE 1. Schematic picture of the comparator.

something fundamentally wrong with our notion of the derivative of the field. The ordinary derivative of a Dirac field in the direction of the vector n^μ is defined by

$$(8) \quad n^\mu \partial_\mu \psi = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\psi(x + \epsilon n) - \psi(x)).$$

Clearly, this derivative has no sensible transformation law under the local $U(1)$ symmetry, since the fields have different transformation laws at different space-time points. In order to obtain a sensible derivative, which transforms in the same way as the field ψ , we introduce a scalar function $U(x, y)$, which is known as a *comparator* (see Fig. 1). We require that $U(x, y)$ transforms as $U(x, y) \rightarrow e^{i\alpha(x)} U(x, y) e^{-i\alpha(y)}$ and that it has the property that $U(x, x) = 1$. Now we can define the following object

$$(9) \quad n^\mu D_\mu \psi = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [\psi(x + \epsilon n) - U(x + \epsilon n, x) \psi(x)].$$

This object will have the same transformation law as ψ because of the way we have defined the comparator.

We shall now write Eq. (9) in a more useful form by expanding in the small parameter ϵ , assuming that the comparator is a continuous function of x and y :

$$(10) \quad U(x + \epsilon n, x) = 1 + \epsilon n^\mu \partial_\mu U(x + \epsilon n, x)|_{\epsilon=0} + \mathcal{O}(\epsilon^2).$$

We rewrite the second term as $\epsilon n^\mu \partial_\mu U(x + \epsilon n, x)|_{\epsilon=0} = i\epsilon n^\mu A_\mu(x)$, where A_μ is a new vector field. If we insert this expression in Eq. (9), then we find the following well-known object

$$(11) \quad D_\mu = \partial_\mu - ieA_\mu.$$

This is of course nothing else but the covariant derivative, defined in section 2. As we have already seen, it has the same transformation law as ψ , which motivates the name *covariant* derivative. The vector field A_μ is in the language of differential geometry known as a *connection*. The transformation law for A_μ follows from the transformation of the comparator. We have that

$$(12) \quad U(x + \epsilon n, x) \rightarrow e^{i\alpha(x + \epsilon n)} U(x + \epsilon n, x) e^{-i\alpha(x)} = 1 - i\epsilon n^\mu \left(A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x) \right) + \mathcal{O}(\epsilon^2).$$

From this follows that the gauge field transforms as

$$(13) \quad A_\mu \rightarrow A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x).$$

In order to complete our construction of the Lagrangian, we need to introduce a gauge invariant kinetic energy term for A_μ . This can be done by constructing a small square of connected comparators. This composite object will be gauge invariant and in infinitesimal form it will reveal that the gauge invariant quantity containing the gauge field is the ordinary field strength tensor

$$(14) \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Thus, we have reproduced the results of the previous section, but we have motivated it by fundamental principles. The covariant derivative, the existence of the gauge field, and its transformation properties all followed from simple geometrical arguments.

4. NON-ABELIAN GENERALIZATIONS

We shall now generalize the discussion from the previous sections to include non-Abelian symmetry groups. To do this we start with a global $SU(2)$ symmetry. In fact this was the symmetry considered by Yang and Mills in their original work [1], which was applied to the isospin symmetries of the proton and neutron.

More generally we have a doublet $\psi = (\psi_1(x), \psi_2(x))^T$, which transforms as

$$(15) \quad \psi(x) \rightarrow \exp\left(i\theta^k \frac{\sigma^k}{2}\right) \psi(x),$$

where σ^k are the Pauli σ -matrices. We now extend this symmetry to a local symmetry by letting $\theta^k \rightarrow \theta^k(x)$. This means that the fields transform according to

$$(16) \quad \psi(x) \rightarrow V(x)\psi(x) = \exp\left(i\theta^k(x) \frac{\sigma^k}{2}\right) \psi(x).$$

We now set out to construct the most general renormalizable Lagrangian invariant under the local $SU(2)$ symmetry transformations in Eq. (16). First we shall determine the covariant derivative. We define it exactly as in Eq. (9) with the difference that now the comparator has to be a 2×2 matrix, since $\psi(x)$ is a two-component object. The comparator has just as before the transformation property that $U(x, y) \rightarrow V(x)U(x, y)V(y)^\dagger$, where $U(x, x) = 1$. By expanding the comparator in ϵ we find

$$(17) \quad U(x + \epsilon n, x) = 1 + i g \epsilon n^\mu A_\mu^k \frac{\sigma^k}{2} + \mathcal{O}(\epsilon^2),$$

where g is a constant, analogous to the electric charge. Thus, the covariant derivative becomes

$$(18) \quad D_\mu = \partial_\mu - i g A_\mu^k \frac{\sigma^k}{2}.$$

In contrast with the $U(1)$ case we now have *three* gauge fields, one for each generator of $SU(2)$.

Next, we find the transformation law for the gauge fields. As in the Abelian case this can be read off from the transformation of the comparator. Thus, we find

$$(19) \quad A_\mu^k \frac{\sigma^k}{2} \rightarrow A_\mu^k \frac{\sigma^k}{2} + \frac{1}{g} (\partial_\mu \theta^k) \frac{\sigma^k}{2} + i [\theta^k \frac{\sigma^k}{2}, A_\mu^l \frac{\sigma^l}{2}] + \dots$$

The new feature here is the commutator, which vanishes in the Abelian case.

Finally, the gauge invariant kinetic terms can be found in the same way as in the Abelian case, paying proper attention to the non commutative properties of the matrices. One then finds the field strength tensor

$$(20) \quad F_{\mu\nu}^k \frac{\sigma^k}{2} = \partial_\mu A_\nu^k \frac{\sigma^k}{2} - \partial_\nu A_\mu^k \frac{\sigma^k}{2} - i g [A_\mu^k \frac{\sigma^k}{2}, A_\nu^l \frac{\sigma^l}{2}].$$

If we use that the σ -matrices satisfy $[\frac{\sigma^k}{2}, \frac{\sigma^l}{2}] = i \epsilon^{klm} \frac{\sigma^m}{2}$, then we obtain

$$(21) \quad F_{\mu\nu}^k = \partial_\mu A_\nu^k - \partial_\nu A_\mu^k + g \epsilon^{klm} A_\mu^l A_\nu^m.$$

Note that this is not a gauge invariant quantity, but only gauge covariant. However, it is easy to construct a gauge invariant term by using the properties of the trace. Thus, a kinetic energy term for the gauge fields is given by $\text{tr}[(F_{\mu\nu}^k \frac{\sigma^k}{2})^2]$. Then, we have that the most general renormalizable Lagrangian invariant under the $SU(2)$ symmetry is

$$(22) \quad \mathcal{L} = \bar{\psi} i \gamma^\mu D_\mu \psi - \frac{1}{4} (F_{\mu\nu}^k)^2 - m \bar{\psi} \psi.$$

This is our first nontrivial example of a *Yang-Mills theory*.

The extension to other continuous symmetry groups is straightforward. Suppose we have fields transforming according to an irreducible representation Ω of some compact Lie group G , $\psi(x) \rightarrow \Omega(g)\psi(x)$. Then one can use all of the arguments above by simply replacing $\frac{\sigma^k}{2} \rightarrow T^a$, where T^a are the generators of G . In deriving the transformation properties and the field strength tensor one has to use the commutation rule for the generators

$$(23) \quad [T^a, T^b] = i f^{abc} T^c,$$

where the numbers f^{abc} are known as the *structure constants*. In this way, one finds the following form for the covariant derivative

$$(24) \quad D_\mu = \partial_\mu - ig A_\mu^a T^a.$$

The field strength tensor is given by

$$(25) \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$$

and the renormalizable Lagrangian will have the form

$$(26) \quad \mathcal{L} = \bar{\psi} i \gamma^\mu D_\mu \psi - m \bar{\psi} \psi - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}.$$

These theories have some general features. We have seen that all of them predict the existence of gauge fields. The number of gauge fields equal the number of generators of the symmetry group. For example, in the case of $SU(N)$ symmetry this means that there will be $N^2 - 1$ gauge fields.

Let us now look at some examples of Yang-Mills theories to see how these theories can be realized in Nature. All of them will have a Lagrangian of the form of Eq. (26) although the expressions for the covariant derivative, the field strength tensor and the transformation properties will depend on the properties of the underlying symmetry group. The first two examples we have already discussed.

4.1. Electrodynamics, $G = U(1)$.

The Lagrangian will be invariant under the field transformations $\psi \rightarrow e^{i\alpha(x)}\psi$ and $A_\mu \rightarrow A_\mu - \frac{1}{e}\partial_\mu\alpha(x)$. There is only one generator and so there will only be one gauge field A_μ , which is the photon. The covariant derivative is given by $D_\mu = \partial_\mu - ieA_\mu$. The field strength tensor is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, since $f_{abc} = 0$.

4.2. Isotopic gauge invariance, $G = SU(2)$.

The fields are $\psi = (p, n)^T$ and the three gauge fields A_μ^k . The covariant derivative is $D_\mu = \partial_\mu - ig A_\mu^k \frac{\sigma^k}{2}$. The fields transform as $\psi \rightarrow \exp\left(i\theta^k(x)\frac{\sigma^k}{2}\right)\psi$ and

$A_\mu^k \rightarrow A_\mu^k + \frac{1}{g}\partial_\mu\theta^k + \epsilon^{klm}A_\mu^l\theta^m$. The field strength tensor is given by $F_{\mu\nu}^k = \partial_\mu A_\nu^k - \partial_\nu A_\mu^k + g\epsilon^{klm}A_\mu^l A_\nu^m$.

4.3. Chromodynamics, $G = SU(3)$.

The fields are the quarks,

$\psi = (\psi_{red}, \psi_{green}, \psi_{blue})^T$ and eight gauge fields, the gluons. The quark fields transform as $\psi \rightarrow \exp(\frac{i}{2}\theta^a\lambda^a)\psi$, where λ^a are the Gell-Mann matrices. The gluon fields transform as $A_\mu^a \rightarrow A_\mu^a + \frac{1}{g}\partial_\mu\theta^a + f^{abc}A_\mu^b\theta^c$. The covariant derivative is $D_\mu = \partial_\mu - ig\frac{1}{2}\lambda^a A_\mu^a$ and the field strength tensor is $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$.

There are several other interesting Yang-Mills theories. For example, it has been suggested that the standard model, based on the group $SU(3) \times SU(2) \times U(1)$, is a subgroup of a larger simple group, such as $SU(5)$. Theories of this kind, which attempt to unify interactions are sometimes known as grand unified theories (GUTs). Another possible GUT is based on the group $SO(10)$. These theories are somewhat speculative and direct measurements are not possible today since the unification energies are too high. However, they often make indirect predictions, such as proton decay or the existence of magnetic monopoles. So far there has been no solid experimental confirmation of GUTs.

5. CONCLUSIONS AND OUTLOOK

We have seen how to construct theories invariant under local non-Abelian symmetries. However, in order to obtain a proper description of Nature we have to *quantize* these theories. In the process one discovers many exotic features like asymptotic freedom [3, 4] and ghost fields. The interested reader should consult the references [5, 6, 7] which contain thorough discussions of these matters. As mentioned in the introduction, quantum Yang-Mills theory is the topic for one of the Millenium Problems. A solution of this problem would require an existence proof for Yang-Mills theory on \mathbb{R}^4 . This means that one should construct an axiomatic formulation of Yang-Mills theory from where all essential physical properties follow. This include for example a proof of the existence of an operator product expansion and of quark confinement. The problem also requires that one proves the existence of a mass gap.

To some extent, general features of Yang-Mills theories have already been seen in computer simulations and simplified models such as lattice QCD. The problem is to obtain a mathematical understanding of these properties.

There has been some progress in proving Yang-Mills theory on \mathbb{T}^4 , that is, on the toroidal compactification of \mathbb{R}^4 . However, even if a proof is given for \mathbb{T}^4 , it is not clear how to make the transition $\mathbb{T}^4 \rightarrow \mathbb{R}^4$.

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