

# Supersymmetric Gauge Theories and the AdS/CFT Correspondence\*

*TASI 2001 Lecture Notes*

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## Abstract

In these lecture notes we first assemble the basic ingredients of supersymmetric gauge theories (particularly N=4 super-Yang-Mills theory), supergravity, and superstring theory. Brane solutions are surveyed. The geometry and symmetries of anti-de Sitter space are discussed. The AdS/CFT correspondence of Maldacena and its application to correlation functions in the the conformal phase of N=4 SYM are explained in considerable detail. A pedagogical treatment of holographic RG flows is given including a review of the conformal anomaly in four-dimensional quantum field theory and its calculation from five-dimensional gravity. Problem sets and exercises await the reader.

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# 1 Introduction

These lecture notes describe one of the most exciting developments in theoretical physics of the past decade, namely Maldacena's bold conjecture concerning the equivalence between superstring theory on certain ten-dimensional backgrounds involving Anti-de Sitter space-time and four-dimensional supersymmetric Yang-Mills theories. This AdS/CFT conjecture was unexpected because it relates a theory of gravity, such as string theory, to a theory with no gravity at all. Additionally, the conjecture related highly non-perturbative problems in Yang-Mills theory to questions in classical superstring theory or supergravity. The promising advantage of the correspondence is that problems that appear to be intractable on one side may stand a chance of solution on the other side. We describe the initial conjecture, the development of evidence that it is correct, and some further applications.

The conjecture has given rise to a tremendous number of exciting directions of pursuit and to a wealth of promising results. In these lecture notes, we shall present a quick introduction to supersymmetric Yang-Mills theory (in particular of  $\mathcal{N} = 4$  theory). Next, we give a concise description of just enough supergravity and superstring theory to allow for an accurate description of the conjecture and for discussions of correlation functions and holographic flows, namely the two topics that constitute the core subject of the lectures.

The notes are based on the loosely coordinated lectures of both authors at the 2001 TASI Summer School. It was decided to combine written versions in order to have a more complete treatment. The bridge between the two sets of lectures is Section 8 which presents a self-contained introduction to the subject and a more detailed treatment of some material from earlier sections.

The AdS/CFT correspondence is a broad principle and the present notes concern one of several pathways through the subject. An effort has been made to cite a reasonably complete set of references on the subjects we discuss in detail, but with less coverage of other aspects and of background material.

Serious readers will take the problem sets and exercises seriously!

## 1.1 Statement of the Maldacena conjecture

The Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence, as originally conjectured by Maldacena, advances a remarkable equivalence between two seemingly unrelated theories. On one side (the *AdS-side*) of the correspondence, we have 10-dimensional Type IIB string theory on the product space  $\text{AdS}_5 \times S^5$ , where the Type IIB 5-form flux through  $S^5$  is an integer  $N$  and the equal radii  $L$  of  $\text{AdS}_5$  and  $S^5$  are given by  $L^4 = 4\pi g_s N \alpha'^2$ , where  $g_s$  is the string coupling. On the other side (the *SYM-side*) of the correspondence, we have 4-dimensional super-Yang-Mills (SYM) theory with maximal  $\mathcal{N} = 4$  supersymmetry, gauge group  $SU(N)$ , Yang-Mills coupling  $g_{YM}^2 = g_s$  in the conformal phase. The AdS/CFT conjecture states that these two theories, including operator observables, states, correlation functions and full dynamics, are equivalent to one another

[1, 2, 3]. Indications of the equivalence had appeared in earlier work, [4, 5, 6]. For a general review of the subject, see [7].

In the strongest form of the conjecture, the correspondence is to hold for all values of  $N$  and all regimes of coupling  $g_s = g_{YM}^2$ . Certain limits of the conjecture are, however, also highly non-trivial. The 't Hooft limit on the SYM-side [8], in which  $\lambda \equiv g_{YM}^2 N$  is fixed as  $N \rightarrow \infty$  corresponds to *classical string theory on*  $\text{AdS}_5 \times \text{S}^5$  (no string loops) on the AdS-side. In this sense, classical string theory on  $\text{AdS}_5 \times \text{S}^5$  provides with a classical Lagrangian formulation of the large  $N$  dynamics of  $\mathcal{N} = 4$  SYM theory, often referred to as the *masterfield equations*. A further limit  $\lambda \rightarrow \infty$  reduces classical string theory to classical Type IIB supergravity on  $\text{AdS}_5 \times \text{S}^5$ . Thus, strong coupling dynamics in SYM theory (at least in the large  $N$  limit) is mapped onto classical low energy dynamics in supergravity and string theory, a problem that offers a reasonable chance for solution.

The conjecture is remarkable because its correspondence is between a 10-dimensional theory of gravity and a 4-dimensional theory without gravity at all, in fact, with spin  $\leq 1$  particles only. The fact that all the 10-dimensional dynamical degrees of freedom can somehow be encoded in a 4-dimensional theory living at the boundary of  $\text{AdS}_5$  suggests that the gravity bulk dynamics results from a *holographic image* generated by the dynamics of the boundary theory [9], see also [10]. Therefore, the correspondence is also often referred to as *holographic*.

## 1.2 Applications of the conjecture

The original correspondence is between a  $\mathcal{N} = 4$  SYM theory in its conformal phase and string theory on  $\text{AdS}_5 \times \text{S}^5$ . The power of the correspondence is further evidenced by the fact that the conjecture may be adapted to situations without conformal invariance and with less or no supersymmetry on the SYM side. The  $\text{AdS}_5 \times \text{S}^5$  space-time is then replaced by other manifold or orbifold solutions to Type IIB theory, whose study is usually more involved than was the case for  $\text{AdS}_5 \times \text{S}^5$  but still reveals useful information on SYM theory.

## 2 Supersymmetry and Gauge Theories

We begin by reviewing the particle and field contents and invariant Lagrangians in 4 dimensions, in preparation for a fuller discussion of  $\mathcal{N} = 4$  super-Yang-Mills (SYM) theory in the next section. Standard references include [11, 12, 13]; our conventions are those of [11].

### 2.1 Supersymmetry algebra in 3+1 dimensions

Poincaré symmetry of flat space-time  $\mathbf{R}^4$  with metric  $\eta_{\mu\nu} = \text{diag}(-+++)$ ,  $\mu, \nu = 0, 1, 2, 3$ , is generated by the translations  $\mathbf{R}^4$  and Lorentz transformations  $SO(1, 3)$ , with generators  $P_\mu$  and  $L_{\mu\nu}$  respectively. The complexified Lorentz algebra is isomorphic to the complexified algebra of  $SU(2) \times SU(2)$ , and its finite-dimensional representations are usually labeled by two positive (or zero) half integers  $(s_+, s_-)$ ,  $s_\pm \in \mathbf{Z}/2$ . Scalar, 4-vector, and rank 2 symmetric tensors transform under  $(0, 0)$ ,  $(\frac{1}{2}, \frac{1}{2})$  and  $(1, 1)$  respectively, while left and right chirality fermions and self-dual and anti-self-dual rank 2 tensors transform under  $(\frac{1}{2}, 0)$  and  $(0, \frac{1}{2})$  and  $(1, 0)$  and  $(0, 1)$  respectively.

Supersymmetry (susy) enlarges the Poincaré algebra by including spinor *supercharges*,

$$a = 1, \dots, \mathcal{N} \quad \begin{cases} Q_\alpha^a & \alpha = 1, 2 \text{ left Weyl spinor} \\ \bar{Q}_{\dot{\alpha}a} = (Q_\alpha^a)^\dagger & \text{right Weyl spinor} \end{cases} \quad (2.1)$$

Here,  $\mathcal{N}$  is the number of independent supersymmetries of the algebra. Two-component spinor notation,  $\alpha = 1, 2$ , is related to 4-component Dirac spinor notation by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad Q^a = \begin{pmatrix} Q_\alpha^a \\ \bar{Q}_{\dot{\alpha}a} \end{pmatrix} \quad (2.2)$$

The supercharges transform as Weyl spinors of  $SO(1, 3)$  and commute with translations. The remaining susy structure relations are

$$\{Q_\alpha^a, \bar{Q}_{\dot{\beta}b}\} = 2\sigma_{\alpha\dot{\beta}}^\mu P_\mu \delta_b^a \quad \{Q_\alpha^a, Q_\beta^b\} = 2\epsilon_{\alpha\beta} Z^{ab} \quad (2.3)$$

By construction, the generators  $Z^{ab}$  are anti-symmetric in the indices  $I$  and  $J$ , and commute with all generators of the supersymmetry algebra. For the last reason, the  $Z^{ab}$  are usually referred to as *central charges*, and we have

$$Z^{ab} = -Z^{ba} \quad [Z^{ab}, \text{anything}] = 0 \quad (2.4)$$

Note that for  $\mathcal{N} = 1$ , the anti-symmetry of  $Z$  implies that  $Z = 0$ .

The supersymmetry algebra is invariant under a global phase rotation of all supercharges  $Q_\alpha^a$ , forming a group  $U(1)_R$ . In addition, when  $\mathcal{N} > 1$ , the different supercharges may be rotated into one another under the unitary group  $SU(\mathcal{N})_R$ . These (automorphism) symmetries of the supersymmetry algebra are called *R-symmetries*. In quantum field theories, part or all of these R-symmetries may be broken by anomaly effects.

## 2.2 Massless Particle Representations

To study massless representations, we choose a Lorentz frame in which the momentum takes the form  $P^\mu = (E, 0, 0, E)$ ,  $E > 0$ . The susy algebra relation (2.3) then reduces to

$$\{Q_\alpha^a, (Q_\beta^b)^\dagger\} = 2(\sigma^\mu P_\mu)_{\alpha\dot{\beta}} \delta_b^a = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \delta_b^a \quad (2.5)$$

We consider only unitary particle representations, in which the operators  $Q_\alpha^a$  act in a positive definite Hilbert space. The relation for  $\alpha = \dot{\beta} = 2$  and  $a = b$  implies

$$\{Q_2^a, (Q_2^a)^\dagger\} = 0 \quad \implies \quad Q_2^a = 0, \quad Z^{ab} = 0 \quad (2.6)$$

The relation  $Q_2^a = 0$  follows because the left hand side of (2.6) implies that the norm of  $Q_2^a|\psi\rangle$  vanishes for any state  $|\psi\rangle$  in the Hilbert space. The relation  $Z^{ab} = 0$  then follows from (2.3) for  $\alpha = 2$  and  $\dot{\beta} = 1$ . The remaining supercharge operators are

- $Q_1^a$  which lowers helicity by 1/2;
- $\bar{Q}_1^a = (Q_1^a)^\dagger$  which raises helicity by 1/2.

Together,  $Q_1^a$  and  $(Q_1^a)^\dagger$ , with  $a = 1, \dots, \mathcal{N}$  form a representation of dimension  $2^\mathcal{N}$  of the Clifford algebra associated with the Lie algebra  $SO(2\mathcal{N})$ . All the states in the representation may be obtained by starting from the highest helicity state  $|h\rangle$  and applying products of  $Q_1^a$  operators for all possible values of  $a$ .

We shall only be interested in CPT invariant theories, such as quantum field theories and string theories, for which the particle spectrum must be symmetric under a sign change in helicity. If the particle spectrum obtained as a Clifford representation in the above fashion is not already CPT self-conjugate, then we shall take instead the direct sum with its CPT conjugate. For helicity  $\leq 1$ , the spectra are listed in table 1. The  $\mathcal{N} = 3$  and  $\mathcal{N} = 4$  particle spectra then coincide, and their quantum field theories are identical.

Helicity $\leq 1$	$\mathcal{N} = 1$ gauge	$\mathcal{N} = 1$ chiral	$\mathcal{N} = 2$ gauge	$\mathcal{N} = 2$ hyper	$\mathcal{N} = 3$ gauge	$\mathcal{N} = 4$ gauge
1	1	0	1	0	1	1
1/2	1	1	2	2	3+1	4
0	0	1+1	1+1	4	3+3	6
-1/2	1	1	2	2	1+3	4
-1	1	0	1	0	1	1
Total #	$2 \times 2$	$2 \times 2$	$2 \times 4$	8	$2 \times 8$	16

Table 1: Numbers of Massless States as a function of  $\mathcal{N}$  and helicity



## 2.3 Massive Particle Representations

For massive particle representations, we choose the rest frame with  $P^\mu = (M, 0, 0, 0)$ , so that the first set of susy algebra structure relations takes the form

$$\{Q_\alpha^a, (Q_\beta^b)^\dagger\} = 2M\delta_\alpha^\beta\delta_b^a \quad (2.7)$$

To deal with the full susy algebra, it is convenient to make use of the  $SU(\mathcal{N})_R$  symmetry to diagonalize in blocks of  $2 \times 2$  the anti-symmetric matrix  $Z^{ab} = -Z^{ba}$ . To do so, we split the label  $a$  into two labels :  $a = (\hat{a}, \bar{a})$  where  $\hat{a} = 1, 2$  and  $\bar{a} = 1, \dots, r$ , where  $\mathcal{N} = 2r$  for  $\mathcal{N}$  even (and we append a further single label when  $\mathcal{N}$  is odd). We then have

$$Z = \text{diag}(\epsilon Z_1, \dots, \epsilon Z_r, \#) \quad \epsilon^{12} = -\epsilon^{21} = 1 \quad (2.8)$$

where  $\#$  equals 0 for  $\mathcal{N} = 2r + 1$  and  $\#$  is absent for  $\mathcal{N} = 2r$ . The  $Z_{\bar{a}}$ ,  $\bar{a} = 1, \dots, r$  are real *central charges*. In terms of linear combinations  $\mathcal{Q}_{\alpha\pm}^{\bar{a}} \equiv \frac{1}{2}(Q_\alpha^{1\bar{a}} \pm \sigma_{\alpha\dot{\beta}}^0(Q_{\dot{\beta}}^{2\bar{a}})^\dagger)$ , the only non-vanishing susy structure relation left is (the  $\pm$  signs below are correlated)

$$\{\mathcal{Q}_{\alpha\pm}^{\bar{a}}, (\mathcal{Q}_{\beta\pm}^{\bar{a}})^\dagger\} = \delta_b^{\bar{a}}\delta_\alpha^\beta(M \pm Z_{\bar{a}}) \quad (2.9)$$

In any *unitary particle representation*, the operator on the left hand side of (2.9) must be positive, and thus we obtain the famous *BPS bound* (for Bogomolnyi-Prasad-Sommerfield, [14]) giving a lower bound on the mass in terms of the central charges,

$$M \geq |Z_{\bar{a}}| \quad \bar{a} = 1, \dots, r = [\mathcal{N}/2] \quad (2.10)$$

Whenever one of the values  $|Z_{\bar{a}}|$  equals  $M$ , the BPS bound is (partially) saturated and either the supercharge  $\mathcal{Q}_{\alpha+}^{\bar{a}}$  or  $\mathcal{Q}_{\alpha-}^{\bar{a}}$  must vanish. The supersymmetry representation then suffers *multiplet shortening*, and is usually referred to as BPS. More precisely, if we have  $M = |Z_{\bar{a}}|$  for  $\bar{a} = 1, \dots, r_o$ , and  $M > |Z_{\bar{a}}|$  for all other values of  $\bar{a}$ , the susy algebra is effectively a Clifford algebra associated with  $SO(4\mathcal{N} - 4r_o)$ , the corresponding representation is said to be  $1/2^{r_o}$  BPS, and has dimension  $2^{2\mathcal{N}-2r_o}$ .

Spin $\leq 1$	$\mathcal{N} = 1$ gauge	$\mathcal{N} = 1$ chiral	$\mathcal{N} = 2$ gauge	$\mathcal{N} = 2$ BPS gauge	$\mathcal{N} = 2$ BPS hyper	$\mathcal{N} = 4$ BPS gauge
1	1	0	1	1	0	1
1/2	2	1	4	2	2	4
0	1	2	5	1	4	5
Total #	8	4	16	8	8	16

Table 2: Numbers of Massive States as a function of  $\mathcal{N}$  and spin

## 2.4 Field Contents and Lagrangians

The analysis of the preceding two subsections has revealed that the supersymmetry particle representations for  $1 \leq \mathcal{N} \leq 4$ , with spin less or equal to 1, simply consist of customary spin 1 vector particles, spin 1/2 fermions and spin 0 scalars. Correspondingly, *the fields in supersymmetric theories with spin less or equal to 1 are customary spin 1 gauge fields, spin 1/2 Weyl fermion fields and spin 0 scalar fields, but these fields are restricted to enter in multiplets of the relevant supersymmetry algebras.*

Let  $\mathcal{G}$  denote the gauge algebra, associated with a compact Lie group  $G$ . For any  $1 \leq \mathcal{N} \leq 4$ , we have a gauge multiplet, which transforms under the adjoint representation of  $\mathcal{G}$ . For  $\mathcal{N} = 4$ , this is the only possible multiplet. For  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$ , we also have *matter multiplets*: for  $\mathcal{N} = 1$ , this is the *chiral multiplet*, and for  $\mathcal{N} = 2$  this is the *hypermultiplet*, both of which may transform under an arbitrary unitary, representation  $\mathcal{R}$  of  $\mathcal{G}$ . Component fields consist of the customary gauge field  $A_\mu$ , left Weyl fermions  $\psi_\alpha$  and  $\lambda_\alpha$  and scalar fields  $\phi$ ,  $H$  and  $X$ .

- $\mathcal{N} = 1$  *Gauge Multiplet* ( $A_\mu \lambda_\alpha$ ), where  $\lambda_\alpha$  is the gaugino Majorana fermion;
- $\mathcal{N} = 1$  *Chiral Multiplet* ( $\psi_\alpha \phi$ ), where  $\psi_\alpha$  is a left Weyl fermion and  $\phi$  a complex scalar, in the representation  $\mathcal{R}$  of  $\mathcal{G}$ .
- $\mathcal{N} = 2$  *Gauge Multiplet* ( $A_\mu \lambda_{\alpha\pm} \phi$ ), where  $\lambda_{\alpha\pm}$  are left Weyl fermions, and  $\phi$  is the complex *gauge scalar*. Under  $SU(2)_R$  symmetry,  $A_\mu$  and  $\phi$  are singlets, while  $\lambda_+$  and  $\lambda_-$  transform as a doublet.
- $\mathcal{N} = 2$  *Hypermultiplet* ( $\psi_{\alpha\pm} H_\pm$ ), where  $\psi_{\alpha\pm}$  are left Weyl fermions and  $H_\pm$  are two complex scalars, transforming under the representation  $\mathcal{R}$  of  $\mathcal{G}$ . Under  $SU(2)_R$  symmetry,  $\psi_\pm$  are singlets, while  $H_+$  and  $H_-$  transform as a doublet.
- $\mathcal{N} = 4$  *Gauge Multiplet* ( $A_\mu \lambda_\alpha^a X^i$ ), where  $\lambda_\alpha^a$ ,  $a = 1, \dots, 4$  are left Weyl fermions and  $X^i$ ,  $i = 1, \dots, 6$  are real scalars. Under  $SU(4)_R$  symmetry,  $A_\mu$  is a singlet,  $\lambda_\alpha^a$  is a 4 and the scalars  $X^i$  are a rank 2 anti-symmetric 6.

Lagrangians invariant under supersymmetry are customary Lagrangians of gauge, spin 1/2 fermion and scalar fields, (arranged in multiplets of the supersymmetry algebra) with certain special relations amongst the coupling constants and masses. We shall restrict attention to local Lagrangians in which each term has a total of no more than two derivatives on all boson fields and no more than one derivative on all fermion fields. All renormalizable Lagrangians are of this form, but all low energy effective Lagrangians are also of this type.

The case of the  $\mathcal{N} = 1$  gauge multiplet ( $A_\mu \lambda_\alpha$ ) by itself is particularly simple. We proceed by writing down all possible gauge invariant polynomial terms of dimension 4 using minimal gauge coupling,

$$\mathcal{L} = -\frac{1}{2g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \frac{\theta_I}{8\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{i}{2} \text{tr} \bar{\lambda} \bar{\sigma}^\mu D_\mu \lambda \quad (2.11)$$

where  $g$  is the gauge coupling,  $\theta_I$  is the instanton angle, the field strength is  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ ,  $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$  is the Poincaré dual of  $F$ , and  $D_\mu = \partial_\mu\lambda + i[A_\mu, \lambda]$ . Remarkably,  $\mathcal{L}$  is automatically invariant under the  $\mathcal{N} = 1$  supersymmetry transformations

$$\begin{aligned}\delta_\xi A_\mu &= i\bar{\xi}\bar{\sigma}_\mu\lambda - i\bar{\lambda}\bar{\sigma}_\mu\xi \\ \delta_\xi\lambda &= \sigma^{\mu\nu}F_{\mu\nu}\xi\end{aligned}\tag{2.12}$$

where  $\xi$  is a spin 1/2 valued infinitesimal supersymmetry parameter. Note that the addition in (2.11) of a Majorana mass term  $m\lambda\lambda$  would spoil supersymmetry.

As soon as scalar fields are to be included, such as is the case for any other multiplet, it is no longer so easy to guess supersymmetry invariant Lagrangians and one is led to the use of superfields. *Superfields* assemble all component fields of a given supermultiplet (together with auxiliary fields) into a supersymmetry multiplet field on which supersymmetry transformations act linearly. Superfield methods provide a powerful tool for producing supersymmetric field equations for any degree of supersymmetry. For  $\mathcal{N} = 1$  there is a standard off-shell superfield formulation as well (see [11, 12, 13] for standard treatments). Considerably more involved off-shell superfield formulations are also available for  $\mathcal{N} = 2$  in terms of harmonic and analytic superspace [15], see also the review of [16]. For  $\mathcal{N} = 4$  supersymmetry, no off-shell formulation is known at present; one is thus forced to work either in components or in the  $\mathcal{N} = 1$  or  $\mathcal{N} = 2$  superfield formulations. A survey of theories with extended supersymmetry may be found in [23].

## 2.5 The $\mathcal{N}=1$ Superfield Formulation

The construction of field multiplets containing all fields that transform linearly into one another under supersymmetry requires the introduction of anti-commuting spin 1/2 coordinates. For  $\mathcal{N} = 1$  supersymmetry, we introduce a (constant) left Weyl spinor coordinate  $\theta_\alpha$  and its complex conjugate  $\bar{\theta}^{\dot{\alpha}} = (\theta_\alpha)^\dagger$ , satisfying  $[x^\mu, \theta_\alpha] = \{\theta_\alpha, \theta_\beta\} = \{\theta_\alpha, \bar{\theta}^{\dot{\beta}}\} = \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = 0$ . Superderivatives are defined by

$$D_\alpha \equiv \frac{\partial}{\partial\theta^\alpha} + i\sigma_{\alpha\dot{\alpha}}^\mu\bar{\theta}^{\dot{\alpha}}\partial_\mu \qquad \bar{D}_{\dot{\alpha}} \equiv -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu\tag{2.13}$$

where differentiation and integration of  $\theta$  coordinates are defined by

$$\frac{\partial}{\partial\theta^\alpha}(1, \theta^\beta, \bar{\theta}^{\dot{\beta}}) \equiv \int d\theta^\alpha(1, \theta^\beta, \bar{\theta}^{\dot{\beta}}) \equiv (0, \delta_\alpha^\beta, 0)\tag{2.14}$$

For general notations and conventions for spinors and their contractions, see [11].

A *general superfield* is defined as a general function of the superspace coordinates  $x^\mu, \theta_\alpha, \bar{\theta}^{\dot{\alpha}}$ . Since the square of each  $\theta^\alpha$  or of each  $\bar{\theta}^{\dot{\alpha}}$  vanishes, superfields admit finite Taylor expansions in powers of  $\theta$  and  $\bar{\theta}$ . A general superfield  $S(x, \theta, \bar{\theta})$  yields the following

component expansion

$$\begin{aligned}
S(x, \theta, \bar{\theta}) &= \phi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \bar{\theta}\bar{\sigma}^\mu\theta A_\mu(x) + \theta\theta f(x) + \bar{\theta}\bar{\theta}g^*(x) \\
&\quad + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\rho(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x)
\end{aligned} \tag{2.15}$$

A bosonic superfield obeys  $[S, \theta^\alpha] = [S, \bar{\theta}_{\dot{\alpha}}] = 0$ , while a fermionic superfield obeys  $\{S, \theta^\alpha\} = \{S, \bar{\theta}_{\dot{\alpha}}\} = 0$ . If  $S$  is bosonic (resp. fermionic), the component fields  $\phi, A_\mu, f, g$  and  $D$  are bosonic (resp. fermionic) as well, while the fields  $\psi, \chi, \lambda$  and  $\rho$  are fermionic (resp. bosonic). The superfields belong to a  $\mathbf{Z}_2$  graded algebra of functions on superspace, with the even grading for bosonic odd grading for fermionic fields. We shall denote the grading by  $g(S)$ , or sometimes just  $S$ . Superderivatives on superfields satisfy the following graded differentiation rule

$$D_\alpha(S_1 S_2) = (D_\alpha S_1)S_2 + (-)^{g(S_1)g(S_2)} S_1(D_\alpha S_2) \tag{2.16}$$

where  $g(S_i)$  is the grading of the field  $S_i$ .

On superfields, supersymmetry transformations are realized in a linear way via super-differential operators. The infinitesimal supersymmetry parameter is still a constant left Weyl spinor  $\xi$ , as in (2.12) and we have

$$\delta_\xi S = (\xi Q + \bar{\xi}\bar{Q})S \tag{2.17}$$

with the supercharges defined by

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu \quad \bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \tag{2.18}$$

The super-differential operators  $D_\alpha$  and  $Q_\alpha$  differ only by a sign change. They generate left and right actions of supersymmetry respectively. Their relevant structure relations are

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma_{\alpha\dot{\beta}}^\mu P_\mu \quad \{D_\alpha, \bar{D}_{\dot{\beta}}\} = -2\sigma_{\alpha\dot{\beta}}^\mu P_\mu \tag{2.19}$$

where  $P_\mu = i\partial_\mu$ . Since left and right actions mutually commute, all components of  $D$  anti-commute with all components of  $Q$ :  $\{Q_\alpha, D_\beta\} = \{Q_\alpha, \bar{D}^{\dot{\beta}}\} = 0$ . Furthermore, the product of any three  $D$ 's or any three  $Q$ 's vanishes,  $D_\alpha D_\beta D_\gamma = Q_\alpha Q_\beta Q_\gamma = 0$ . The general superfield is reducible; the irreducible components are as follows.

(a) The *Chiral Superfield*  $\Phi$  is obtained by imposing the condition

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{2.20}$$

This condition is invariant under the supersymmetry transformations of (2.17) since  $D$  and  $Q$  anti-commute. Equation (2.20) may be solved in terms of the composite coordinates  $x_\pm^\mu = x^\mu \pm i\theta\sigma^\mu\bar{\theta}$  which satisfy  $\bar{D}_{\dot{\alpha}}x_+^\mu = 0$  and  $D_\alpha x_-^\mu = 0$ . We have

$$\Phi(x, \theta, \bar{\theta}) = \phi(x_+) + \sqrt{2}\theta\psi(x_+) + \theta\theta F(x_+) \tag{2.21}$$

The component fields  $\phi$  and  $\psi$  are the scalar and left Weyl spinor fields of the chiral multiplet respectively, as discussed previously. The field equation for  $F$  is a non-dynamical or *auxiliary field* of the chiral multiplet.

(b) The *Vector Superfield* is obtained by imposing on a general superfield of the type (2.15) the condition

$$V = V^\dagger \quad (2.22)$$

thereby setting  $\chi = \psi$ ,  $g = f$  and  $\rho = \lambda$  and requiring  $\phi$ ,  $A_\mu$  and  $D$  to be real.

(c) The *Gauge Superfield* is a special case of a vector superfield, where  $V$  takes values in the gauge algebra  $\mathcal{G}$ . The reality condition  $V = V^\dagger$  is preserved by the gauge transformation

$$e^V \longrightarrow e^{V'} = e^{-i\Lambda^\dagger} e^V e^{i\Lambda}. \quad (2.23)$$

where  $\Lambda$  is a chiral superfield taking values also in  $\mathcal{G}$ . Under the above gauge transformation law, the component fields  $\phi$ ,  $\psi = \chi$ , and  $f = g$  of a general real superfield may be gauged away in an algebraic way. The gauge in which this is achieved is the *Wess-Zumino gauge*, where the gauge superfield is given by

$$V(x, \theta, \bar{\theta}) = \bar{\theta}\bar{\sigma}^\mu\theta A_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \quad (2.24)$$

The component fields  $A_\mu$  and  $\lambda$  are the gauge and gaugino fields of the gauge multiplet respectively, as discussed previously. The field  $D$  has not appeared previously and is an *auxiliary field*, just as  $F$  was an auxiliary field for the chiral multiplet.

## 2.6 General $\mathcal{N}=1$ Susy Lagrangians via Superfields

Working out the supersymmetry transformation (2.17) on chiral and vector superfields in terms of components, we see that the only contribution to the auxiliary fields is from the  $\theta\partial$  term of  $Q$  and thus takes the form of a total derivative. However, because the form (2.24) was restricted to Wess-Zumino gauge,  $F$  and  $D$  transform by a total derivative only if  $F$  and  $D$  are themselves gauge singlets, in which case we have

$$\begin{aligned} \delta_\xi F &= i\sqrt{2}\partial_\mu(\bar{\xi}\bar{\sigma}^\mu\psi) \\ \delta_\xi D &= \partial_\mu(i\bar{\xi}\bar{\sigma}^\mu\lambda - i\lambda\bar{\sigma}^\mu\xi) \end{aligned} \quad (2.25)$$

These transformation properties guarantee that the  $F$  and  $D$  auxiliary fields yield supersymmetric invariant Lagrangian terms,

$$\begin{aligned} \text{F - terms} & \quad \mathcal{L}_F = F = \int d^2\theta \Phi \\ \text{D - terms} & \quad \mathcal{L}_D = \frac{1}{2}D = \int d^4\theta V \end{aligned} \quad (2.26)$$

The  $F$  and  $D$  terms used to construct invariants need not be elementary fields, and may be gauge invariant composites of elementary fields instead. Allowing for this possibility, we may now derive the most general possible  $\mathcal{N} = 1$  invariant Lagrangian in terms of superfields. To do so, we need the following ingredients  $\mathcal{L}_U$ ,  $\mathcal{L}_G$  and  $\mathcal{L}_K$ . Putting together contributions from these terms, we have the most general  $\mathcal{N} = 1$  supersymmetric Lagrangian with the restrictions of above.

(1) Any complex analytic function  $U$  depending only on left chiral superfields  $\Phi^i$  (but not on their complex conjugates) is itself a left chiral superfield, since  $\bar{D}_\alpha \Phi^i = 0$  implies that  $\bar{D}_\alpha U(\Phi^i) = 0$ . Thus, for any complex analytic function  $U$ , called the *superpotential*, we may construct an invariant contribution to the Lagrangian by forming an  $F$ -term

$$\mathcal{L}_U = \int d^2\theta U(\Phi^i) + \int d^2\theta \overline{U(\Phi^i)} = \sum_i F^i \frac{\partial U}{\partial \phi^i} - \frac{1}{2} \sum_{i,j} \psi^i \psi^j \frac{\partial^2 U}{\partial \phi^i \partial \phi^j} + \text{cc} \quad (2.27)$$

(2) The gauge field strength is a fermionic left chiral spinor superfield  $W_\alpha$ , which is constructed out of the gauge superfield  $V$  by

$$4W_\alpha = -\bar{D}\bar{D}(e^{-V}D_\alpha e^{+V}) \quad (2.28)$$

The gauge field strength may be used as a chiral superfield along with elementary (scalar) chiral superfields to build up  $\mathcal{N} = 1$  supersymmetric Lagrangians via  $F$ -terms. In view of our restriction to Lagrangians with no more than two derivatives on Bose fields,  $W$  can enter at most quadratically. Thus, the most general gauge kinetic and self-interaction term is from the  $F$ -term of the gauge field strength  $W_\alpha$  and the elementary (scalar) chiral superfields  $\Phi^i$  as follows,

$$\mathcal{L}_G = \int d^2\theta \tau_{cc'}(\Phi^i) W^c W^{c'} + \text{complex conjugate} \quad (2.29)$$

Here,  $c$  and  $c'$  stand for the gauge index running over the adjoint representation of  $\mathcal{G}$ , and are contracted in a gauge invariant way. The functions  $\tau_{cc'}(\Phi^i)$  are complex analytic.

(3) The left and right chiral superfields  $\Phi^i$  and  $(\Phi^i)^\dagger$ , as well as the gauge superfield  $V$ , may be combined into a gauge invariant vector superfield  $K(e^V \Phi^i, (\Phi^i)^\dagger)$ , provided the gauge algebra is realized linearly on the fields  $\Phi^i$ . The function  $K$  is called the *Kähler potential*. Assuming that the gauge transformations  $\Lambda$  act on  $V$  by (2.23), the chiral superfields  $\Phi$  transform as  $\Phi \longrightarrow \Phi' = e^{-i\Lambda} \Phi$ , so that  $e^V \Phi$  transforms as  $\Phi$ . An invariant Lagrangian may be constructed via a  $D$ -term,

$$\mathcal{L}_K = \int d^4\theta K(e^V \Phi^i, (\Phi^i)^\dagger) \quad (2.30)$$

Upon expanding  $\mathcal{L}_K$  in components, one sees immediately that the leading terms already generates an action with two derivatives on boson fields. As a result,  $K$  must be a function only of the superfields  $\Phi^i$  and  $(\Phi^i)^\dagger$  and  $V$ , but not of their derivatives.

## 2.7 $\mathcal{N} = 1$ non-renormalization theorems

Non-renormalization theorems provide very strong results on the structure of the effective action at low energy as a function of the bare action. Until recently, their validity was restricted to perturbation theory and the proof of the theorems was based on supergraph methods [17]. Now, however, non-renormalization theorems have been extended to the non-perturbative regime, including the effects of instantons [18]. The assumptions

underlying the theorems are that (1) a supersymmetric renormalization is carried out, (2) the effective action is constructed by Wilsonian renormalization (see [19] for a review). The last assumption allows one to circumvent any possible singularities resulting from the integration over massless states.

The non-renormalization theorems state that the superpotential  $\mathcal{L}_U$  is *unrenormalized*, or more precisely that it receives no quantum corrections, infinite or finite. Furthermore, the gauge field term  $\mathcal{L}_G$  is renormalized only through the gauge coupling  $\tau_{cc'}$ , such that its complex analytic dependence on the chiral superfields is preserved. In perturbation theory,  $\tau_{cc'}$  receives quantum contributions only through 1-loop graphs, essentially because the  $U(1)_R$  axial anomaly is a 1-loop effect in view of the Adler-Bardeen theorem. Non-perturbatively, instanton corrections also enter, but in a very restricted way. The special renormalization properties of these two  $F$ -terms are closely related to their holomorphicity [18]. The Kähler potential term  $\mathcal{L}_K$  on the other hand does receive renormalizations both at the perturbative and non-perturbative levels.

### 3 $\mathcal{N} = 4$ Super Yang-Mills

The Lagrangian for the  $\mathcal{N} = 4$  super-Yang Mills theory is unique and given by [20]

$$\begin{aligned} \mathcal{L} = \text{tr} \left\{ & -\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta_I}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - \sum_a i \bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a - \sum_i D_\mu X^i D^\mu X^i \right. \\ & \left. + \sum_{a,b,i} g C_i^{ab} \lambda_a [X^i, \lambda_b] + \sum_{a,b,i} g \bar{C}_{iab} \bar{\lambda}^a [X^i, \bar{\lambda}^b] + \frac{g^2}{2} \sum_{i,j} [X^i, X^j]^2 \right\} \end{aligned} \quad (3.1)$$

The constants  $C_i^{ab}$  and  $C_{iab}$  are related to the Clifford Dirac matrices for  $SO(6)_R \sim SU(4)_R$ . This is evident when considering  $\mathcal{N} = 4$  SYM in  $D = 4$  as the dimensional reduction on  $T^6$  of  $D = 10$  super-Yang-Mills theory (see problem set 4.1). By construction, the Lagrangian is invariant under  $\mathcal{N} = 4$  Poincaré supersymmetry, whose transformation laws are given by

$$\begin{aligned} \delta X^i &= [Q_\alpha^a, X^i] = C^{iab} \lambda_{\alpha b} \\ \delta \lambda_b &= \{Q_\alpha^a, \lambda_{\beta b}\} = F_{\mu\nu}^+ (\sigma^{\mu\nu})^\alpha{}_\beta \delta_b^a + [X^i, X^j] \epsilon_{\alpha\beta} (C_{ij})^a{}_b \\ \delta \bar{\lambda}_\beta^b &= \{Q_\alpha^a, \bar{\lambda}_\beta^b\} = C_i^{ab} \bar{\sigma}_{\alpha\beta}^\mu D_\mu X^i \\ \delta A_\mu &= [Q_\alpha^a, A_\mu] = (\sigma_\mu)_\alpha{}^\beta \bar{\lambda}_\beta^a \end{aligned} \quad (3.2)$$

The constants  $(C_{ij})^a{}_b$  are related to bilinears in Clifford Dirac matrices of  $SO(6)_R$ .

Classically,  $\mathcal{L}$  is *scale invariant*. This may be seen by assigning the standard mass-dimensions to the fields and couplings

$$[A_\mu] = [X^i] = 1 \quad [\lambda_a] = \frac{3}{2} \quad [g] = [\theta_I] = 0 \quad (3.3)$$

All terms in the Lagrangian are of dimension 4, from which scale invariance follows. Actually, in a relativistic field theory, scale invariance and Poincaré invariance combine into a larger *conformal symmetry*, forming the group  $SO(2,4) \sim SU(2,2)$ . Furthermore, the combination of  $\mathcal{N} = 4$  Poincaré supersymmetry and conformal invariance produces an even larger *superconformal symmetry* given by the supergroup  $SU(2,2|4)$ .

Remarkably, upon perturbative quantization,  $\mathcal{N} = 4$  SYM theory exhibits no ultraviolet divergences in the correlation functions of its canonical fields. Instanton corrections also lead to finite contributions and is believed that the theory is UV finite. As a result, the renormalization group  $\beta$ -function of the theory vanishes identically (since no dependence on any renormalization scale is introduced during the renormalization process). The theory is exactly scale invariant at the quantum level, and the superconformal group  $SU(2,2|4)$  is a fully quantum mechanical symmetry.

The *Montonen-Olive or S-duality conjecture* in addition posits a discrete global symmetry of the theory [21]. To state this invariance, it is standard to combine the real coupling



$g$  and the real instanton angle  $\theta_I$  into a single complex coupling

$$\tau \equiv \frac{\theta_I}{2\pi} + \frac{4\pi i}{g^2} \quad (3.4)$$

The quantum theory is invariant under  $\theta_I \rightarrow \theta_I + 2\pi$ , or  $\tau \rightarrow \tau + 1$ . The *Montonen-Olive conjecture* states that the quantum theory is also invariant under the  $\tau \rightarrow -1/\tau$ . The combination of both symmetries yields the S-duality group  $\text{SL}(2, \mathbf{Z})$ , generated by

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \quad ad - bc = 1, \quad a, b, c, d \in \mathbf{Z} \quad (3.5)$$

Note that when  $\theta_I = 0$ , the S-duality transformation amounts to  $g \rightarrow 1/g$ , thereby exchanging strong and weak coupling.

### 3.1 Dynamical Phases

To analyze the dynamical behavior of  $\mathcal{N} = 4$  theory, we look at the potential energy term,

$$-g^2 \sum_{i,j} \int \text{tr}[X^i, X^j]^2$$

In view of the positive definite behavior of the Cartan - Killing form on the compact gauge algebra  $\mathcal{G}$ , each term in the sum is positive or zero. When the full potential is zero, a minimum is thus automatically attained corresponding to a  $\mathcal{N} = 4$  supersymmetric ground state. In turn, any  $\mathcal{N} = 4$  supersymmetric ground state is of this form,

$$[X^i, X^j] = 0, \quad i, j = 1, \dots, 6 \quad (3.6)$$

There are two classes of solutions to this equation,

- The *superconformal phase*, for which  $\langle X^i \rangle = 0$  for all  $i = 1, \dots, 6$ . The gauge algebra  $\mathcal{G}$  is unbroken. The superconformal symmetry  $SU(2, 2|4)$  is also unbroken. The physical states and operators are gauge invariant (i.e.  $\mathcal{G}$ -singlets) and transform under unitary representations of  $SU(2, 2|4)$ .
- The *spontaneously broken or Coulomb phase*, where  $\langle X^i \rangle \neq 0$  for at least one  $i$ . The detailed dynamics will depend upon the degree of residual symmetry. Generically,  $\mathcal{G} \rightarrow U(1)^r$  where  $r = \text{rank } \mathcal{G}$ , in which case the low energy behavior is that of  $r$  copies of  $\mathcal{N} = 4$   $U(1)$  theory. Superconformal symmetry is spontaneously broken since the non-zero vacuum expectation value  $\langle X^i \rangle$  sets a scale.

### 3.2 Isometries and Conformal Transformations

In the first part of these lectures, we shall consider the SYM theory in the conformal phase and therefore make heavy use of superconformal symmetry. In the present subsection,

we begin by reviewing conformal symmetry first. Let  $M$  be a Riemannian (or pseudo-Riemannian) manifold of dimension  $D$  with metric  $G_{\mu\nu}$ ,  $\mu, \nu = 0, 1, \dots, D-1$ . We shall now review the notions of diffeomorphisms, isometries and conformal transformations.

- A *diffeomorphism* of  $M$  is a differentiable map of local coordinates  $x^\mu$ ,  $\mu = 1, \dots, D$ , of  $M$  given either globally by  $x^\mu \rightarrow x'^\mu(x)$  or infinitesimally by a vector field  $v^\mu(x)$  so that  $\delta_v x^\mu = -v^\mu(x)$ . Under a general diffeomorphism, the metric on  $M$  transforms as

$$\begin{aligned} G'_{\mu\nu}(x') dx'^\mu dx'^\nu &= G_{\mu\nu} dx^\mu dx^\nu & (3.7) \\ \delta_v G_{\mu\nu} &= \nabla_\mu v_\nu + \nabla_\nu v_\mu & \nabla_\mu v_\nu \equiv \partial_\mu v_\nu - \Gamma_{\mu\nu}^\rho v_\rho \end{aligned}$$

- An *isometry* is a diffeomorphism under which the metric is invariant,

$$G'_{\mu\nu}(x) = G_{\mu\nu}(x) \quad \text{or} \quad \delta_v G_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu = 0 \quad (3.8)$$

- A *conformal transformation* is a diffeomorphism that preserves the metric up to an overall (in general  $x$ -dependent) scale factor, thereby preserving all angles,

$$G'_{\mu\nu}(x) = w(x) G_{\mu\nu}(x) \quad \text{or} \quad \delta_v G_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu = \frac{2}{D} G_{\mu\nu} \nabla_\rho v^\rho \quad (3.9)$$

The case of  $M = \mathbf{R}^D$ ,  $D \geq 3$ , flat Minkowski space-time with flat metric  $\eta_{\mu\nu} = \text{diag}(-+\dots+)$  is an illuminating example. (When  $D = 2$ , the conformal algebra is isomorphic to the infinite-dimensional Virasoro algebra.) Since now  $\nabla_\mu = \partial_\mu$ , the equations for isometries reduce to  $\partial_\mu v_\nu + \partial_\nu v_\mu = 0$ , while those for conformal transformations become  $\partial_\mu v_\nu + \partial_\nu v_\mu - 2/D \eta_{\mu\nu} \partial_\rho v^\rho = 0$ . The solutions are

isometries	(1)	$v^\mu$ constant	: translations	
	(2)	$v_\mu = \omega_{\mu\nu} x^\nu$	: Lorentz	
conformal	(3)	$v^\mu = \lambda x^\mu$	: dilations	
	(4)	$v_\mu = 2c_\rho x^\rho x_\mu - x_\rho x^\rho c_\mu$	: special conformal	(3.10)

In a local field theory, continuous symmetries produce conserved currents, according to Noether's Theorem. Currents associated with isometries and conformal transformations may be expressed in terms of the stress tensor  $T_{\mu\nu}$ . This is because the stress tensor for any local field theory encodes the response of the theory to a change in metric; as a result, it is automatically symmetric  $T^{\mu\nu} = T^{\nu\mu}$ . We have

$$j_v^\mu \equiv T^{\mu\nu} v_\nu \quad (3.11)$$

Covariant conservation of this current requires that  $\nabla_\mu j_v^\mu = (\nabla_\mu T^{\mu\nu}) v_\nu + T^{\mu\nu} \nabla_\mu v_\nu = 0$ . For an *isometry*, conservation thus requires that  $\nabla_\mu T^{\mu\nu} = 0$ . For a *conformal transformation*, conservation in addition requires that  $T_\mu{}^\mu = 0$ . Starting out with Poincaré and scale invariance, all of the above conditions will have to be met so that special conformal invariance will be automatic.

### 3.3 (Super) Conformal $\mathcal{N}=4$ Super Yang-Mills

In this subsection, we show that the global continuous symmetry group of  $\mathcal{N} = 4$  SYM is given by the supergroup  $SU(2, 2|4)$ , see [22]. The ingredients are as follows.

- *Conformal Symmetry*, forming the group  $SO(2, 4) \sim SU(2, 2)$  is generated by translations  $P^\mu$ , Lorentz transformations  $L_{\mu\nu}$ , dilations  $D$  and special conformal transformations  $K^\mu$ ;
- *R-symmetry*, forming the group  $SO(6)_R \sim SU(4)_R$ , generated by  $T^A$ ,  $A = 1, \dots, 15$ ;
- *Poincaré supersymmetries* generated by the supercharges  $Q_\alpha^a$  and their complex conjugates  $\bar{Q}_{\dot{\alpha}a}$ ,  $a = 1, \dots, 4$ . The presence of these charges results immediately from  $\mathcal{N} = 4$  Poincaré supersymmetry;
- *Conformal supersymmetries* generated by the supercharges  $S_{\alpha a}$  and their complex conjugates  $\bar{S}_{\dot{\alpha}}^a$ . The presence of these symmetries results from the fact that the Poincaré supersymmetries and the special conformal transformations  $K_\mu$  do not commute. Since both are symmetries, their commutator must also be a symmetry, and these are the  $S$  generators.

The two bosonic subalgebras  $SO(2, 4)$  and  $SU(4)_R$  commute. The supercharges  $Q_\alpha^a$  and  $\bar{S}_{\dot{\alpha}}^a$  transform under the  $\mathbf{4}$  of  $SU(4)_R$ , while  $\bar{Q}_{\dot{\alpha}a}$  and  $S_{\alpha a}$  transform under the  $\mathbf{4}^*$ . From these data, it is not hard to see how the various generators fit into a super algebra,

$$\begin{pmatrix} P_\mu & K_\mu & L_{\mu\nu} & D & Q_\alpha^a & \bar{S}_{\dot{\alpha}}^a \\ & \bar{Q}_{\dot{\alpha}a} & S_{\alpha a} & & T^A & \end{pmatrix} \quad (3.12)$$

All structure relations are rather straightforward, except the relations between the supercharges, which we now spell out. To organize the structure relations, it is helpful to make use of a natural grading of the algebra given by the dimension of the generators,

$$\begin{aligned} [D] = [L_{\mu\nu}] = [T^A] = 0 & & [P^\mu] = +1 & & [K_\mu] = -1 \\ [Q] = +1/2 & & [S] = -1/2 & & \end{aligned} \quad (3.13)$$

Thus, we have

$$\begin{aligned} \{Q_\alpha^a, Q_\beta^b\} &= \{S_{\alpha a}, S_{\beta b}\} = \{Q_\alpha^a, \bar{S}_{\dot{\beta}}^b\} = 0 \\ \{Q_\alpha^a, \bar{Q}_{\dot{\beta}b}\} &= 2\sigma_{\alpha\dot{\beta}}^\mu P_\mu \delta_b^a \\ \{S_{\alpha a}, \bar{S}_{\dot{\beta}}^b\} &= 2\sigma_{\alpha\dot{\beta}}^\mu K_\mu \delta_a^b \\ \{Q_\alpha^a, S_{\beta b}\} &= \epsilon_{\alpha\beta}(\delta_b^a D + T^a{}_b) + \frac{1}{2}\delta_b^a L_{\mu\nu} \sigma_{\alpha\beta}^{\mu\nu} \end{aligned} \quad (3.14)$$

### 3.4 Superconformal Multiplets of Local Operators

We shall be interested in constructing and classifying all local, gauge invariant operators in the theory that are polynomial in the canonical fields. The restriction to polynomial operators stems from the fact that it is those operators that will have definite dimension.

The canonical fields  $X^i$ ,  $\lambda_a$  and  $A_\mu$  have unrenormalized dimensions, given by 1, 3/2 and 1 respectively. Gauge invariant operators will be constructed rather from the gauge covariant objects  $X^i$ ,  $\lambda_a$ ,  $F_{\mu\nu}^\pm$  and the covariant derivative  $D_\mu$ , whose dimensions are

$$[X^i] = [D_\mu] = 1 \quad [F_{\mu\nu}^\pm] = 2 \quad [\lambda_a] = \frac{3}{2} \quad (3.15)$$

Here,  $F_{\mu\nu}^\pm$  stands for the (anti) self-dual gauge field strength. Thus, if we temporarily ignore the renormalization effects of composite operators, we see that all operator dimensions will be positive and that the number of operators whose dimension is less than a given number is finite. The only operator with dimension 0 will be the unit operator.

Next, we introduce the notion of *superconformal primary operator*. Since the conformal supercharges  $S$  have dimension  $-1/2$ , successive application of  $S$  to any operator of definite dimension must at some point yield 0 since otherwise we would start generating operators of negative dimension, which is impossible in a unitary representation. Therefore one defines a *superconformal primary operator*  $\mathcal{O}$  to be a non-vanishing operator such that

$$[S, \mathcal{O}]_\pm = 0 \quad \mathcal{O} \neq 0 \quad (3.16)$$

An equivalent way of defining a superconformal primary operator is as the lowest dimension operator in a given superconformal multiplet or representation. It is important to distinguish a superconformal primary operator from a *conformal primary operator*, which is instead annihilated by the special conformal generators  $K^\mu$ , and is thus defined by a weaker condition. Therefore, every superconformal primary is also a conformal primary operator, but the converse is not, in general, true.

Finally, an operator  $\mathcal{O}$  is called a *superconformal descendant operator* of an operator  $\mathcal{O}'$  when it is of the form,

$$\mathcal{O} = [Q, \mathcal{O}' ]_\pm \quad (3.17)$$

for some well-defined local polynomial gauge invariant operator  $\mathcal{O}'$ . If  $\mathcal{O}$  is a descendant of  $\mathcal{O}'$ , then these two operators belong to the same superconformal multiplet. Since the dimensions are related by  $\Delta_{\mathcal{O}} = \Delta_{\mathcal{O}'} + 1/2$ , the operator  $\mathcal{O}$  can never be a conformal primary operator, because there is in the same multiplet at least one operator  $\mathcal{O}'$  of dimension lower than  $\mathcal{O}$ . As a result, in a given irreducible superconformal multiplet, there is a single superconformal primary operator (the one of lowest dimension) and all others are superconformal descendants of this primary.

It is instructive to have explicit forms for the superconformal primary operators in  $\mathcal{N} = 4$  SYM. The construction is most easily carried out by using the fact that a superconformal primary operator is NOT the  $Q$ -commutator of another operator. Thus, a key ingredient

in the construction is the  $Q$  transforms of the canonical fields. We shall need these here only schematically,

$$\begin{aligned} \{Q, \lambda\} &= F^+ + [X, X] & [Q, X] &= \lambda \\ \{Q, \bar{\lambda}\} &= DX & [Q, F] &= D\lambda \end{aligned} \quad (3.18)$$

A local polynomial operator containing any of the elements on the rhs of the above structure relations cannot be primary. As a result, chiral primary operators can involve neither the gauginos  $\lambda$  nor the gauge field strengths  $F^\pm$ . Being thus only functions of the scalars  $X$ , they can involve neither derivatives nor commutators of  $X$ . As a result, superconformal primary operators are gauge invariant scalars involving only  $X$  in a symmetrized way.

The simplest are the *single trace operators*, which are of the form

$$\text{str}\left(X^{i_1} X^{i_2} \dots X^{i_n}\right) \quad (3.19)$$

where  $i_j, j = 1, \dots, n$  stand for the  $SO(6)_R$  fundamental representation indices. Here, “str” denotes the symmetrized trace over the gauge algebra and as a result of this operation, the above operator is totally symmetric in the  $SO(6)_R$ -indices  $i_j$ . In general, the above operators transform under a reducible representation (namely the symmetrized product of  $n$  fundamentals) and irreducible operators may be obtained by isolating the traces over  $SO(6)_R$  indices. Since  $\text{tr}X^i = 0$ , the simplest operators are

$$\begin{aligned} \sum_i \text{tr}X^i X^i &\sim \text{Konishi multiplet} \\ \text{tr}X^{\{i} X^{j\}} &\sim \text{supergravity multiplet} \end{aligned} \quad (3.20)$$

where  $\{ij\}$  stands for the traceless part only. The reasons for these nomenclatures will become clear once we deal with the AdS/CFT correspondence.

More complicated are the *multiple trace operators*, which are obtained as products of single trace operators. Upon taking the tensor product of the individual totally symmetric representations, we may now also encounter (partially) anti-symmetrized representations of  $SO(6)_R$ . There is a one-to-one correspondence between chiral primary operators and unitary superconformal multiplets, and so all state and operator multiplets may be labeled in terms of the superconformal chiral primary operators.

### 3.5 $\mathcal{N} = 4$ Chiral or BPS Multiplets of Operators

The unitary representations of the superconformal algebra  $SU(2, 2|4)$  may be labeled by the quantum numbers of the bosonic subgroup, listed below,

$$\begin{aligned} SO(1, 3) \times SO(1, 1) \times SU(4)_R \\ (s_+, s_-) \quad \Delta \quad [r_1, r_2, r_3] \end{aligned} \quad (3.21)$$

where  $s_\pm$  are positive or zero half integers,  $\Delta$  is the positive or zero dimension and  $[r_1, r_2, r_3]$  are the Dynkin labels of the representations of  $SU(4)_R$ . It is sometimes preferable to refer

to  $SU(4)_R$  representations by their dimensions, given in terms of  $\bar{r}_i \equiv r_i + 1$  by

$$\dim[r_1, r_2, r_3] = \frac{1}{12} \bar{r}_1 \bar{r}_2 \bar{r}_3 (\bar{r}_1 + \bar{r}_2) (\bar{r}_2 + \bar{r}_3) (\bar{r}_1 + \bar{r}_2 + \bar{r}_3) \quad (3.22)$$

instead of their Dynkin labels. The complex conjugation relation is  $[r_1, r_2, r_3]^* = [r_3, r_2, r_1]$ .

In unitary representations, the dimensions  $\Delta$  of the operators are bounded from below by the spin and  $SU(4)_R$  quantum numbers. To see this, it suffices to restrict to primary operators since they have the lowest dimension in a given irreducible multiplet. As shown previously, such operators are scalars, so that the spin quantum numbers vanish, and the dimension is bounded from below by the  $SU(4)_R$  quantum numbers. A systematic analysis of [24], (see also [25, 26]) for this case reveals the existence of 4 distinct series,

1.  $\Delta = r_1 + r_2 + r_3$ ;
2.  $\Delta = \frac{3}{2}r_1 + r_2 + \frac{1}{2}r_3 \geq 2 + \frac{1}{2}r_1 + r_2 + \frac{3}{2}r_3$     this requires  $r_1 \geq r_3 + 2$ ;
3.  $\Delta = \frac{1}{2}r_1 + r_2 + \frac{3}{2}r_3 \geq 2 + \frac{3}{2}r_1 + r_2 + \frac{1}{2}r_3$     this requires  $r_3 \geq r_1 + 2$ ;
4.  $\Delta \geq \text{Max} \left[ 2 + \frac{3}{2}r_1 + r_2 + \frac{1}{2}r_3; 2 + \frac{1}{2}r_1 + r_2 + \frac{3}{2}r_3 \right]$

Clearly, cases 2. and 3. are complex conjugates of one another.

Cases 1. 2. and 3. correspond to *discrete series of representations*, for which at least one supercharge  $Q$  commutes with the primary operator. Such representations are shortened and usually referred to as *chiral multiplets* or *BPS multiplets*. The term BPS multiplet arises from the analogy with the BPS multiplets of Poincaré supersymmetry discussed in subsections §2.3. Since these representations are shortened, their dimension is *unrenormalized* or protected from receiving quantum corrections.

Case 4. corresponds to *continuous series of representations*, for which no supercharges  $Q$  commute with the primary operator. Such representations are referred to as *non-BPS*. Notice that the dimensions of the operators in the continuous series is separated from the dimensions in the discrete series by a gap of at least 2 units of dimension.

The BPS multiplets play a special role in the AdS/CFT correspondence. In Table 3 below, we give a summary of properties of various BPS and non-BPS multiplets. In the column labeled by  $\#Q$  is listed the number of Poincaré supercharges that leave the primary invariant.

### Half-BPS operators

It is possible to give an explicit description of all 1/2 BPS operators. The simplest series is given by single-trace operators of the form

$$\mathcal{O}_k(x) \equiv \frac{1}{n_k} \text{str} \left( X^{\{i_1}(x) \cdots X^{i_k}(x) \right) \quad (3.23)$$

where “str” stands for the symmetrized trace introduced previously,  $\{i_1 \cdots i_k\}$  stands for the  $SO(6)_R$  traceless part of the tensor, and  $n_k$  stands for an overall normalization of the

Operator type	# $Q$	spin range	$SU(4)_R$ primary	dimension $\Delta$
identity	16	0	$[0, 0, 0]$	0
1/2 BPS	8	2	$[0, k, 0], k \geq 2$	$k$
1/4 BPS	4	3	$[\ell, k, \ell], \ell \geq 1$	$k + 2\ell$
1/8 BPS	2	7/2	$[\ell, k, \ell + 2m]$	$k + 2\ell + 3m, m \geq 1$
non-BPS	0	4	any	unprotected

Table 3: Characteristics of BPS and Non-BPS multiplets

operator which will be fixed by normalizing its 2-point function. The dimension of these operators is unrenormalized, and thus equal to  $k$ .

However, it is also possible to have multiple trace 1/2 BPS operators. They are built as follows. The tensor product of  $n$  representations  $[0, k_1, 0] \otimes \cdots \otimes [0, k_n, 0]$ , always contains the representation  $[0, k, 0]$ ,  $k = k_1 + \cdots + k_n$ , with multiplicity 1. (The highest weight of the representation  $[0, k, 0]$  is then the sum of the highest weights of the component representations.) The most general 1/2 BPS gauge invariant operators are given by the projection onto the representation  $[0, k, 0]$  of the corresponding product of operators,

$$\mathcal{O}_{(k_1, \dots, k_n)}(x) \equiv \left[ \mathcal{O}_{k_1}(x) \cdots \mathcal{O}_{k_n}(x) \right]_{[0, k, 0]} \quad k = k_1 + \cdots + k_n \quad (3.24)$$

Here the brackets  $[ ]$  stand for the operators product of the operators inside. This product is in general singular and thus ambiguous, but the projection onto the representation  $[0, k, 0]$  is singularity free and thus unique.

### 1/4 and 1/8 BPS Operators

There are no single-trace 1/4 BPS operators. The simplest construction is in terms of double trace operators. It is easiest to list all possibilities in a single expression, using the notations familiar already from the 1/2 BPS case. The operators are of the form

$$\left[ \mathcal{O}_{k_1}(x) \cdots \mathcal{O}_{k_n}(x) \right]_{[\ell, k, \ell]} \quad k + 2\ell = k_1 + \cdots + k_n \quad (3.25)$$

In the free theory, the above operators will be genuinely 1/4 BPS, but in the interacting theory, the operators will also contain an admixture of descendants of non-BPS operators [27]. The series of 1/8 BPS operators starts with triple trace operators, and are generally of the form

$$\left[ \mathcal{O}_{k_1}(x) \cdots \mathcal{O}_{k_n}(x) \right]_{[\ell, k, \ell + 2m]} \quad k + 2\ell + 3m = k_1 + \cdots + k_n \quad (3.26)$$

In the interacting theory, admixtures with descendants again have to be included.

### 3.6 Problem Sets

- (3.1) Show that the 1-loop renormalization group  $\beta$ -function for  $\mathcal{N} = 4$  SYM vanishes.
- (3.2) Express the  $\mathcal{N} = 4$  SYM Lagrangian in terms of  $\mathcal{N} = 1$  superfields.
- (3.3) Work out the full conformal  $SO(2, 4) \sim SU(2, 2)$  and superconformal  $SU(2, 2|4)$  structure relations (commutators and anti-commutators of the generators).
- (3.4) Derive the Noether currents associated with the Poincaré  $Q_\alpha^a$  and conformal  $\bar{S}_{\alpha a}$  supercharges (and complex conjugates) in terms of the canonical fields of  $\mathcal{N} = 4$  SYM.
- (3.5) In the Abelian Coulomb phase of  $\mathcal{N} = 4$  SYM, where the gauge algebra  $\mathcal{G}$  is spontaneously broken to  $U(1)^r$ ,  $r = \text{rank } \mathcal{G}$ , the global superconformal algebra  $SU(2, 2|4)$  is also spontaneously broken. To simplify matters, you may take  $\mathcal{G} = SU(2)$ . (a) Identify the generators of  $SU(2, 2|4)$  which are preserved and (b) those which are spontaneously broken, thus producing Goldstone bosons and fermions. (c) Express the Goldstone boson and fermion fields in terms of the canonical fields of  $\mathcal{N} = 4$  SYM.



## 4 Supergravity and Superstring Theory

In this section, we shall review the necessary supergravity and superstring theory to develop the theory of D-branes and D3-branes in particular.

### 4.1 Spinors in general dimensions

Consider  $D$ -dimensional Minkowski space-time  $M_D$  with flat metric  $\eta_{\mu\nu} = \text{diag}(-+\cdots+)$ ,  $\mu, \nu = 0, 1, \dots, D-1$ . The Lorentz group is  $SO(1, D-1)$  and the generators of the Lorentz algebra  $J_{\mu\nu}$  obey the standard structure relations

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i\eta_{\mu\rho}J_{\nu\sigma} + i\eta_{\nu\rho}J_{\mu\sigma} - i\eta_{\nu\sigma}J_{\mu\rho} + i\eta_{\mu\sigma}J_{\nu\rho} \quad (4.1)$$

The *Dirac spinor* representation, denoted  $S_D$ , is defined in terms of the standard Clifford-Dirac matrices  $\Gamma_\mu$ ,

$$J_{\mu\nu} = \frac{i}{4}[\Gamma_\mu, \Gamma_\nu] \quad \{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu} \quad (4.2)$$

Its (complex) dimension is given by  $\dim_{\mathbb{C}} S_D = 2^{\lfloor D/2 \rfloor}$ .

For  $D$  even, the Dirac spinor representation is always reducible because in that case there exists a chirality matrix  $\bar{\Gamma}$ , with square  $\bar{\Gamma}^2 = I$ , which anti-commutes with all  $\Gamma_\mu$  and therefore commutes with  $J_{\mu\nu}$ ,

$$\bar{\Gamma} \equiv i^{\frac{1}{2}D(D-1)+1} \Gamma_0 \Gamma_1 \cdots \Gamma_{d-1} \quad \{\bar{\Gamma}, \Gamma_\mu\} = 0 \quad \Rightarrow \quad [\bar{\Gamma}, J_{\mu\nu}] = 0 \quad (4.3)$$

As a result, the Dirac spinor is the direct sum of two *Weyl spinors*  $S_D = S_+ \oplus S_-$ . The reality properties of the Weyl spinors depends on  $D \pmod{8}$ , and is given as follows,

$$\begin{array}{lll} D \equiv 0, 4 \pmod{8} & S_- = S_+^* & \text{both complex} \\ D \equiv 2, 6 \pmod{8} & S_+ = S_- & \text{self-conjugate} \end{array} \quad (4.4)$$

For both even and odd  $D$ , the *charge conjugate*  $\psi^c$  of a Dirac spinor  $\psi$  is defined by

$$\psi^c \equiv C\Gamma_0\psi^* \quad C\Gamma_\mu C^{-1} = -(\Gamma_\mu)^T \quad (4.5)$$

Requiring that a spinor be real is a basis dependent condition and thus not properly Lorentz covariant. The proper Lorentz invariant condition for reality is that a spinor be its own charge conjugate  $\psi^c = \psi$ ; such a spinor is called a *Majorana spinor*. The Majorana condition requires that  $(\psi^c)^c = \psi$ , or  $C\Gamma_0(C\Gamma_0)^* = I$ , which is possible only in dimensions  $D \equiv 0, 1, 2, 3, 4 \pmod{8}$ . In dimensions  $D \equiv 0, 4 \pmod{8}$ , a Majorana spinor is equivalent to a Weyl spinor, while in dimension  $D \equiv 2 \pmod{8}$  it is possible to impose the Majorana and Weyl conditions at the same time, resulting in *Majorana-Weyl spinors*. In dimensions  $D \equiv 5, 6, 7 \pmod{8}$ , one may group spinors into doublets  $\Psi_\pm$  and it is possible to impose a *symplectic Majorana condition* given by  $\Psi_\pm^c = \mp\Psi_\mp$ . Useful reviews are in [12, 28].

## 4.2 Supersymmetry in general dimensions

The basic Poincaré supersymmetry algebra in  $M_D$  is obtained by supplementing the Poincaré algebra with  $\mathcal{N}$  supercharges  $Q_\alpha^I$ ,  $I = 1, \dots, \mathcal{N}$ . Here  $Q$  transforms in the spinor representation  $S$ , which could be a Dirac spinor, a Weyl spinor, a Majorana spinor or a Majorana-Weyl spinor, depending on  $D$ . Thus,  $\alpha$  runs over the spinor indices  $\alpha = 1, \dots, \dim S$ . Whatever the spinor is, we shall always write it as a Dirac spinor. The fundamental supersymmetry algebra could include central charges just as was the case for  $D = 4$ . However, we shall here be interested mostly in a restricted class of supersymmetry representations in which we have a massless graviton, such as we have in supergravity and in superstring theory. Therefore, we may ignore the central charges.

A general result, valid in dimension  $D \geq 4$ , states that *interacting massless fields of spin  $> 2$  cannot be causal, and are excluded on physical grounds*. Considering theories with a massless graviton, and assuming that supersymmetry is realized linearly, the massless graviton must be part of a massless supermultiplet of states and fields. By the above general result, this multiplet cannot contain fields and states of spin  $> 2$ . This fact puts severe restrictions on which supersymmetry algebras can be realized in various dimensions.

The existence of massless unitary representations of the supersymmetry algebra requires vanishing central charges, just as was the case in  $d = 4$ . Thus, we shall consider the Poincaré supersymmetry algebras of the form (useful reviews are in [12, 28], see also [29] and [30]),

$$\{Q_\alpha^I, (Q_\beta^J)^\dagger\} = 2\delta^I_J (\Gamma_\mu)_\alpha^\beta P^\mu \quad \{Q_\alpha^I, Q_\beta^J\} = 0 \quad (4.6)$$

To analyze massless representations, choose  $P^\mu = (E, 0, \dots, 0, E)$ ,  $E > 0$ , so that the supersymmetry algebra in this representation simplifies and becomes

$$\{Q_\alpha^I, (Q_\beta^J)^\dagger\} = 2\delta^I_J \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}_\alpha^\beta \quad (4.7)$$

On this unitary massless representation, half of the supercharges effectively vanish  $Q_\alpha^I = 0$ ,  $\alpha = \frac{1}{2} \dim S + 1, \dots, \dim S$ . Half of the remaining supercharges may be viewed as lowering operators for the Clifford algebra, while the other half may be viewed as raising operators. Thus, the total number of raising operators is  $1/4 \cdot \mathcal{N} \cdot \dim_{\mathbf{R}} S$ . Each operator raising helicity by  $1/2$ , and total helicity ranging at most from  $-2$  to  $+2$ , we should have at most 8 raising operators and this produces an important bound,

$$\mathcal{N} \dim_{\mathbf{R}} S \leq 32 \quad (4.8)$$

In other words, the maximum number of Poincaré supercharges is always 32.

The largest dimension  $D$  for which the bound may be satisfied is  $D = 11$  and  $\mathcal{N} = 1$ , for which there are precisely 32 Majorana supercharges. In  $D = 10$ , the bound is saturated for  $\mathcal{N} = 2$  and 16-dimensional Majorana-Weyl spinors. There is indeed a unique  $D = 11$  supergravity theory discovered by Cremmer, Julia and Scherk [31]. Many of the lower dimensional theories may be constructed by Kaluza-Klein compactification on a circle or on a torus of the  $D = 11$  theory and we shall therefore treat this method first [32].

### 4.3 Kaluza-Klein compactification on a circle

We wish to compactify one space dimension on a circle  $S_R^1$  of radius  $R$ . Accordingly, we decompose the coordinates  $x^\mu$  of  $\mathbf{R}^D$  into a coordinate  $y$  on the circle and the remaining coordinates  $x^{\bar{\mu}}$ . The wave operator with flat metric in  $D$  dimensions  $\square_D$  then becomes

$$\square_D = \square_{D-1} + \frac{\partial^2}{\partial y^2} \quad (4.9)$$

We shall be interested in finding out how various fields behave, in particular in the limit  $R \rightarrow 0$ , referred to as *dimensional reduction*.

We begin with a scalar field  $\phi(x^\mu)$  obeying periodic boundary conditions on  $S_R^1$ , which has the following Fourier decomposition,

$$\phi(x^{\bar{\mu}}, y) = \sum_{n \in \mathbf{Z}} \phi_n(x^{\bar{\mu}}) e^{2\pi i n y / R} \quad (4.10)$$

The  $d$ -dimensional kinetic term of a scalar field with mass  $m$  then decomposes as follows,

$$\int d^d x \phi (-\square_d + m^2) \phi = \sum_{n \in \mathbf{Z}} 2\pi R \int d^{d-1} x \phi_n \left( -\square_{d-1} + m^2 + \frac{4\pi^2 n^2}{R^2} \right) \phi_n \quad (4.11)$$

As  $R \rightarrow 0$ , all modes except  $n = 0$  acquire an infinitely heavy mass and decouple. The zero mode  $n = 0$  is the unique mode invariant under translations on  $S_R^1$ . Thus, *the dimensional reduction on a circle of a scalar field with periodic boundary conditions is again a scalar field*. Under dimensional reduction with any other boundary condition, there will be no zero mode left and thus the scalar field will completely decouple.

Next, consider a bosonic field with periodic boundary conditions transforming under an arbitrary tensor representation of the Lorentz group  $SO(1, D-1)$  on  $M_D$ . Let us begin with a vector field  $A_\mu(x^\nu)$  in the fundamental of  $SO(1, D-1)$ . The index  $\mu$  must now also be split into a component along the direction  $y$  and the remaining  $D-1$  directions  $\bar{\mu}$ . The first results in a scalar  $A_y(x^{\bar{\nu}})$ , while the second results in a vector  $A_{\bar{\mu}}(x^{\bar{\nu}})$  of the  $D-1$  dimensional Lorentz group  $SO(1, D-2)$ . We notice that this decomposition is nothing but the branching rule for the fundamental representation of  $SO(1, D-1)$  decomposing under the subgroup  $SO(1, D-2)$ . For a field  $A$  obeying period boundary conditions and transforming under a general tensor representation  $T$  of  $SO(1, D-1)$ , dimensional reduction on a circle will produce a direct sum of representations  $T_i$  of  $SO(1, D-2)$ , which is the restriction of  $T$  to the subgroup  $SO(1, D-2)$ .

For a spinor field obeying periodic boundary conditions and transforming under a general spinor representation  $S$  of  $SO(1, D-1)$ , dimensional reduction will produce a direct sum of representations  $S_i$  of  $SO(1, D-2)$  which is the restriction of  $S$  to the subgroup  $SO(1, D-2)$ . Finally, assembling bosons and fermions with periodic boundary conditions in a supersymmetry multiplet, we see that dimensional reduction will preserve all Poincaré supersymmetries, and that the supercharges will behave as the spinor fields described above under this reduction.

An important example is the rank 2 symmetric tensor, i.e. the metric  $G_{\mu\nu}$ ,

$$G_{\mu\nu} \rightarrow \begin{cases} G_{yy} & \text{scalar mixing with dilaton} \\ G_{\bar{\mu}y} & \text{graviphoton} \\ G_{\bar{\mu}\bar{\nu}} & \text{metric} \end{cases} \quad (4.12)$$

Again, fields obeying boundary conditions other than periodic will completely decouple.

#### 4.4 D=11 and D=10 Supergravity Particle and Field Contents

In this subsection, we begin by listing the field contents and the number of physical degrees of freedom of the  $\mathcal{N} = 1$ ,  $D = 11$  supergravity theory. By dimensional reduction on a circle, we find the  $\mathcal{N} = 2$ ,  $D = 10$  Type IIA theory, which is parity conserving and has two Majorana-Weyl gravitini of opposite chiralities. Finally, we list the field and particle contents for the  $\mathcal{N} = 2$ ,  $D = 10$  Type IIB theory, which is chiral and has two Majorana-Weyl gravitini of the same chirality.

The  $\mathcal{N} = 1$ ,  $D = 11$  supergravity theory has the following field and particle contents,

$$D = 11 \begin{cases} G_{\mu\nu} & SO(9) & 44_B & \text{metric} - \text{graviton} \\ A_{\mu\nu\rho} & & 84_B & \text{antisymmetric rank 3} \\ \psi_{\mu\alpha} & & 128_F & \text{Majorana gravitino} \end{cases} \quad (4.13)$$

Here and below, the numbers following the little group (for the massless representations)  $SO(9)$  represent the number of physical degrees of freedom in the multiplet. For example, the graviton in  $D = 11$  is given by the rank 2 symmetric traceless representation of  $SO(9)$ , of dimension  $9 \times 10/2 - 1 = 44$ . The Majorana spinor  $\psi_{\mu\alpha}$  as a vector has 9 physical components, but it also satisfies the  $\Gamma$ -tracelessness condition  $(\Gamma^\mu)^{\beta\alpha}\psi_{\mu\alpha} = 0$ , which cuts the number down to 8. The 32 component spinor satisfies a Dirac equation, which cuts its number of physical components down to 16, yielding a total of  $8 \times 16 = 128$ . The subscripts  $B$  and  $F$  refer to the bosonic or fermionic nature of the state.

The  $\mathcal{N} = 2$ ,  $D = 10$  Type IIA theory is obtained by dimensional reduction on a circle,

$$\text{Type IIA} \begin{cases} G_{\mu\nu} & SO(8) & 35_B & \text{metric} - \text{graviton} \\ \Phi & & 1_B & \text{dilaton} \\ B_{\mu\nu} & & 28_B & \text{NS} - \text{NS rank 2 antisymmetric} \\ A_{3\mu\nu\rho} & & 56_B & \text{antisymmetric rank 3} \\ A_{1\mu} & & 8_B & \text{graviphoton} \\ \psi_{\mu\alpha}^\pm & & 112_F & \text{Majorana} - \text{Weyl gravitinos} \\ \lambda_\alpha^\pm & & 16_F & \text{Majorana} - \text{Weyl dilatinos} \end{cases} \quad (4.14)$$

Here, the gravitinos are again  $\Gamma$ -traceless. The two gravitinos  $\psi_{\mu\alpha}^\pm$  as well as the two dilatinos  $\lambda_\alpha^\pm$  have opposite chiralities and the theory is parity conserving.

The  $\mathcal{N} = 2$ ,  $D = 10$  Type IIB theory has the following field and particle contents,

$$\text{Type IIB} \left\{ \begin{array}{llll} G_{\mu\nu} & SO(8) & 35_B & \text{metric} - \text{graviton} \\ C + i\Phi & & 2_B & \text{axion} - \text{dilaton} \\ B_{\mu\nu} + iA_{2\mu\nu} & & 56_B & \text{rank 2 antisymmetric} \\ A_{4\mu\nu\rho\sigma}^+ & & 35_B & \text{antisymmetric rank 4} \\ \psi_{\mu\alpha}^I \quad I=1,2 & & 112_F & \text{Majorana} - \text{Weyl gravitinos} \\ \lambda_\alpha^I \quad I=1,2 & & 16_F & \text{Majorana} - \text{Weyl dilatinos} \end{array} \right. \quad (4.15)$$

The rank 4 antisymmetric tensor  $A_{\mu\nu\rho\sigma}^+$  has self-dual field strength, a fact that is indicated with the + superscript. The gravitinos are again  $\Gamma$ -traceless. The two gravitinos  $\psi_{\mu\alpha}^I$  have the same chirality, while the two dilatinos  $\lambda_\alpha^I$  also have the same chirality but opposite to that of the gravitinos. The theory is chiral or parity violating.

## 4.5 D=11 and D=10 Supergravity Actions

Remarkably, the  $D = 11$  supergravity theory has a relatively simple action. It is convenient to use exterior differential notation for all anti-symmetric tensor fields, such as the rank 3 tensor  $A_3 \equiv 1/3! A_{3\mu\nu\rho} dx^\mu dx^\nu dx^\rho$ , with field strength  $F_4 \equiv dA_3$ ,

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int \left[ \sqrt{G}(R_G - \frac{1}{2}|F_4|^2) - \frac{1}{6}A_3 \wedge F_4 \wedge F_4 \right] + \text{fermions} \quad (4.16)$$

where  $\kappa_{11}^2$  is the 11-dimensional Newton constant. The action for the Type IIA theory may be deduced from this action by dimensional reduction, but we shall not need it here. There are also  $D = 10$  supergravities with only  $\mathcal{N} = 1$  supersymmetry, which in particular may couple to  $D = 10$  super-Yang-Mills theory.

There exists no completely satisfactory action for the Type IIB theory, since it involves an antisymmetric field  $A_4^+$  with self-dual field strength. However, one may write an action involving both dualities of  $A_4$  and then impose the self-duality as a supplementary field equation. Doing so, one obtains<sup>i</sup> (see for example [33, 28])

$$S_{IIB} = +\frac{1}{4\kappa_B^2} \int \sqrt{G} e^{-2\Phi} (2R_G + 8\partial_\mu \Phi \partial^\mu \Phi - |H_3|^2) - \frac{1}{4\kappa_B^2} \int \left[ \sqrt{G} (|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2) + A_4^+ \wedge H_3 \wedge F_3 \right] + \text{fermions} \quad (4.17)$$

where the field strengths are defined by

$$\left\{ \begin{array}{l} F_1 = dC \\ H_3 = dB \\ F_3 = dA_2 \\ F_5 = dA_4^+ \end{array} \right. \quad \left\{ \begin{array}{l} \tilde{F}_3 = F_3 - CH_3 \\ \tilde{F}_5 = F_5 - \frac{1}{2}A_2 \wedge H_3 + \frac{1}{2}B \wedge F_3 \end{array} \right. \quad (4.18)$$

and we have the supplementary self-duality condition  $*\tilde{F}_5 = \tilde{F}_5$ .

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<sup>i</sup>We use the notation  $G \equiv -\det G_{\mu\nu}$  and  $\int \sqrt{G}|F_p|^2 \equiv \frac{1}{p!} \int \sqrt{G} G^{\mu_1\nu_1} \dots G^{\mu_p\nu_p} \bar{F}_{\mu_1\dots\mu_p} F_{\nu_1\dots\nu_p}$  where  $\bar{F}$  denotes the complex conjugate of  $F$ . For real fields, this definition coincides with that of [28].

The above form of the action naturally arises from the string low energy approximation. The first line in (4.17) originates from the NS-NS sector while the second line (except for the fermions) originates from the RR sector, as we shall see shortly. Type IIB supergravity is invariant under the non-compact symmetry group  $SU(1, 1) \sim SL(2, \mathbf{R})$ , but this symmetry is not manifest in (4.17). To render the symmetry manifest, we redefine fields from the *string metric*  $G_{\mu\nu}$  used in (4.17) to the *Einstein metric*  $G_{E\mu\nu}$ , along with expressing the tensor fields in terms of complex fields,<sup>ii</sup>

$$\begin{aligned} G_{E\mu\nu} &\equiv e^{-\Phi/2} G_{\mu\nu} & \tau &\equiv C + ie^{-\Phi} \\ G_3 &\equiv (F_3 - \tau H_3) / \sqrt{\text{Im}\tau} \end{aligned} \quad (4.19)$$

The action may then be written simply as,

$$\begin{aligned} S_{\text{IIB}} &= \frac{1}{4\kappa_B^2} \int \sqrt{G_E} \left( 2R_{G_E} - \frac{\partial_\mu \bar{\tau} \partial^\mu \tau}{(\text{Im}\tau)^2} - \frac{1}{2} |F_1|^2 - |G_3|^2 - \frac{1}{2} |\tilde{F}_5|^2 \right) \\ &\quad - \frac{1}{4i\kappa_B^2} \int A_4 \wedge \bar{G}_3 \wedge G_3 \end{aligned} \quad (4.20)$$

Under the  $SU(1, 1) \sim SL(2, \mathbf{R})$  symmetry of Type IIB supergravity, the metric and  $A_4^+$  fields are left invariant. The dilaton-axion field  $\tau$  changes under a Möbius transformation,

$$\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d} \quad ad - bc = 1, \quad a, b, c, d \in \mathbf{R} \quad (4.21)$$

Finally, the  $B_{\mu\nu}$  and  $A_{2\mu\nu}$  fields rotate into one another under the linear transformation associated with the above Möbius transformation, and this may most easily be re-expressed in terms of the complex 3-form field  $G_3$ ,

$$G_3 \rightarrow G'_3 = \frac{c\bar{\tau} + d}{|c\tau + d|} G_3 \quad (4.22)$$

The susy transformation laws of Type IIB supergravity [33, 34] on the fermion fields – the dilatino  $\lambda$  and the gravitino  $\psi_M$  – are of the form, (we shall not need the transformation laws on bosons),

$$\begin{aligned} \delta\lambda &= \frac{i}{\kappa_B} \Gamma^\mu \eta^* \frac{\partial_\mu \tau}{\text{Im}\tau} - \frac{i}{24} \Gamma^{\mu\nu\rho} \eta G_{3\mu\nu\rho} + (\text{Fermi})^2 \\ \delta\psi_\mu &= \frac{1}{\kappa_B} D_\mu \eta + \frac{i}{480} \Gamma^{\mu_1 \dots \mu_5} \Gamma_\mu \eta F_{5\mu_1 \dots \mu_5} + \frac{1}{96} (\Gamma_\mu^{\rho\sigma\tau} G_{3\rho\sigma\tau} - 9\Gamma^{\nu\rho} G_{3\mu\nu\rho}) \eta^* + (\text{Fermi})^2 \end{aligned} \quad (4.23)$$

Note that in the  $SU(1, 1)$  formulation, the supersymmetry transformation parameter  $\eta$  has  $U(1)$  charge 1/2, so that  $\lambda$  has charge 3/2 and  $\psi_\mu$  has charge 1/2.

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<sup>ii</sup>The detailed relation with the  $SU(1, 1)$  formulation of Type IIB supergravity is given as follow : the  $SU(1, 1)$  frame  $V_\pm^\alpha$ ,  $\alpha = 1, 2$  is given by  $V_+^1 = \tau/\sqrt{\text{Im}\tau}$ ,  $V_-^1 = \bar{\tau}/\sqrt{\text{Im}\tau}$ , and  $V_\pm^2 = 1/\sqrt{\text{Im}\tau}$ . The frame transforms as a  $SU(1, 1)$  doublet and satisfied  $V_-^1 V_+^2 - V_-^2 V_+^1 = 1$ . The complex 3-form is defined by  $G_3 = V_+^2 F_3 - V_+^1 H_3$  and is a  $SU(1, 1)$  singlet. The complex variable  $\tau$  parametrizes the coset  $SU(1, 1)/U(1)$ ; under this local  $U(1)$  group,  $V_\pm$  have charge  $\pm 1$  while  $G_3$  has charge +1.

## 4.6 Superstrings in $D = 10$

The geometrical data of superstring theory in the Ramond-Neveu-Schwarz (RNS) formulation are the bosonic worldsheet field  $x^\mu$  and the fermionic worldsheet fields  $\psi_\pm^\mu$ , which may both be viewed as functions of local worldsheet coordinates  $\xi^1, \xi^2$ . The subscript  $\pm$  indicates the two worldsheet chiralities. Both  $x^\mu$  and  $\psi_\pm^\mu$  transform under the vector representation of the space-time Lorentz group. The theory has two sectors, the Neveu-Schwarz (NS) and Ramond (R) sectors. The NS ground state is a space-time boson, while the R ground state is a space-time fermion. The full space-time bosonic (resp. fermionic) spectrum of the theory is obtained by applying  $x^\mu$  and  $\psi^\mu$  fields to the NS (resp. R) ground states. Space-time supersymmetry is achieved by imposing a suitable Gliozzi-Scherk-Olive (GSO) projection [29]. For simplicity, we shall only consider theories with orientable strings; the Type II and heterotic string theories fit in this category. Interactions arise from the joining and splitting of the worldsheets, so that the number of handles (which equals the genus for orientable worldsheets) corresponds to the number of loops in a field theory reinterpretation of the string diagram. (Standard references on superstring theory include [34, 28], lecture notes [35] and a review on perturbation theory [36].)

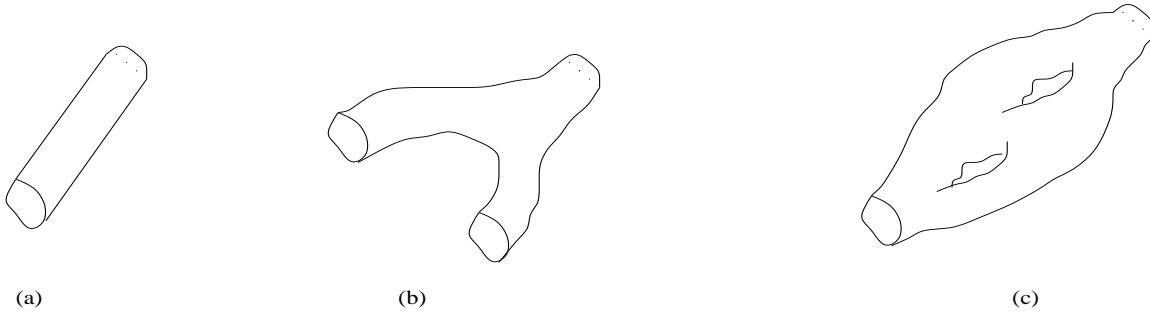


Figure 1: Propagating closed strings (a) free, (b) interaction, (c) two-loop

One aspect of string theory that we shall make use of in these lectures is the fact that (1) the low energy limit of string theory is supergravity and that (2) string theory produces definite and calculable higher derivative corrections to the supergravity action and field equations. To explain these facts, it is easiest to concentrate on the space-time bosonic fields, since space-time fermionic fields require the use of the more complicated fermion vertex operator. For Type II theories, the space-time bosons arise from two sectors in turn; the NS-NS sector and the R-R sector. Fields in the R-R sector again couple to the string worldsheet through the use of the fermion vertex operator, and for simplicity we shall ignore also these fields here (even though they will of course be very important for the AdS/CFT conjecture). The remaining fields are now the same for all four closed orientable string theories, Type IIA, Type IIB and the two heterotic strings, namely the metric  $G_{\mu\nu}$ , the NS-NS antisymmetric rank 2 tensor  $B_{\mu\nu}$  and the dilaton  $\Phi$ . The full worldsheet action for the coupling of these fields is still very complicated on a worldsheet with general worldsheet metric and worldsheet gravitino fields  $\chi_m$ . The contribution from

the worldsheet bosonic field  $x^\mu$  gives rise to a generalized non-linear sigma model,

$$S_x = \frac{1}{4\pi\alpha'} \int_\Sigma \sqrt{\gamma} \left[ \{ \gamma^{mn} G_{\mu\nu}(x) + \epsilon^{mn} B_{\mu\nu}(x) \} \partial_m x^\mu \partial_n x^\nu + \alpha' R_\gamma^{(2)} \Phi(x) \right] \quad (4.24)$$

where  $\alpha'$  is the square of the Planck length,  $\gamma_{mn}$  is the worldsheet metric,  $\gamma^{mn}$  its inverse and  $R_\gamma^{(2)}$  its associated Gaussian curvature. The contribution from the worldsheet fermionic field  $\psi_\pm^\mu$  gives rise to a worldsheet supersymmetric completion of the above non-linear sigma model. Here, we quote its form only for a flat worldsheet metric and vanishing worldsheet gravitino field,

$$S_\psi = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\xi \left( G_{\mu\nu}(x) (\psi_+^\mu D_{\bar{z}} \psi_+^\nu + \psi_-^\mu D_z \psi_-^\nu) + \frac{1}{2} R_{\mu\nu\rho\sigma} \psi_+^\mu \psi_+^\nu \psi_-^\rho \psi_-^\sigma \right) \quad (4.25)$$

where  $R_{\mu\nu\rho\sigma}$  is the Riemann tensor for the metric  $G_{\mu\nu}$  and the covariant derivatives are given by

$$\begin{aligned} D_{\bar{z}} \psi_+^\mu &= \partial_{\bar{z}} \psi_+^\mu + \left( \Gamma_{\rho\sigma}^\mu(x) + \frac{1}{2} H_{3\mu\rho\sigma}(x) \right) \partial_{\bar{z}} x^\rho \psi_+^\sigma \\ D_z \psi_-^\mu &= \partial_z \psi_-^\mu + \left( \Gamma_{\rho\sigma}^\mu(x) - \frac{1}{2} H_{3\mu\rho\sigma}(x) \right) \partial_z x^\rho \psi_-^\sigma \end{aligned} \quad (4.26)$$

where  $H_{3\mu\rho\sigma}$  is the field strength of  $B_{\mu\nu}$  and  $\Gamma_{\rho\sigma}^\mu$  is the Levi-Civita connections for  $G$ .

The non-chiral scattering amplitudes are given by the functional integral over all  $x^\mu$  and  $\psi_\pm$  as well as over all worldsheet metrics  $\gamma_{mn}$  and all worldsheet gravitini fields  $\chi_m$  by

$$\text{amplitude} = \sum_{\text{topologies}} \int D\gamma_{mn} D\chi_m \int Dx^\mu D\psi e^{-S_x + S_\psi} \quad (4.27)$$

The full amplitudes must then be obtained by first chirally splitting [36, 37] the non-chiral amplitudes in terms of the conformal blocks of the corresponding conformal field theories of the left and right movers and imposing the GSO projection.

The quantization prescription given by the above formula for the amplitude is in the first quantized formulation of string theory. There, a given string configuration (a given worldsheet topology) is quantized in the presence of external background fields, such as the metric  $G_{\mu\nu}$ , the rank 2 anti-symmetric tensor field  $B_{\mu\nu}$  and the dilaton  $\Phi$ . The quantization of the string produces excitations of these very fields as well as of all the other string modes. In comparison with the first quantized formulation of particles in field theory, the background fields may be interpreted as vacuum expectation values of the corresponding field operators.

If the vacuum expectation value of the dilaton field is  $\phi = \langle \Phi \rangle$ , then the contribution of the vacuum expectation value to the string amplitude is governed by the Euler number  $\chi(\Sigma)$  of the worldsheet  $\Sigma$ ,

$$\frac{1}{2\pi} \int_\Sigma \sqrt{\gamma} R_\gamma^{(2)} = \chi(\Sigma) = 2 - 2h - b \quad (4.28)$$

where  $h$  is the genus or number of handles and  $b$  is the number of boundaries or punctures. Therefore, a genus  $h$  worldsheet (without boundary) will receive a multiplicative contribution of  $e^{-(2-2h)\phi} = g_s^{2h-2}$  which gives reason to identify  $g_s = e^\phi$  with the (closed) string



coupling constant. For open string theories, the expansion is rather in integer powers of the open string coupling constant  $g_o = e^{\phi/2}$ .

## 4.7 Conformal Invariance and Supergravity Field Equations

As a two-dimensional quantum field theory, the generalized non-linear sigma model makes sense for any background field assignment. However, when the non-linear sigma model is to define a consistent string theory, further physical conditions need to be satisfied. The most crucial one is that the single string spectrum be free of negative norm states. Such states always appear because Poincaré invariance of the theory forces the string map  $x^\mu$  to obey the following canonical relations  $[x^\mu, \dot{x}^\nu] \sim G^{\mu\nu}$ , so that  $x^0$  creates negative norm states.

The decoupling of negative norm states out of the Fock space construction occurs via *worldsheet conformal invariance* of the non-linear sigma model. In particular, conformal invariance requires worldsheet scale invariance of the full quantum mechanical non-linear sigma model. Transformations of the worldsheet scale  $\Lambda$  are broken by quantum mechanical anomalies whose form is encoded by the  $\beta$ -functions of the renormalization group (RG). As will be explained in the next paragraph, each background field has a  $\beta$ -function, and worldsheet scale and conformal invariance thus require the vanishing of these  $\beta$ -functions.

The background fields  $G_{\mu\nu}(x)$ ,  $B_{\mu\nu}(x)$  and  $\Phi(x)$  may be viewed as generating functions for an infinite series of coupling constants. For example, for the metric we have,

$$G_{\mu\nu}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} (x - x_0)^{\mu_1} \cdots (x - x_0)^{\mu_n} \partial_{\mu_1} \cdots \partial_{\mu_n} G_{\mu\nu}(x_0) \quad (4.29)$$

where each of the Taylor expansion coefficients  $\partial_{\mu_1} \cdots \partial_{\mu_n} G_{\mu\nu}(x_0)$  may be viewed as an independent set of couplings. Under renormalization, and thus under RG flow, this infinite number of couplings flows into itself, and the corresponding flows may again be described by generating functions  $\beta_{\mu\nu}^G(x)$ ,  $\beta_{\mu\nu}^B(x)$  and  $\beta^\Phi(x)$  defined, for example, for the metric by

$$\beta_{\mu\nu}^G(x) = \frac{\partial G_{\mu\nu}}{\partial \ln \Lambda} \equiv \sum_{n=0}^{\infty} \frac{1}{n!} (x - x_0)^{\mu_1} \cdots (x - x_0)^{\mu_n} \frac{\partial \partial_{\mu_1} \cdots \partial_{\mu_n} G_{\mu\nu}(x_0)}{\partial \ln \Lambda} \quad (4.30)$$

Customarily, when an infinite number of couplings occur in a quantum field theory, it is termed *non-renormalizable*, because the prediction of any physical observable would require an infinite number of input data to be specified at the renormalization point. In string theory, however, this infinite number of couplings is exactly what is required to describe the dynamics of a string in a consistent background. We now explain how this comes about.

First, we assume that the whole renormalization process of the non-linear sigma model will preserve space-time diffeomorphism invariance. The number of terms that can appear in the RG flow is then finite, order by order in the  $\alpha'$  expansion [38]. Second, the presence

of an infinite number of couplings makes it possible to have the string propagate in an infinite family of space-times. The leading order  $\beta$ -functions are given by [39]

$$\begin{aligned}\beta_{\mu\nu}^G &= \frac{1}{2}R_{\mu\nu} - \frac{1}{8}H_{\mu\rho\sigma}H_{\nu}{}^{\rho\sigma} + \partial_\mu\Phi\partial_\nu\Phi + \mathcal{O}(\alpha') \\ \beta_{\mu\nu}^B &= -\frac{1}{2}D_\rho H^\rho{}_{\mu\nu} + \partial_\rho H^\rho{}_{\mu\nu} + \mathcal{O}(\alpha') \\ \beta^\Phi &= \frac{1}{6}(D-10) + \alpha' \left( 2\partial_\mu\Phi\partial^\mu\Phi - 2\nabla^\mu\partial_\mu\Phi + \frac{1}{2}R_G - \frac{1}{24}H_{\mu\nu\rho}H^{\mu\nu\rho} \right) + \mathcal{O}(\alpha')^2\end{aligned}\tag{4.31}$$

To leading order in  $\alpha'$ , the requirement of scale invariance reduces precisely to the supergravity field equations for the Type II theory where all RR  $A$ -fields have been (consistently) set to 0. String theory provides higher  $\alpha'$  corrections to the supergravity field equations, which by dimensional analysis must be also terms with higher derivatives in  $x^\mu$ .

## 4.8 Branes in Supergravity

A rank  $p+1$  antisymmetric tensor field  $A_{\mu_1\dots\mu_{p+1}}$  may be identified with a  $(p+1)$ -form,

$$A_{p+1} \equiv \frac{1}{(p+1)!} A_{\mu_1\dots\mu_{p+1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}}\tag{4.32}$$

A  $(p+1)$ -form naturally couples to geometrical objects  $\Sigma_{p+1}$  of space-time dimension  $p+1$ , because a diffeomorphism invariant action may be constructed as follows

$$S_{p+1} = T_{p+1} \int_{\Sigma_{p+1}} A_{p+1}\tag{4.33}$$

The action is invariant under Abelian gauge transformations  $\rho_p(x)$  of rank  $p$

$$A_{p+1} \rightarrow A_{p+1} + d\rho_p\tag{4.34}$$

because  $S_{p+1}$  transforms with a total derivative. The field  $A_{p+1}$  has a gauge invariant field strength  $F_{p+2}$ , which is a  $p+2$  form whose flux is conserved. *Solutions to supergravity with non-trivial  $A_{p+1}$  charge are referred to as  $p$ -branes, after the space-dimension of their geometry.*

Each  $A_{p+1}$  gauge field has a *magnetic dual*  $A_{D-3-p}^{\text{magn}}$  which is a differential form field of rank  $D-3-p$ , whose field strength is related to that of  $A_{p+1}$  by Poincaré duality

$$dA_{D-3-p}^{\text{magn}} \equiv *dA_{p+1}\tag{4.35}$$

Accordingly, each  $p$ -brane also has a magnetic dual, which is a  $(D-4-p)$  brane and which now couples to the field  $A_{D-3-p}^{\text{magn}}$ .

The possible branes in  $D=11$  supergravity are very restricted because the only antisymmetric tensor field in the theory is  $A_{\mu\nu\rho}$  of rank 3, so that we have a 2-brane, denoted

$M2$  and its magnetic dual  $M5$ . The branes in Type IIA/B theory are further distinguished as follows. When the antisymmetric field whose charge they carry is in the R-R sector, the brane is referred to as a  $D$ -brane. D-branes were introduced first in string theory in [40]. On the other hand, the 1-brane that couples to the NS-NS field  $B_{\mu\nu}$  is nothing but the fundamental string, denoted F1, whose magnetic dual is NS5 [41]. Below we present a Table of the branes occurring for various  $p$  in the  $D = 11$  supergravity and in the Type IIA/B supergravities in  $D = 10$ .

## 4.9 Brane Solutions in Supergravity

Each brane is realized as a 1/2 BPS solution in supergravity. The geometry of these solutions will be important, and we describe it now. A  $p$ -brane has a  $(p + 1)$ -dimensional flat hypersurface, with Poincaré invariance group  $\mathbf{R}^{p+1} \times SO(1, p)$ . The transverse space is then of dimension  $D - p - 1$  and solutions may always be found with maximal rotational symmetry  $SO(D - p - 1)$  in this transverse space. Thus,  $p$ -branes in supergravity may be thought of as solutions with symmetry groups

$$\begin{cases} D=11 & \mathbf{R}^{p+1} \times SO(1, p) \times SO(10 - p) \\ D=10 & \mathbf{R}^{p+1} \times SO(1, p) \times SO(9 - p) \end{cases} \quad (4.36)$$

For example the  $M2$  brane has symmetry group  $\mathbf{R}^3 \times SO(1, 2) \times SO(8)$  while the  $D3$  brane has instead  $\mathbf{R}^4 \times SO(1, 3) \times SO(6)$ . We shall denote the coordinates as follows

$$\begin{aligned} \text{Coordinates } // \text{ to brane} & \quad x^\mu & \quad \mu = 0, 1, \dots, p \\ \text{Coordinates } \perp \text{ to brane} & \quad y^u = x^{p+u} & \quad u = 1, 2, \dots, D - p - 1 \end{aligned}$$

Poincaré invariance in  $p+1$  dimensions forces the metric in those directions to be a rescaling of the Minkowski flat metric, while rotation invariance in the transverse directions forces the metric in those directions to be a rescaling of the Euclidean metric in those dimensions. Furthermore, the metric rescaling functions should be independent of  $x^\mu$ ,  $\mu = 0, 1, \dots, p$ . Substituting an Ansatz with the above restrictions into the field equations, one finds that

name	$D = 11$	Type IIA	Type IIB	Magnetic Dual
D(-1) instanton	—	—	$A_0 = C + ie^{-\Phi}$	D7
D0 particle	—	$A_{1\mu}$	—	D6
F1 string	—	$B_{\mu\nu}$	$B_{\mu\nu}$	NS5
D1 string	—	—	$A_{2\mu\nu}$	D5
M2 membrane	$A_{\mu\nu\rho}$	—	—	M5
D2 brane	—	$A_{3\mu\nu\rho}$	—	D4
D3 brane	—	—	$A_{4\mu\nu\rho\sigma}^+$	D3

Table 4: Branes in various theories

the solution may be expressed in terms of a single function  $H$  as follows, [42]

$$\begin{aligned}
\text{Dp} & \quad ds^2 = H(\vec{y})^{-1/2} dx^\mu dx_\mu + H(\vec{y})^{1/2} d\vec{y}^2 & \quad e^\Phi = H(\vec{y})^{(3-p)/4} \\
\text{NS5} & \quad ds^2 = dx^\mu dx_\mu + H(\vec{y}) d\vec{y}^2 & \quad e^{2\Phi} = H(\vec{y}) \\
\text{M2} & \quad ds^2 = H(\vec{y})^{-2/3} dx^\mu dx_\mu + H(\vec{y})^{1/3} d\vec{y}^2 \\
\text{M5} & \quad ds^2 = H(\vec{y})^{-1/3} dx^\mu dx_\mu + H(\vec{y})^{2/3} d\vec{y}^2
\end{aligned} \tag{4.37}$$

Here, the  $Dp$  metric is expressed in the string frame. The single function  $H$  must be harmonic with respect to  $\vec{y}$ .

Assuming maximal rotational symmetry by  $SO(D-p-1)$  in the transversal dimensions, and using the fact that the metric should tend to flat space-time as  $y \rightarrow \infty$ , the most general solution is parametrized by a single scale factor  $L$  and is given by

$$H(y) = 1 + \frac{L^{D-p-3}}{y^{D-p-3}} \tag{4.38}$$

Since  $\alpha'$  is the only dimensionful parameter of the theory,  $L$  must be a numerical constant (possibly dependent on the dimensionless string couplings) times the above  $\alpha'$  dependence. Of particular interest will be the solution of  $N$  coincident branes, for which we have  $L^{D-p-3} = N\rho_p$ . For  $Dp$  branes, we have  $\rho_p = g_s (4\pi)^{(5-p)/2} \Gamma((7-p)/2) (\alpha')^{(D-p-3)/2}$ .

It is easy to see that one still has a solution when  $H$  is harmonic without insisting on rotation invariance in the transverse space, so that the general solution is of the form,

$$H(\vec{y}) = 1 + \sum_{I=1}^N \frac{C_I}{|\vec{y} - \vec{y}_I|^{D-3-p}} \quad C_I = N_I \rho_p, \quad N_I \in \mathbb{N} \tag{4.39}$$

for any array of  $N$  points  $\vec{y}$ .

It is very important in the theory of branes in Type IIA/B string theory to understand the dependence of the string coupling  $g_s$  of the various brane solutions, in particular of  $c_p$ . To do so, we return to the supergravity field equations, (omitting derivative terms in the dilaton and axion fields for simplicity),

$$\begin{aligned}
\text{IIA} & \quad R_{\mu\nu} = \frac{1}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} + e^{2\Phi} \left( F_{2\mu\rho} F_{2\nu}{}^\rho + \frac{1}{6} F_{4\mu\sigma\rho\tau} F_{4\nu}{}^{\rho\sigma\tau} \right) \\
\text{IIB} & \quad R_{\mu\nu} = \frac{1}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} + e^{2\Phi} \left( F_{1\mu} F_{1\nu} + \frac{1}{4} \tilde{F}_{3\mu\sigma\rho} \tilde{F}_{3\nu}{}^{\rho\sigma} + \frac{1}{24} \tilde{F}_{5\mu\rho\sigma\tau\nu}^+ \tilde{F}_{5\nu}{}^{\rho\sigma\tau\nu} \right)
\end{aligned} \tag{4.40}$$

Recall that the string coupling is given by  $g_s = e^\phi$  where  $\phi = \langle \Phi \rangle$ . In both Type IIA and Type IIB, the fundamental string F1 and the NS5 brane have non-vanishing  $H_{\mu\rho\sigma}$  fields, but *vanishing RR fields*  $F_i$ . Therefore, these brane solutions do not involve the string coupling constant  $g_s$  and  $\rho_p$  is independent of  $g_s$ . D-brane solutions on the other hand will have  $H_{\mu\rho\sigma} = 0$ , but have at least one of the R-R antisymmetric fields  $F_i \neq 0$ . Such solutions will involve the string coupling explicitly and therefore  $\rho_p \sim g_s$ . This leads for example to the expression given for  $\rho_p$  above. Each brane solution breaks precisely half of the supersymmetries of the corresponding theory, as is shown in Problem Set (4.1).

## 4.10 Branes in Superstring Theory

While originally found as solutions to supergravity field equations, the  $p$ -branes of Type IIA/B supergravity are expected to extend to solutions of the full Type IIA/B string equations. These solutions will then break precisely half of the supersymmetries of the string theory. As compared to the supergravity solutions, the full string solutions may, of course, be subject to  $\alpha'$  corrections of their metric and other fields. Often, it is useful to compare these semi-classical solutions of string theory with solitons in quantum field theory, such as the familiar 't Hooft–Polyakov magnetic monopole. The fundamental string F1 and the NS5 brane indeed very much behave as large size semi-classical solitons, whose energy depends on the string coupling via  $1/g_s^2$ , as is familiar from solitons in quantum field theory.

Besides its supergravity low energy limit, the only other well-understood limit of string theory is that of weak coupling where  $g_s \rightarrow 0$ . It is in this approximation that string theory may be defined in terms of a genus expansion in string worldsheets. Remarkably, D-branes (but not the F1 string or NS5 branes) admit a special limit as well. As may be seen from (4.39), in the limit where  $g_s \rightarrow 0$ , the metric becomes flat everywhere, except on the  $(p+1)$ -dimensional hyperplane characterized by  $\vec{y} = 0$ , where the metric appears to be singular. Thus, in the weak-coupling limit, the D-brane solution of supergravity reduces to a localized defect in flat space-time. Strings propagating in this background are moving in flat space-time, except when the string reaches the D-brane. The interaction of the string with the D-brane is summarized by a boundary condition on the string dynamics. The correct conditions turn out to be Dirichlet boundary conditions in the directions perpendicular to the brane and Neumann conditions parallel to the brane. The  $Dp$ -brane may alternatively be described in string perturbation theory as a  $(p+1)$ -dimensional hypersurface in flat 10-dimensional space-time on which open strings end with the above boundary conditions. The open string end points are thus tied to be on the brane, but can move freely along the brane. This was indeed the original formulation [40]; see also [43].

## 4.11 The Special Case of D3 branes

The D3-brane solution is of special interest for a variety of reasons : (1) its worldbrane has 4-dimensional Poincaré invariance; (2) it has constant axion and dilaton fields; (3) it is regular at  $y = 0$ ; (4) it is self-dual. Given its special importance, we shall present here a more complete description of the D3-brane. The solution is characterized by

$$\begin{cases} g_s = e^\phi, C \text{ constant} \\ B_{\mu\nu} = A_{2\mu\nu} = 0 \\ ds^2 = H(y)^{-1/2} dx^\mu dx_\mu + H(y)^{1/2} (dy^2 + y^2 d\Omega_5^2) \\ F_{5\mu\nu\rho\sigma\tau}^+ = \epsilon_{\mu\nu\rho\sigma\tau\nu} \partial^\nu H \end{cases} \quad (4.41)$$

Here,  $\epsilon_{\mu\nu\rho\sigma\tau\nu}$  is the volume element transverse to the 4-dimensional Minkowski D3-brane in  $D = 10$ . The  $N$ -brane solution with general locations of  $N_I$  parallel D3-branes located

at transverse position  $\vec{y}_i$  is given by

$$H(\vec{y}) = 1 + \sum_{I=1}^N \frac{4\pi g_s N_I (\alpha')^2}{|\vec{y} - \vec{y}_I|^4} \quad (4.42)$$

where the total number of D3-branes is  $N = \sum_I N_I$ . The fact that the geometry is regular as  $\vec{y} \rightarrow \vec{y}_I$  despite the apparent singularity in the metric will be shown in the next section.

It is useful to compare the scales involved in the D3 brane solution and their relations with the coupling constant.<sup>iii</sup> The radius  $L$  of the D3 brane solution to string theory is a scale that is not necessarily of the same order of magnitude as the Planck length  $\ell_P$ , which is defined by  $\ell_P^2 = \alpha'$ . Their ratio is given instead by  $L^4 = 4\pi g_s N \ell_P^4$ . For  $g_s N \ll 1$ , the radius  $L$  is much smaller than the string length  $\ell_P$ , and thus the supergravity approximation is not expected to be a reliable approximation to the full string solution. In this regime we have  $g_s \ll 1$ , so that string perturbation theory is expected to be reliable and the D3 brane may be treated using conformal field theory techniques. For  $g_s N \gg 1$ , the radius  $L$  is much larger than the string length  $\ell_P$ , and thus the supergravity approximation is expected to be a good approximation to the full string solution. It is possible to have at the same time  $g_s \ll 1$  provided  $N$  is very large, so string perturbation theory may be *simultaneously* a good approximation.

The D3 brane solution is more properly a two-parameter family of solutions, labeled by the string coupling  $g_s$  and the instanton angle  $\theta_I = 2\pi C$ , or the single complex parameter  $\tau = C + ie^{-\phi}$ . The  $SU(1, 1) \sim SL(2, \mathbf{R})$  symmetry of Type IIB supergravity acts transitively on  $\tau$ , so all solutions lie in a single orbit of this group. In superstring theory, however, the range of  $\theta_I$  is quantized so that the identification  $\theta_I \sim \theta_I + 2\pi$  may be made, and as a result also  $\tau \sim \tau + 1$ . Therefore, the allowed Möbius transformations must be elements of the  $SL(2, \mathbf{Z})$  subgroup of  $SL(2, \mathbf{R})$ , for which  $a, b, c, d \in \mathbf{Z}$ . These transformations map between equivalent solutions in string theory. Thus, the string theories defined on D3 backgrounds which are related by an  $SL(2, \mathbf{Z})$  duality will be equivalent to one another. This property will be of crucial importance in the AdS/CFT correspondence where it will emerge as the reflection of Montonen-Olive duality in  $\mathcal{N} = 4$  SYM theory.

## 4.12 Problem Sets

(4.1) The Lagrangian for  $D = 10$  super-Yang-Mills theory (which is constructed to be invariant under  $\mathcal{N} = 1$  supersymmetry) is given by

$$\mathcal{L} = -\frac{1}{2g^2} \text{tr}(F_{\mu\nu} F^{\mu\nu} - 2i\bar{\lambda}\Gamma^\mu D_\mu \lambda) \quad (4.43)$$

The supersymmetry transformations are given by  $(\Gamma^{\mu\nu} \equiv \frac{1}{2}[\Gamma^\mu, \Gamma^\nu])$

$$\delta A_\mu = -i\bar{\zeta}\Gamma_\mu \lambda \quad \delta \lambda = \frac{1}{2} F_{\mu\nu} \Gamma^{\mu\nu} \zeta \quad (4.44)$$

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<sup>iii</sup>The discussion given here may be extended to  $Dp$  branes to some extent. However, when  $p \neq 3$ , the dilaton is not constant and the strength of the coupling will depend upon the distance to the brane.

for a Majorana-Weyl spinor gaugino  $\lambda$ . Show that under dimensional reduction on a flat 6-dimensional torus, (with periodic boundary conditions on all fields), the theory reduces to  $D = 4$ ,  $\mathcal{N} = 4$  super-Yang-Mills. Use this reduction to relate the matrices  $C_i$  in the Lagrangian for the  $D = 4$  theory to the Clifford Dirac matrices of  $SO(6)$ , and to derive the supersymmetry transformations of the theory.

(4.2) Assume the following Ansatz for a D3 brane solution to the Type IIB sugra field equations : constant dilaton  $\phi$ , vanishing axion  $C = 0$ , vanishing two-forms  $A_{2\mu\nu} = B_{\mu\nu} = 0$ ,  $F_{5\mu\nu\rho\sigma\tau} \sim \epsilon_{\mu\nu\rho\sigma\tau\nu} \partial^\nu H$  and metric of the form

$$ds^2 = H^{-\frac{1}{2}}(\vec{y}) dx^\mu dx_\mu + H^{\frac{1}{2}}(\vec{y}) d\vec{y}^2$$

Here,  $x^\mu$ ,  $\mu = 0, \dots, 3$  are the coordinates along the brane, while  $\vec{y} \in \mathbf{R}^6$  are the coordinates perpendicular to the brane. Show that the sugra equations hold provided  $H$  is harmonic in the transverse directions (i.e. satisfies  $\square_y H = 0$ , except at the position of the brane, where a pole will occur).

(4.3) Continuing with the set-up of (4.2), show that regularity of the solution requires the poles of  $H$  to have integer strength.

(4.4) Show that the D3 brane solution preserves 16 supersymmetries (i.e. half of the total number).

## 5 The Maldacena AdS/CFT Correspondence

In the preceding sections, we have provided descriptions of  $D = 4$ ,  $\mathcal{N} = 4$  super-Yang-Mills theory on the one hand and of D3 branes in supergravity and superstring theory on the other hand. We are now ready to exhibit the *Maldacena or near-horizon limit* close to the D3 branes and formulate precisely the Maldacena or AdS/CFT correspondence which conjectures the identity or duality between  $\mathcal{N} = 4$  SYM and Type IIB superstring theory on  $\text{AdS}_5 \times \text{S}^5$ . We shall also present the three different forms of the conjecture, the first being a correspondence with the full quantum string theory, the second being with classical string theory and finally the weakest form being with classical supergravity on  $\text{AdS}_5 \times \text{S}^5$ . In this section, the precise mapping between both sides of the conjecture will be made for the global symmetries as well as for the fields and operators. The mapping between the correlation functions will be presented in the next section. For a general review see [7]; see also [44] and [45].

### 5.1 Non-Abelian Gauge Symmetry on D3 branes

Open strings whose both end points are attached to a single brane can have arbitrarily short length and must therefore be massless. This excitation mode induces a massless  $U(1)$  gauge theory on the worldbrane which is effectively 4-dimensional flat space-time [46]. Since the brane breaks half of the total number of supersymmetries (it is 1/2 BPS), the  $U(1)$  gauge theory must have  $\mathcal{N} = 4$  Poincaré supersymmetry. In the low energy approximation (which has at most two derivatives on bosons and one derivative on fermions in this case), the  $\mathcal{N} = 4$  supersymmetric  $U(1)$  gauge theory is free.

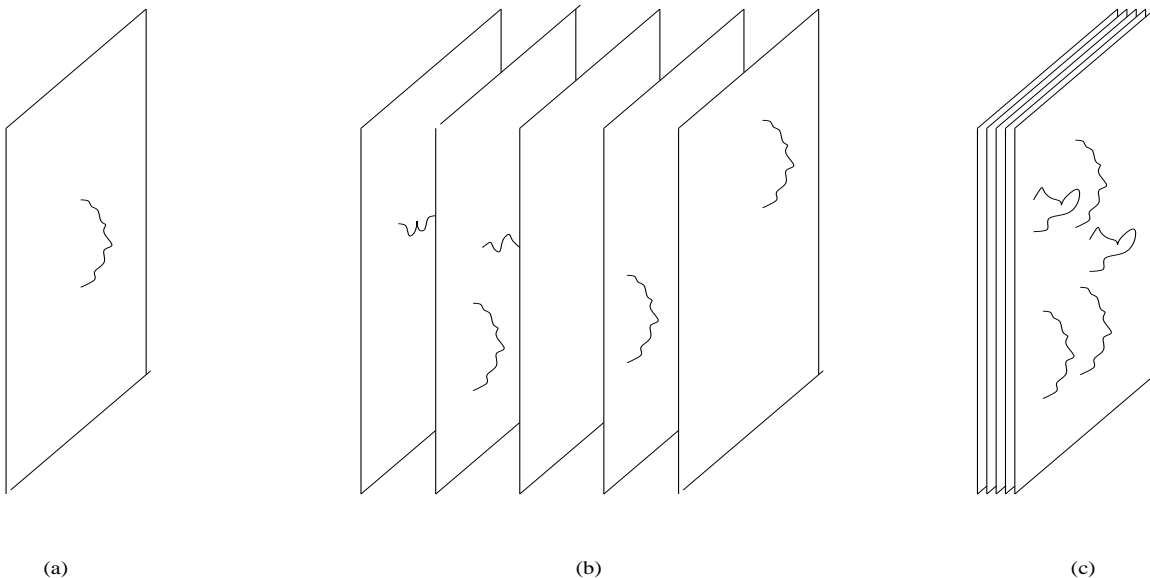


Figure 2: D-branes : (a) single, (b) well-separated, (c) (almost) coincident



With a number  $N > 1$  of parallel separated D3-branes, the end points of an open string may be attached to the same brane. For each brane, these strings can have arbitrarily small length and must therefore be massless. These excitation modes induce a massless  $U(1)^N$  gauge theory with  $\mathcal{N} = 4$  supersymmetry in the low energy limit. An open string can also, however, have one of its ends attached to one brane while the other end is attached to a different brane. The mass of such a string cannot get arbitrarily small since the length of the string is bounded from below by the separation distance between the branes (see however problem set (5.4)). There are  $N^2 - N$  such possible strings. In the limit where the  $N$  branes all tend to be coincident, all string states would be massless and the  $U(1)^N$  gauge symmetry is enhanced to a full  $U(N)$  gauge symmetry. Separating the branes should then be interpreted as Higgsing the gauge theory to the Coulomb branch where the gauge symmetry is spontaneously broken (generically to  $U(1)^N$ ). The overall  $U(1) = U(N)/SU(N)$  factor actually corresponds to the overall position of the branes and may be ignored when considering dynamics on the branes, thereby leaving only a  $SU(N)$  gauge symmetry [47]. These various configurations are depicted in Fig. 2.

In the low energy limit,  $N$  coincident branes support an  $\mathcal{N} = 4$  super-Yang-Mills theory in 4-dimensions with gauge group  $SU(N)$ .

## 5.2 The Maldacena limit

The space-time metric of  $N$  coincident D3-branes may be recast in the following form,<sup>iv</sup>

$$ds^2 = \left(1 + \frac{L^4}{y^4}\right)^{-\frac{1}{2}} \eta_{ij} dx^i dx^j + \left(1 + \frac{L^4}{y^4}\right)^{\frac{1}{2}} (dy^2 + y^2 d\Omega_5^2) \quad (5.1)$$

where the *radius*  $L$  of the D3-brane is given by

$$L^4 = 4\pi g_s N (\alpha')^2 \quad (5.2)$$

To study this geometry more closely, we consider its limit in two regimes.

As  $y \gg L$ , we recover flat space-time  $\mathbf{R}^{10}$ . When  $y < L$ , the geometry is often referred to as the *throat* and would at first appear to be singular as  $y \ll L$ . A redefinition of the coordinate

$$u \equiv L^2/y \quad (5.3)$$

and the large  $u$  limit, however, transform the metric into the following asymptotic form

$$ds^2 = L^2 \left[ \frac{1}{u^2} \eta_{ij} dx^i dx^j + \frac{du^2}{u^2} + d\Omega_5^2 \right] \quad (5.4)$$

which corresponds to a product geometry. One component is the five-sphere  $S^5$  with metric  $L^2 d\Omega_5^2$ . The remaining component is the hyperbolic space  $\text{AdS}_5$  with constant negative

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<sup>iv</sup>In this section, we shall denote 10-dimensional indices by  $M, N, \dots$ , 5-dimensional indices by  $\mu, \nu, \dots$  and 4-dimensional Minkowski indices by  $i, j, \dots$ , and the Minkowski metric by  $\eta_{ij} = \text{diag}(-+++)$ .

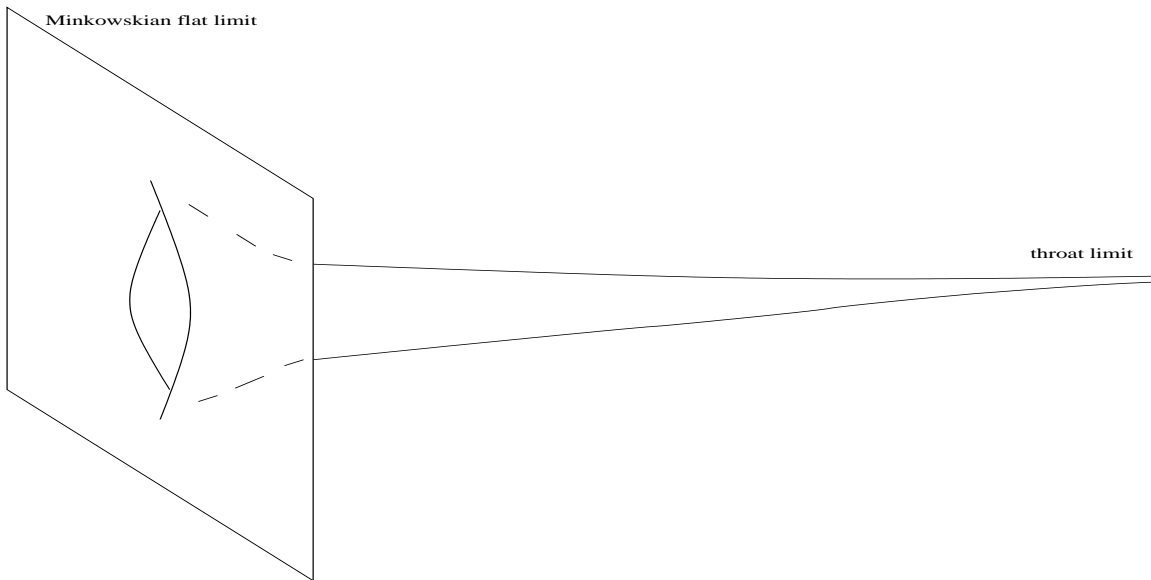


Figure 3: Minkowski region of AdS (a), and throat region of AdS (b)

curvature metric  $L^2 u^{-2} (du^2 + \eta_{ij} dx^i dx^j)$ . In conclusion, the geometry close to the brane ( $y \sim 0$  or  $u \sim \infty$ ) is regular and highly symmetrical, and may be summarized as  $\text{AdS}_5 \times \text{S}^5$  where both components have identical radius  $L$ .

The Maldacena limit [1] corresponds to keeping fixed  $g_s$  and  $N$  as well as all physical length scales, while letting  $\alpha' \rightarrow 0$ . Remarkably, this limit of string theory exists and is (very !) interesting. In the Maldacena limit, only the  $\text{AdS}_5 \times \text{S}^5$  region of the D3-brane geometry survives the limit and contributes to the string dynamics of physical processes, while the dynamics in the asymptotically flat region decouples from the theory.

To see this decoupling in an elementary way, consider a physical quantity, such as the effective action  $\mathcal{L}$  and carry out its  $\alpha'$  expansion in an arbitrary background with Riemann tensor, symbolically denoted by  $R$ . The expansion takes on the schematic form

$$\mathcal{L} = a_1 \alpha' R + a_2 (\alpha')^2 R^2 + a_3 (\alpha')^3 R^3 + \dots \quad (5.5)$$

Now physical objects and length scales in the asymptotically flat region are characterized by a scale  $y \gg L$ , so that by simple scaling arguments we have  $R \sim 1/y^2$ . Substitution this behavior into (5.5) yields the following expansion of the effective action,

$$\mathcal{L} = a_1 \alpha' \frac{1}{y^2} + a_2 (\alpha')^2 \frac{1}{y^4} + a_3 (\alpha')^3 \frac{1}{y^6} + \dots \quad (5.6)$$

Keeping the physical size  $y$  fixed, the entire contribution to the effective action from the limit  $\alpha' \rightarrow 0$  is then seen to vanish.

A more precise way of establishing this decoupling is by taking the Maldacena limit directly on the string theory non-linear sigma model in the D3 brane background. We shall

concentrate here on the metric part, thereby ignoring the contributions from the tensor field  $F_5^+$ . We denote the  $D = 10$  coordinates by  $x^M$ ,  $M = 0, 1, \dots, 9$ , and the metric by  $G_{MN}(x)$ . The first 4 coordinates coincide with  $x^\mu$  of the Poincaré invariant D3 worldvolume, while the coordinates on the 5-sphere are  $x^M$  for  $M = 5, \dots, 9$  and  $x^4 = u$ . The full D3 brane metric of (5.1) takes the form  $ds^2 = G_{MN}dx^M dx^N = L^2 \bar{G}_{MN}(x; L) dx^M dx^N$ , where the rescaled metric  $\bar{G}_{MN}$  is given by

$$\bar{G}_{MN}(x; L) dx^M dx^N = \left(1 + \frac{L^4}{u^4}\right)^{\frac{1}{2}} \left(\frac{du^2}{u^2} + d\Omega_5^2\right) + \left(1 + \frac{L^4}{u^4}\right)^{-\frac{1}{2}} \frac{1}{u^2} \eta_{ij} dx^i dx^j \quad (5.7)$$

Inserting this metric into the non-linear sigma model, we obtain

$$S_G = \frac{1}{4\pi\alpha'} \int_{\Sigma} \sqrt{\gamma} \gamma^{mn} G_{MN}(x) \partial_m x^M \partial_n x^N = \frac{L^2}{4\pi\alpha'} \int_{\Sigma} \sqrt{\gamma} \gamma^{mn} \bar{G}_{MN}(x; L) \partial_m x^M \partial_n x^N \quad (5.8)$$

The overall coupling constant for the sigma model dynamics is given by

$$\frac{L^2}{4\pi\alpha'} = \sqrt{\frac{\lambda}{4\pi}} \quad \lambda \equiv g_s N \quad (5.9)$$

Keeping  $g_s$  and  $N$  fixed but letting  $\alpha' \rightarrow 0$  implies that  $L \rightarrow 0$ . Under this limit the sigma model action admits a smooth limit, given by

$$S_G = \sqrt{\frac{\lambda}{4\pi}} \int_{\Sigma} \sqrt{\gamma} \gamma^{mn} \bar{G}_{MN}(x; 0) \partial_m x^M \partial_n x^N \quad (5.10)$$

where the metric  $\bar{G}_{MN}(x; 0)$  is the metric on  $\text{AdS}_5 \times \text{S}^5$ ,

$$\bar{G}_{MN}(x; L) dx^M dx^N = \frac{1}{u^2} \eta_{ij} dx^i dx^j + \frac{du^2}{u^2} + d\Omega_5^2 \quad (5.11)$$

rescaled to unit radius. Manifestly, the coupling  $1/\sqrt{\lambda}$  has taken over the role of  $\alpha'$  as the non-linear sigma model coupling constant and the radius  $L$  has cancelled out.

### 5.3 Geometry of Minkowskian and Euclidean AdS

Before moving on to the actual Maldacena conjecture, we clarify the geometry of AdS space-time, both with Minkowskian and Euclidean signatures. Minkowskian  $\text{AdS}_{d+1}$  (of unit radius) may be defined in  $\mathbf{R}^{d+1}$  with coordinates  $(Y_{-1}, Y_0, Y_1, \dots, Y_d)$  as the  $d + 1$  dimensional connected hyperboloid with isometry  $SO(2, d)$  given by the equation

$$-Y_{-1}^2 - Y_0^2 + Y_1^2 + \dots + Y_d^2 = -1 \quad (5.12)$$

with induced metric  $ds^2 = -dY_{-1}^2 - dY_0^2 + dY_1^2 + \dots + dY_d^2$ . The topology of the manifold is that of the cylinder  $S^1 \times \mathbf{R}$  times the sphere  $S^{d-1}$ , and is therefore not simply connected. The topology of the boundary is consequently given by  $\partial\text{AdS}_{d+1} = S^1 \times S^{d-1}$ . The manifold

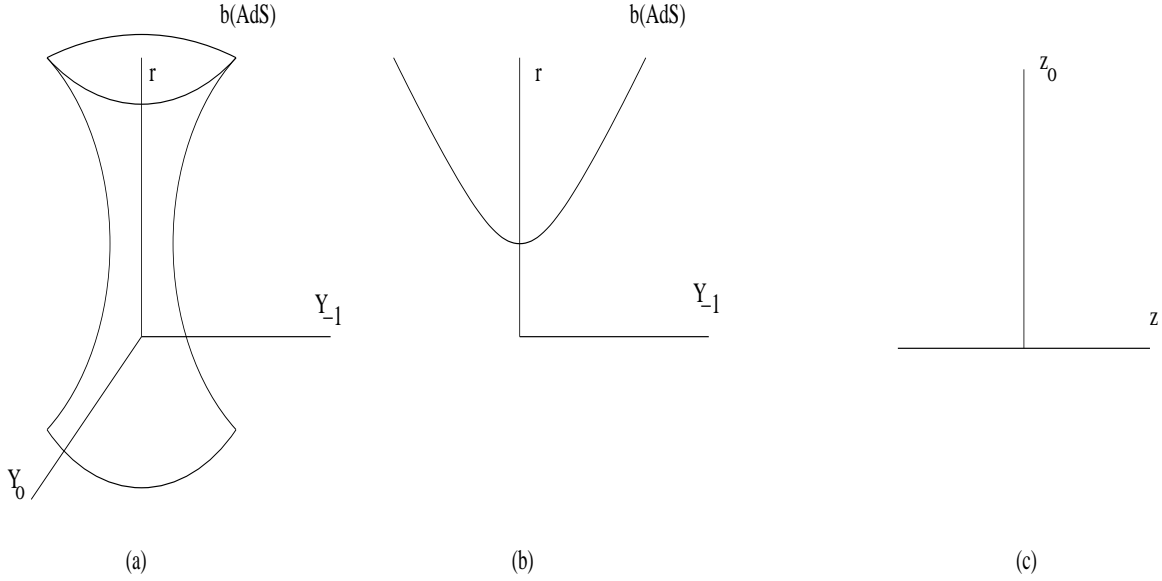


Figure 4: Anti-de Sitter Space (a) Euclidean, (b) Minkowskian, (c) upper half space

may be represented by the coset  $SO(2, d)/SO(1, d)$ . A schematic rendition of the manifold is given in Fig. 4 (a), with  $r^2 = Y_1^2 + \dots + Y_d^2$ .

Euclidean  $\text{AdS}_{d+1}$  (of unit radius) may be defined in Minkowski flat space  $\mathbf{R}^{d+1}$  with coordinates  $(Y_{-1}, Y_0, Y_1, \dots, Y_d)$  as the  $d + 1$  dimensional disconnected hyperboloid with isometry  $SO(1, d)$  given by the equation

$$-Y_{-1}^2 + Y_0^2 + Y_1^2 + \dots + Y_d^2 = -1 \quad (5.13)$$

with induced metric  $ds^2 = -dY_{-1}^2 + dY_0^2 + dY_1^2 + \dots + dY_d^2$ . The topology of the manifold is that of  $\mathbf{R}^{d+1}$ . The topology of the boundary is that of the  $d$ -sphere,  $\partial\text{AdS}_{d+1} = S^d$ . The manifold may be represented by the coset  $SO(1, d + 1)/SO(d + 1)$ . A schematic rendition of the manifold is given in Fig. 4 (b), with  $r^2 = Y_0^2 + Y_1^2 + \dots + Y_d^2$ . Introducing the coordinates  $Y_{-1} + Y_0 = \frac{1}{z_0}$  and  $z_i = z_0 Y_i$  for  $i = 1, \dots, d$ , we may map Euclidean  $\text{AdS}_{d+1}$  onto the upper half space  $H_{d+1}$  with Poincaré metric  $ds^2$ , defined by

$$H_{d+1} = \{(z_0, \vec{z}), z_0 \in \mathbf{R}^+, \vec{z} \in \mathbf{R}^d\} \quad ds^2 = \frac{1}{z_0^2}(dz_0^2 + d\vec{z}^2) \quad (5.14)$$

A schematic rendition is given in Fig. 4 (c). A standard stereographic transformation may be used to map  $H_{d+1}$  onto the unit ball.

## 5.4 The AdS/CFT Conjecture

The AdS/CFT or Maldacena conjecture states the *equivalence* (also referred to as *duality*) between the following theories [1]

- Type IIB superstring theory on  $\text{AdS}_5 \times S^5$  where both  $\text{AdS}_5$  and  $S^5$  have the same radius  $L$ , where the 5-form  $F_5^+$  has integer flux  $N = \int_{S^5} F_5^+$  and where the string coupling is  $g_s$ ;
- $\mathcal{N} = 4$  super-Yang-Mills theory in 4-dimensions, with gauge group  $SU(N)$  and Yang-Mills coupling  $g_{YM}$  in its (super)conformal phase;

with the following identifications between the parameters of both theories,

$$g_s = g_{YM}^2 \quad L^4 = 4\pi g_s N (\alpha')^2 \quad (5.15)$$

and the axion expectation value equals the SYM instanton angle  $\langle C \rangle = \theta_I$ . Precisely what is meant by *equivalence* or *duality* will be the subject of the remainder of this section, as well as of the next one. In brief, *equivalence* includes a precise map between the states (and fields) on the superstring side and the local gauge invariant operators on the  $\mathcal{N} = 4$  SYM side, as well as a correspondence between the correlators in both theories.

The above statement of the conjecture is referred to as the *strong form*, as it is to hold for all values of  $N$  and of  $g_s = g_{YM}^2$ . String theory quantization on a general curved manifold (including  $\text{AdS}_5 \times S^5$ ), however, appears to be very difficult and is at present out of reach. Therefore, it is natural to seek limits in which the Maldacena conjecture becomes more tractable but still remains non-trivial.

#### 5.4.1 The ‘t Hooft Limit

The ‘t Hooft limit consists in keeping the ‘t Hooft coupling  $\lambda \equiv g_{YM}^2 N = g_s N$  fixed and letting  $N \rightarrow \infty$ . In Yang-Mills theory, this limit is well-defined, at least in perturbation theory, and corresponds to a topological expansion of the field theory’s Feynman diagrams. On the AdS side, one may interpret the ‘t Hooft limit as follows. The string coupling may be re-expressed in terms of the ‘t Hooft coupling as  $g_s = \lambda/N$ . Since  $\lambda$  is being kept fixed, the ‘t Hooft limit corresponds to *weak coupling string perturbation theory*.

This form of the conjecture, though weaker than the original version is still a very powerful correspondence between classical string theory and the large  $N$  limit of gauge theories. The problem of finding an action built out of classical fields to which the large  $N$  limit of gauge theories are classical solutions is a challenge that had been outstanding since ‘t Hooft’s original paper [8]. The above correspondence gives a concrete, though still ill-understood, realization of this “large  $N$  master-equation”.

#### 5.4.2 The Large $\lambda$ Limit

In taking the ‘t Hooft limit,  $\lambda = g_s N$  is kept fixed while  $N \rightarrow \infty$ . Once this limit has been taken, the only parameter left is  $\lambda$ . Quantum field theory perturbation theory corresponds to  $\lambda \ll 1$ . On the AdS side of the correspondence, it is actually natural to take  $\lambda \gg 1$  instead. It is very instructive to establish the meaning of an expansion around  $\lambda$  large.

<ul style="list-style-type: none"> <li>• <math>\mathcal{N} = 4</math> conformal SYM</li> <li>all <math>N, g_{YM}</math></li> <li>• <math>g_s = g_{YM}^2</math></li> </ul>	$\Leftrightarrow$	<ul style="list-style-type: none"> <li>• Full Quantum Type IIB string theory on <math>\text{AdS}_5 \times S^5</math></li> <li>• <math>L^4 = 4\pi g_s N \alpha'^2</math></li> </ul>
<ul style="list-style-type: none"> <li>• 't Hooft limit of <math>\mathcal{N} = 4</math> SYM</li> <li><math>\lambda = g_{YM}^2 N</math> fixed, <math>N \rightarrow \infty</math></li> <li>• <math>1/N</math> expansion</li> </ul>	$\Leftrightarrow$	<ul style="list-style-type: none"> <li>• Classical Type IIB string theory on <math>\text{AdS}_5 \times S^5</math></li> <li>• <math>g_s</math> string loop expansion</li> </ul>
<ul style="list-style-type: none"> <li>• Large <math>\lambda</math> limit of <math>\mathcal{N} = 4</math> SYM (for <math>N \rightarrow \infty</math>)</li> <li>• <math>\lambda^{-1/2}</math> expansion</li> </ul>	$\Leftrightarrow$	<ul style="list-style-type: none"> <li>• Classical Type IIB supergravity on <math>\text{AdS}_5 \times S^5</math></li> <li>• <math>\alpha'</math> expansion</li> </ul>

Table 5: The three forms of the AdS/CFT conjecture in order of decreasing strength

To do, we expand in powers of  $\alpha'$  a physical quantity such as the effective action, as we already did in (5.5),

$$\mathcal{L} = a_1 \alpha' R + a_2 (\alpha')^2 R^2 + a_3 (\alpha')^3 R^3 + \dots \quad (5.16)$$

The distance scales in which we are now interested are those typical of the throat, whose scale is set by the AdS radius  $L$ . Thus, the scale of the Riemann tensor is set by

$$R \sim 1/L^2 = (g_s N)^{-\frac{1}{2}} / \alpha' = \lambda^{-\frac{1}{2}} / \alpha' \quad (5.17)$$

and therefore, the expansion of the effective action in powers of  $\alpha'$  effectively becomes an expansion in powers of  $\lambda^{-\frac{1}{2}}$ ,

$$\mathcal{L} = a_1 \lambda^{-\frac{1}{2}} + a_2 \lambda^{-1} + a_3 (\alpha')^3 \lambda^{-\frac{3}{2}} + \dots \quad (5.18)$$

The interchange of the roles of  $\alpha'$  and  $\lambda^{-1/2}$  may also be seen directly from the worldsheet non-linear sigma model action of (5.10). Clearly, any  $\alpha'$  dependence has disappeared from the string theory problem and the role of  $\alpha'$  as a scale has been replaced by the parameter  $\lambda^{-1/2}$ .

## 5.5 Mapping Global Symmetries

A key necessary ingredient for the AdS/CFT correspondence to hold is that the global unbroken symmetries of the two theories be identical. The continuous global symmetry of  $\mathcal{N} = 4$  super-Yang-Mills theory in its conformal phase was previously shown to be the superconformal group  $SU(2, 2|4)$ , whose maximal bosonic subgroup is  $SU(2, 2) \times SU(4)_R \sim SO(2, 4) \times SO(6)_R$ . Recall that the bosonic subgroup arises as the product of the conformal group  $SO(2, 4)$  in 4-dimensions by the  $SU(4)_R$  automorphism group of the  $\mathcal{N} = 4$  Poincaré supersymmetry algebra. This bosonic group is immediately recognized on the AdS side as the isometry group of the  $\text{AdS}_5 \times S^5$  background. The completion into the full supergroup  $SU(2, 2|4)$  was discussed for the SYM theory in subsection §3.3, and arises on the AdS side

because 16 of the 32 Poincaré supersymmetries are preserved by the array of  $N$  parallel D3-branes, and in the AdS limit, are supplemented by another 16 conformal supersymmetries (which are broken in the full D3-brane geometry). Thus, the global symmetry  $SU(2, 2|4)$  matches on both sides of the AdS/CFT correspondence.

$\mathcal{N} = 4$  super-Yang-Mills theory also has Montonen-Olive or S-duality symmetry, realized on the complex coupling constant  $\tau$  by Möbius transformations in  $SL(2, \mathbf{Z})$ . On the AdS side, this symmetry is a global discrete symmetry of Type IIB string theory, which is unbroken by the D3-brane solution, in the sense that it maps non-trivially only the dilaton and axion expectation values, as was shown earlier. Thus, S-duality is also a symmetry of the AdS side of the AdS/CFT correspondence. It must be noted, however, that S-duality is a useful symmetry only in the strongest form of the AdS/CFT conjecture. As soon as one takes the 't Hooft limit  $N \rightarrow \infty$  while keeping  $\lambda = g_{YM}^2 N$  fixed, S-duality no longer has a consistent action. This may be seen for  $\theta_I = 0$ , where it maps  $g_{YM} \rightarrow 1/g_{YM}$  and thus  $\lambda \rightarrow N^2/\lambda$ .

## 5.6 Mapping Type IIB Fields and CFT Operators

Given that we have established that the global symmetry groups on both sides of the AdS/CFT correspondence coincide, it remains to show that the actual representations of the supergroup  $SU(2, 2|4)$  also coincide on both sides. The spectrum of operators on the SYM side was explained already in subsection §3.5. Suffice it to recall here the special significance of the short multiplet representations, namely 1/2 BPS representations with a span of spin 2, 1/4 BPS representations with a span of spin 3 and 1/8 BPS representations with a span of spin 7/2. Non-BPS representations in general have a span of spin 4.

A special role is played by the *single color trace operators* because out of them, all higher trace operators may be constructed using the OPE. Thus one should expect single trace operators on the SYM side to correspond to single particle states (or canonical fields) on the AdS side [1]; see also [48]. Multiple trace states should then be interpreted as bound states of these one particle states. Multiple trace BPS operators have the property that their dimension on the AdS side is simply the sum of the dimensions of the BPS constituents. Such bound states occur in the spectrum at the lower edge of the continuum threshold and are therefore called *threshold bound states*. A good example to keep in mind when thinking of threshold bound states in ordinary quantum field theory is another case of BPS objects : magnetic monopoles [49] in the Prasad-Sommerfield limit [14] (or exactly in the Coulomb phase of  $\mathcal{N} = 4$  SYM). A collection of  $N$  magnetic monopoles with like charges forms a static solution of the BPS equations and therefore form a threshold bound state. Very recently, a direct coupling of double-trace operators to AdS supergravity has been studied in [50].

To identify the contents of irreducible representations of  $SU(2, 2|4)$  on the AdS side, we describe all Type IIB massless supergravity and massive string degrees of freedom by fields  $\varphi$  living on  $AdS_5 \times S^5$ . We introduce coordinates  $z^\mu$ ,  $\mu = 0, 1, \dots, 4$  for  $AdS_5$  and  $y^u$ ,

$u = 1, \dots, 5$  for  $S^5$ , and decompose the metric as

$$ds^2 = g_{\mu\nu}^{AdS} dz^\mu dz^\nu + g_{uv}^S dy^u dy^v \quad (5.19)$$

The fields then become functions  $\varphi(z, y)$  associated with the various  $D = 10$  degrees of freedom. It is convenient to decompose  $\varphi(z, y)$  in a series on  $S^5$ ,

$$\varphi(z, y) = \sum_{\Delta=0}^{\infty} \varphi_{\Delta}(z) Y_{\Delta}(y) \quad (5.20)$$

where  $Y_{\Delta}$  stands for a basis of spherical harmonics on  $S^5$ . For scalars for example,  $Y_{\Delta}$  are labelled by the rank  $\Delta$  of the totally symmetric traceless representations of  $SO(6)$ . Just as fields on a circle received a mass contribution from the momentum mode on the circle, so also do fields compactified on  $S^5$  receive a contribution to the mass. From the eigenvalues of the Laplacian on  $S^5$ , for various spins, we find the following relations between mass and scaling dimensions,

scalars	$m^2 = \Delta(\Delta - 4)$	
spin 1/2, 3/2	$ m  = \Delta - 2$	
$p - \text{form}$	$m^2 = (\Delta - p)(\Delta + p - 4)$	
spin2	$m^2 = \Delta(\Delta - 4)$	(5.21)

The complete correspondence between the representations of  $SU(2, 2|4)$  on both sides of the correspondence is given in Table 6. The mapping of the descendant states is also very interesting. For the  $D = 10$  supergravity multiplet, this was worked out in [51], and is given in Table 7. Generalizations to  $AdS_4 \times S^7$  were discussed in [52, 53, 54] while those to  $AdS_7 \times S^4$  were discussed in [55, 56], with recent work on AdS/CFT for M-theory on these spaces in [57, 58, 59, 60]. General reviews may be found in [61], [62]. Recently, conjectures involving also de Sitter space-times have been put forward in [63] and references therein. Finally, we point out that the existence of singleton and doubleton representations of the conformal group  $SO(2,4)$  is closely related with the AdS/CFT correspondence; for recent accounts, see [64], [65], [66] and [67] and references therein. Additional references on the (super)symmetries of AdS are in [68], [69] and [70].

## 5.7 Problem Sets

(5.1) The Poincaré upper half space is defined by  $H_{d+1} = \{(z_0, \vec{z}) \in \mathbf{R}^{d+1}, z_0 > 0\}$  with metric  $ds^2 = (dz_0^2 + d\vec{z}^2)/z_0^2$ . (a) Show – by solving the geodesic equations – that the geodesics of  $H_{d+1}$  are the half-circles of arbitrary radius  $R$ , centered at an arbitrary point  $(0, \vec{c})$  on the boundary of  $H_{d+1}$ . Compute the geodesic distance between any two arbitrary points.

(5.2) We now represent Euclidean  $AdS_{d+1}$  as the manifold in  $\mathbf{R}^{d+2}$  given by the equation  $-Y_{-1}^2 + Y_0^2 + \vec{Y}^2 = -1$ , with induced metric  $ds^2 = -dY_{-1}^2 + dY_0^2 + d\vec{Y}^2$ . Show that the



Type IIB string theory	$\mathcal{N} = 4$ conformal super-Yang-Mills
Supergravity Excitations 1/2 BPS, spin $\leq 2$	Chiral primary + descendants $\mathcal{O}_2 = \text{tr} X^{\{i} X^{j\}} + \text{desc.}$
Supergravity Kaluza-Klein 1/2 BPS, spin $\leq 2$	Chiral primary + Descendants $\mathcal{O}_\Delta = \text{tr} X^{\{i_1 \dots X^{i_\Delta\}} + \text{desc.}$
Type IIB massive string modes non-chiral, long multiplets	Non-Chiral operators, dimensions $\sim \lambda^{1/4}$ e.g. Konishi $\text{tr} X^i X^i$
Multiparticle states	products of operators at distinct points $\mathcal{O}_{\Delta_1}(x_1) \dots \mathcal{O}_{\Delta_n}(x_n)$
Bound states	product of operators at same point $\mathcal{O}_{\Delta_1}(x) \dots \mathcal{O}_{\Delta_n}(x)$

Table 6: Mapping of String and Sugra states onto SYM Operators

geodesics found in problem (5.1) above are simply the sections by planes through the origin, given by the equation

$$Y_{-1} - Y_0 = (R^2 - \vec{c}^2)(Y_{-1} + Y_0) + 2\vec{c} \cdot \vec{Y}$$

(You may wish to explore the analogy with the geometry and geodesics of the sphere  $S^{d+1}$ .)

(5.3) The geodesic distance between two separate D3 branes is actually infinite, as may be seen by integrating the infinitesimal distance  $ds$  of the D3 metric. Using the worldsheet action of a string suspended between the two D3 branes, explain why this string still has a finite mass spectrum.

(5.4) Consider a classical bosonic string in  $\text{AdS}_{d+1}$  space-time, with its dynamics governed by the Polyakov action, namely in the presence of the  $\text{AdS}_{d+1}$  metric  $G_{\mu\nu}(x)$ . (We ignore the anti-symmetric tensor fields for simplicity.)

$$S[x] = \int_{\Sigma} d^2\xi \sqrt{\gamma} \gamma^{mn} \partial_m x^\mu \partial_n x^\nu G_{\mu\nu}(x)$$

Solve the string equations assuming a special Ansatz that the solution be spherically symmetric, i.e. invariant under the  $SO(d)$  subgroup of  $SO(2, d)$ .

SYM Operator	desc	SUGRA	dim	spin	$Y$	$SU(4)_R$	lowest reps
$\mathcal{O}_k \sim \text{tr} X^k, k \geq 2$	–	$h_\alpha^\alpha a_{\alpha\beta\gamma\delta}$	$k$	$(0, 0)$	0	$(0, k, 0)$	<b>20', 50, 105</b>
$\mathcal{O}_k^{(1)} \sim \text{tr} \lambda X^k, k \geq 1$	$Q$	$\psi_{(\alpha)}$	$k + \frac{3}{2}$	$(\frac{1}{2}, 0)$	$\frac{1}{2}$	$(1, k, 0)$	<b>20, 60, 140'</b>
$\mathcal{O}_k^{(2)} \sim \text{tr} \lambda \lambda X^k$	$Q^2$	$A_{\alpha\beta}$	$k + 3$	$(0, 0)$	1	$(2, k, 0)$	<b>10<sub>c</sub>, 45<sub>c</sub>, 126<sub>c</sub></b>
$\mathcal{O}_k^{(3)} \sim \text{tr} \lambda \bar{\lambda} X^k$	$Q\bar{Q}$	$h_{\mu\alpha} a_{\mu\alpha\beta\gamma}$	$k + 3$	$(\frac{1}{2}, \frac{1}{2})$	0	$(1, k, 1)$	<b>15, 64, 175</b>
$\mathcal{O}_k^{(4)} \sim \text{tr} F_+ X^k, k \geq 1$	$Q^2$	$A_{\mu\nu}$	$k + 2$	$(1, 0)$	1	$(0, k, 0)$	<b>6<sub>c</sub>, 20<sub>c</sub>, 50<sub>c</sub></b>
$\mathcal{O}_k^{(5)} \sim \text{tr} F_+ \bar{\lambda} X^k$	$Q^2\bar{Q}$	$\psi_\mu$	$k + \frac{7}{2}$	$(1, \frac{1}{2})$	$\frac{1}{2}$	$(0, k, 1)$	<b>4*, 20*, 60*</b>
$\mathcal{O}_k^{(6)} \sim \text{tr} F_+ \lambda X^k$	$Q^3$	“ $\lambda$ ”	$k + \frac{7}{2}$	$(\frac{1}{2}, 0)$	$\frac{3}{2}$	$(1, k, 0)$	<b>4, 20, 60</b>
$\mathcal{O}_k^{(7)} \sim \text{tr} \lambda \lambda \bar{\lambda} X^k$	$Q^2\bar{Q}$	$\psi_{(\alpha)}$	$k + \frac{9}{2}$	$(0, \frac{1}{2})$	$\frac{1}{2}$	$(2, k, 1)$	<b>36, 140, 360</b>
$\mathcal{O}_k^{(8)} \sim \text{tr} F_+^2 X^k$	$Q^4$	$B$	$k + 4$	$(0, 0)$	2	$(0, k, 0)$	<b>1<sub>c</sub>, 6<sub>c</sub>, 20'<sub>c</sub></b>
$\mathcal{O}_k^{(9)} \sim \text{tr} F_+ F_- X^k$	$Q^2\bar{Q}^2$	$h'_{\mu\nu}$	$k + 4$	$(1, 1)$	0	$(0, k, 0)$	<b>1, 6, 20'</b>
$\mathcal{O}_k^{(10)} \sim \text{tr} F_+ \lambda \bar{\lambda} X^k$	$Q^3\bar{Q}$	$A_{\mu\alpha}$	$k + 5$	$(\frac{1}{2}, \frac{1}{2})$	1	$(1, k, 1)$	<b>15, 64, 175</b>
$\mathcal{O}_k^{(11)} \sim \text{tr} F_+ \bar{\lambda} \bar{\lambda} X^k$	$Q^2\bar{Q}^2$	$a_{\mu\nu\alpha\beta}$	$k + 5$	$(1, 0)$	0	$(0, k, 2)$	<b>10<sub>c</sub>, 45<sub>c</sub>, 126<sub>c</sub></b>
$\mathcal{O}_k^{(12)} \sim \text{tr} \lambda \lambda \bar{\lambda} \bar{\lambda} X^k$	$Q^2\bar{Q}^2$	$h_{(\alpha\beta)}$	$k + 6$	$(0, 0)$	0	$(2, k, 2)$	<b>84, 300, 2187</b>
$\mathcal{O}_k^{(13)} \sim \text{tr} F_+^2 \bar{\lambda} X^k$	$Q^4\bar{Q}$	“ $\lambda$ ”	$k + \frac{11}{2}$	$(0, \frac{1}{2})$	$\frac{3}{2}$	$(0, k, 1)$	<b>4*, 20*, 60*</b>
$\mathcal{O}_k^{(14)} \sim \text{tr} F_+ \lambda \bar{\lambda} \bar{\lambda} X^k$	$Q^3\bar{Q}^2$	$\psi_{(\alpha)}$	$k + \frac{13}{2}$	$(\frac{1}{2}, 0)$	$\frac{1}{2}$	$(1, k, 2)$	<b>36*, 140*, 360*</b>
$\mathcal{O}_k^{(15)} \sim \text{tr} F_+ F_- \lambda X^k$	$Q^3\bar{Q}^2$	$\psi_\mu$	$k + \frac{11}{2}$	$(\frac{1}{2}, 1)$	$\frac{1}{2}$	$(1, k, 0)$	<b>4, 20, 60</b>
$\mathcal{O}_k^{(16)} \sim \text{tr} F_+ F_-^2 X^k$	$Q^4\bar{Q}^2$	$A_{\mu\nu}$	$k + 6$	$(1, 0)$	1	$(0, k, 0)$	<b>1<sub>c</sub>, 6<sub>c</sub>, 20'<sub>c</sub></b>
$\mathcal{O}_k^{(17)} \sim \text{tr} F_+ F_- \lambda \bar{\lambda} X^k$	$Q^3\bar{Q}^3$	$h_{\mu\alpha} a_{\mu\alpha\beta\gamma}$	$k + 7$	$(\frac{1}{2}, \frac{1}{2})$	0	$(1, k, 1)$	<b>15, 64, 175</b>
$\mathcal{O}_k^{(18)} \sim \text{tr} F_+^2 \bar{\lambda} \bar{\lambda} X^k$	$Q^4\bar{Q}^2$	$A_{\alpha\beta}$	$k + 7$	$(0, 0)$	1	$(0, k, 2)$	<b>10<sub>c</sub>, 45<sub>c</sub>, 126<sub>c</sub></b>
$\mathcal{O}_k^{(19)} \sim \text{tr} F_+^2 F_- \bar{\lambda} X^k$	$Q^4\bar{Q}^3$	$\psi_{(\alpha)}$	$k + \frac{15}{2}$	$(0, \frac{1}{2})$	$\frac{1}{2}$	$(0, k, 1)$	<b>4*, 20*, 60*</b>
$\mathcal{O}_k^{(20)} \sim \text{tr} F_+^2 F_-^2 X^k$	$Q^4\bar{Q}^4$	$h_\alpha^\alpha a_{\alpha\beta\gamma\delta}$	$k + 8$	$(0, 0)$	0	$(0, k, 0)$	<b>1, 6, 20'</b>

Table 7: Super-Yang-Mills Operators, Supergravity Fields and  $SO(2, 4) \times U(1)_Y \times SU(4)_R$  Quantum Numbers. The range of  $k$  is  $k \geq 0$ , unless otherwise specified.

## 6 AdS/CFT Correlation Functions

In the preceding section, evidence was presented for the Maldacena correspondence between  $\mathcal{N} = 4$  super-conformal Yang-Mills theory with  $SU(N)$  gauge group and Type IIB superstring theory on  $\text{AdS}_5 \times S^5$ . The evidence was based on the precise matching of the global symmetry group  $SU(2, 2|4)$ , as well as of the specific representations of this group. In particular, the single trace 1/2 BPS operators in the SYM theory matched in a one-to-one way with the canonical fields of supergravity, compactified on  $\text{AdS}_5 \times S^5$ . In the present section, we present a more detailed version of the AdS/CFT correspondence by mapping the correlators on both sides of the correspondence.

### 6.1 Mapping Super Yang-Mills and AdS Correlators

We work with Euclidean  $\text{AdS}_5$ , or  $H = \{(z_0, \vec{z}), z_0 > 0, \vec{z} \in \mathbf{R}^4\}$  with Poincaré metric  $ds^2 = z_0^{-2}(dz_0^2 + d\vec{z}^2)$ , and boundary  $\partial H = \mathbf{R}^4$ . (Often, this space will be graphically represented as a disc, whose boundary is a circle; see Fig. 5.) The metric diverges at the boundary  $z_0 = 0$ , because the overall scale factor blows up there. This scale factor may be removed by a Weyl rescaling of the metric, but such rescaling is not unique. *A unique well-defined limit to the boundary of  $\text{AdS}_5$  can only exist if the boundary theory is scale invariant* [3]. For finite values of  $z_0 > 0$ , the geometry will still have 4-dimensional Poincaré invariance but need not be scale invariant.

Superconformal  $\mathcal{N} = 4$  Yang-Mills theory is scale invariant and may thus consistently live at the boundary  $\partial H$ . The dynamical observables of  $\mathcal{N} = 4$  SYM are the local gauge invariant polynomial operators described in section 3; they naturally live on the boundary  $\partial H$ , and are characterized by their dimension, Lorentz group  $SO(1, 3)$  and  $SU(4)_R$  quantum numbers [3].

On the AdS side, we shall decompose all 10-dimensional fields onto Kaluza-Klein towers on  $S^5$ , so that effectively all fields  $\varphi_\Delta(z)$  are on  $\text{AdS}_5$ , and labeled by their dimension  $\Delta$  (other quantum number are implicit). Away from the bulk interaction region, it is assumed that the bulk fields are free asymptotically (just as this is assumed in the derivation of the LSZ formalism in flat space-time quantum field theory). The free field then satisfies  $(\square + m_\Delta^2)\varphi_\Delta^0 = 0$  with  $m_\Delta^2 = \Delta(\Delta - 4)$  for scalars. The two independent solutions are characterized by the following asymptotics as  $z_0 \rightarrow 0$ ,

$$\varphi_\Delta^0(z_0, \vec{z}) = \begin{cases} z_0^\Delta & \text{normalizable} \\ z_0^{4-\Delta} & \text{non-normalizable} \end{cases} \quad (6.1)$$

Returning to the interacting fields in the fully interacting theory, solutions will have the same asymptotic behaviors as in the free case. It was argued in [71] that the normalizable modes determine the vacuum expectation values of operators of associated dimensions and quantum numbers. The non-normalizable solutions on the other hand do not correspond to bulk excitations because they are not properly square normalizable. Instead, they represent

the coupling of external sources to the supergravity or string theory. The precise correspondence is as follows [3]. The non-normalizable solutions  $\varphi_\Delta$  define *associated boundary fields*  $\bar{\varphi}_\Delta$  by the following relation

$$\bar{\varphi}_\Delta(\vec{z}) \equiv \lim_{z_0 \rightarrow 0} \varphi_\Delta(z_0, \vec{z}) z_0^{4-\Delta} \quad (6.2)$$

Given a set of boundary fields  $\bar{\varphi}_\Delta(\vec{z})$ , it is assumed that a complete and unique bulk solution to string theory exists. We denote the fields of the associated solution  $\varphi_\Delta$ .

The mapping between the correlators in the SYM theory and the dynamics of string theory is given as follows [3, 2]. First, we introduce a generating functional  $\Gamma[\bar{\varphi}_\Delta]$  for all the correlators of single trace operators  $\mathcal{O}_\Delta$  on the SYM side in terms of the source fields  $\bar{\varphi}_\Delta$ ,

$$\exp\{-\Gamma[\bar{\varphi}_\Delta]\} \equiv \langle \exp\left\{ \int_{\partial H} \bar{\varphi}_\Delta \mathcal{O}_\Delta \right\} \rangle \quad (6.3)$$

This expression is understood to hold order by order in a perturbative expansion in the number of fields  $\bar{\varphi}_\Delta$ . On the AdS side, we assume that we have an action  $S[\varphi_\Delta]$  that summarizes the dynamics of Type IIB string theory on  $\text{AdS}_5 \times S^5$ . In the supergravity approximation,  $S[\varphi_\Delta]$  is just the Type IIB supergravity action on  $\text{AdS}_5 \times S^5$ . Beyond the supergravity approximation,  $S[\varphi_\Delta]$  will also include  $\alpha'$  corrections due to massive string effects. The mapping between the correlators is given by

$$\Gamma[\bar{\varphi}_\Delta] = \text{extr } S[\varphi_\Delta] \quad (6.4)$$

where the extremum on the rhs is taken over all fields  $\varphi_\Delta$  that satisfy the asymptotic behavior (6.2) for the boundary fields  $\bar{\varphi}_\Delta$  that are the sources to the SYM operators  $\mathcal{O}_\Delta$  on the lhs. Additional references on the field-state-operator mapping may be found in [72], [73], [74], [77], [75] and [76].

## 6.2 Quantum Expansion in $1/N$ – Witten Diagrams

The actions of interest to us will have an overall coupling constant factor. For example, the part of the Type IIB supergravity action for the dilaton  $\Phi$  and the axion  $C$  in the presence of a metric  $G_{\mu\nu}$  in the Einstein frame, is given by

$$S[G, \Phi, C] = \frac{1}{2\kappa_5^2} \int_H \sqrt{G} \left[ -R_G + \Lambda + \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2} e^{2\Phi} \partial_\mu C \partial^\mu C \right] \quad (6.5)$$

and the 5-dimensional Newton constant  $\kappa_5^2$  is given by  $\kappa_5^2 = 4\pi^2/N^2$ , a relation that will be explained and justified in (8.3). For large  $N$ ,  $\kappa_5$  will be small and one may perform a small  $\kappa_5$ , i.e. a semi-classical expansion of the correlators generated by this action. The result is a set of rules, analogous to Feynman rules, which may be summarized by *Witten diagrams*. The Witten diagram is represented by a disc, whose interior corresponds to the interior of AdS while the boundary circle corresponds to the boundary of AdS [3]. The graphical rules are as follows,

- Each external source to  $\bar{\varphi}_\Delta(\vec{x}_I)$  is located at the boundary circle of the Witten diagram at a point  $\vec{x}_I$ .
- From the external source at  $\vec{x}_I$  departs a propagator to either another boundary point, or to an interior interaction point via a *boundary-to-bulk propagator*.
- The structure of the interior interaction points is governed by the interaction vertices of the action  $S$ , just as in Feynman diagrams.
- Two interior interaction points may be connected by *bulk-to-bulk propagators*, again following the rules of ordinary Feynman diagrams.

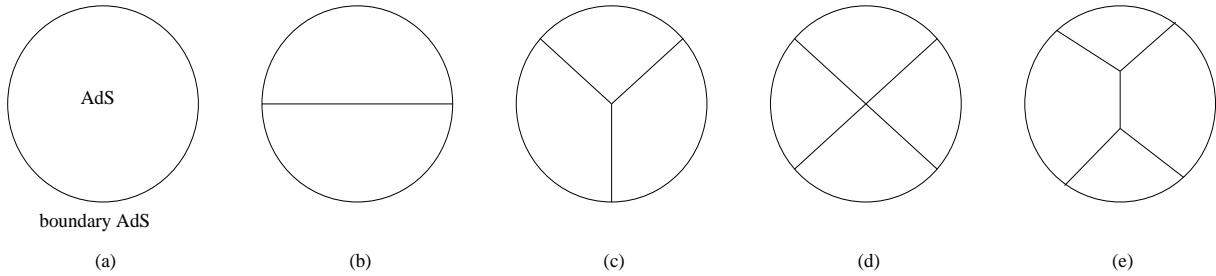


Figure 5: Witten diagrams (a) empty, (b) 2-pt, (c) 3-pt, (d) 4-pt contact, (e) exchange

Tree-level 2-, 3- and 4-point function contributions are given as an example in figure 5. The approach that will be taken here is based on the component formulation of sugra. It is possible however to make progress directly in superspace [78], but we shall not discuss this here.

### 6.3 AdS Propagators

We shall define and list the solution for the propagators of general scalar fields, of massless gauge fields and massless gravitons. The propagators are considered in Euclidean  $\text{AdS}_{d+1}$ , a space that we shall denote by  $H$ . Recall that the Poincaré metric is given by

$$ds^2 = g_{\mu\nu} dz^\mu dz^\nu = z_0^{-2} (dz_0^2 + d\vec{z}^2) \quad (6.6)$$

Here, we have set the  $\text{AdS}_{d+1}$  radius to unity. By  $SO(1, d + 1)$  isometry of  $H$ , the Green functions essentially depend upon the  $SO(1, d + 1)$ -invariant distance between two points in  $H$ . The *geodesic distance* is given by (see problem 5.1)

$$d(z, w) = \int_w^z ds = \ln\left(\frac{1 + \sqrt{1 - \xi^2}}{\xi}\right) \quad \xi \equiv \frac{2z_0 w_0}{z_0^2 + w_0^2 + (\vec{z} - \vec{w})^2} \quad (6.7)$$

Given its algebraic dependence on the coordinates, it is more convenient to work with the object  $\xi$  than with the geodesic distance. The *chordal distance* is given by  $u = \xi^{-1} - 1$ . The distance relation may be inverted to give  $u = \xi^{-1} - 1 = \cosh d - 1$ .

## The massive scalar bulk-to-bulk propagator

Let  $\varphi_\Delta(z)$  be a scalar field of conformal weight  $\Delta$  and mass<sup>2</sup>  $m^2 = \Delta(\Delta - d)$  whose linearized dynamics is given by a coupling to a scalar source  $J$  via the action

$$S_{\varphi_\Delta} = \int_H d^{d+1}z \sqrt{g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi_\Delta \partial_\nu \varphi_\Delta + \frac{1}{2} m^2 \varphi_\Delta^2 - \varphi_\Delta J \right] \quad (6.8)$$

The field is then given in response to the source by

$$\varphi_\Delta(z) = \int_H d^{d+1}z' \sqrt{g} G_\Delta(z, z') J(z') \quad (6.9)$$

where the scalar Green function satisfies the differential equation

$$(\square_g + m^2)G_\Delta(z, z') = \delta(z, z') \quad \delta(z, z') \equiv \frac{1}{\sqrt{g}} \delta(z - z') \quad (6.10)$$

The (positive) scalar Laplacian is given by

$$\square_g = -\frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu = -z_0^2 \partial_0^2 + (d-1)z_0 \partial_0 - z_0^2 \sum_{i=1}^d \partial_i^2 \quad (6.11)$$

The scalar Green function is the solution to a hypergeometric equation, given by [85],<sup>v</sup>

$$G_\Delta(z, w) = G_\Delta(\xi) = \frac{2^{-\Delta} C_\Delta}{2\Delta - d} \xi^\Delta F\left(\frac{\Delta}{2}, \frac{\Delta}{2} + \frac{1}{2}; \Delta - \frac{d}{2} + 1; \xi^2\right) \quad (6.12)$$

where the overall normalization constant is defined by

$$C_\Delta = \frac{\Gamma(\Delta)}{\pi^{d/2} \Gamma(\Delta - \frac{d}{2})} \quad (6.13)$$

Since  $0 \leq \xi \leq 1$ , the hypergeometric function is defined by its convergent Taylor series for all  $\xi$  except at the coincident point  $\xi = 1$  where  $z = w$ .

## The massive scalar boundary-to-bulk propagator

An important limiting case of the scalar bulk-to-bulk propagator is when the source is on the boundary of  $H$ . The action to linearized order is given by

$$S_{\varphi_\Delta} = \int_H d^{d+1}z \sqrt{g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi_\Delta \partial_\nu \varphi_\Delta + \frac{1}{2} m^2 \varphi_\Delta^2 \right] - \int_{\partial H} d^d \vec{z} \bar{\varphi}_\Delta(\vec{z}) \bar{J}(\vec{z}) \quad (6.14)$$

where the bulk field  $\varphi_\Delta$  is related to the boundary field  $\bar{\varphi}_\Delta$  by the limiting relation,

$$\bar{\varphi}_\Delta(\vec{z}) = \lim_{z_0 \rightarrow \infty} z_0^{\Delta-d} \varphi_\Delta(z_0, \vec{z}) \quad (6.15)$$

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<sup>v</sup>The study of quantum Liouville theory with a  $SO(2,1)$  invariant vacuum [86] is closely related to the study of  $AdS_2$ , as was shown in [87]. Propagators and amplitudes were studied there long ago [86] and the  $\mathcal{N} = 1$  supersymmetric generalization is also known [88].

The corresponding *boundary-to-bulk propagator* is the Poisson kernel, [3],

$$K_{\Delta}(z, \vec{x}) = C_{\Delta} \left( \frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2} \right)^{\Delta} \quad (6.16)$$

The bulk field generated in response to the boundary source  $\bar{J}$  is given by

$$\varphi_{\Delta}(z) = \int_{\partial H} d^d \vec{z} K_{\Delta}(z, \vec{x}) \bar{J}(\vec{x}) \quad (6.17)$$

This propagator will be especially important in the AdS/CFT correspondence.

### The gauge propagator

Let  $A_{\mu}(z)$  be a massless or massive gauge field, whose linearized dynamics is given by a coupling to a covariantly conserved bulk current  $j^{\mu}$  via the action

$$S_A = \int_H d^{d+1}z \sqrt{g} \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_{\mu} A^{\mu} - A_{\mu} j^{\mu} \right] \quad (6.18)$$

It would be customary to introduce a gauge fixing term, such as Feynman gauge, to render the second order differential operator acting on  $A_{\mu}$  invertible when  $m = 0$ . A more convenient way to proceed is to remark that the differential operator needs to be inverted only on the subspace of all  $j^{\mu}$  that are covariantly conserved. The gauge propagator is a bivector  $G_{\mu\nu'}(z, z')$  which satisfies

$$-\frac{1}{\sqrt{g}} \partial_{\sigma} \left( \sqrt{g} g^{\sigma\rho} \partial_{[\rho} G_{\mu]\nu'}(z, z') \right) + m^2 G_{\mu\nu'}(z, z') = g_{\mu\nu} \delta(z, z') + \partial_{\mu} \partial_{\nu'} \Lambda(u) \quad (6.19)$$

The term in  $\Lambda$  is immaterial when integrated against a covariantly conserved current. For the massless case, the gauge propagator is given by, [79, 80], see also [89],

$$G_{\mu\nu'}(z, z') = -(\partial_{\mu} \partial_{\nu'} u) F(u) + \partial_{\mu} \partial_{\nu'} S(u) \quad (6.20)$$

where  $S$  is a gauge transformation function, while the physical part of the propagator takes the form,

$$F(u) = \frac{\Gamma((d-1)/2)}{4\pi^{(d+1)/2}} \frac{1}{[u(u+2)]^{(d-1)/2}} \quad (6.21)$$

### The massless graviton propagator

The action for matter coupled to gravity in an AdS background is given by

$$S_g = \frac{1}{2} \int_H d^{d+1}z \sqrt{g} (-R_g + \Lambda) + S_m \quad (6.22)$$

where  $R_g$  is the Ricci scalar for the metric  $g$  and  $\Lambda$  is the ‘‘cosmological constant’’.  $S_m$  is the matter action, whose variation with respect to the metric is, by definition, the stress

tensor  $T_{\mu\nu}$ . The stress tensor is covariantly conserved  $\nabla_\mu T^{\mu\nu} = 0$ . Einstein's equations read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R_g - \Lambda) = T_{\mu\nu} \quad T_{\mu\nu} = \frac{\delta S_m}{\sqrt{g}\delta g^{\mu\nu}} \quad (6.23)$$

We take  $\Lambda = -d(d-1)$ , so that in the absence of matter sources, we obtain Euclidean AdS=  $H$  with  $R_g = -d(d+1)$  as the maximally symmetric solution. To obtain the equation for the graviton propagator  $G_{\mu\nu;\mu'\nu'}(z, w)$ , it suffices to linearize Einstein's equations around the AdS metric in terms of small deviations  $h_{\mu\nu} = \delta g_{\mu\nu}$  of the metric. One find

$$h_{\mu\nu}(z) = \int_H d^{d+1}w \sqrt{g} G_{\mu\nu;\mu'\nu'}(z, w) T^{\mu'\nu'}(w) \quad (6.24)$$

where the graviton propagator satisfies

$$W_{\mu\nu}{}^{\kappa\lambda} G_{\kappa\lambda\mu'\nu'} = \left( g_{\mu\mu'} g_{\nu\nu'} + g_{\mu\nu'} g_{\nu\mu'} - \frac{2g_{\mu\nu} g_{\mu'\nu'}}{d-1} \right) \delta(z, w) + \nabla_{\mu'} \Lambda_{\mu\nu;\nu'} + \nabla_{\nu'} \Lambda_{\mu\nu;\mu'}$$

and the differential operator  $W$  is defined by

$$\begin{aligned} W_{\mu\nu}{}^{\kappa\lambda} G_{\kappa\lambda\mu'\nu'} \equiv & -\nabla^\sigma \nabla_\sigma G_{\mu\nu;\mu'\nu'} - \nabla_\mu \nabla_\nu G^\sigma{}_{\sigma;\mu'\nu'} + \nabla_\mu \nabla^\sigma G_{\sigma\nu;\mu'\nu'} \\ & + \nabla_\nu \nabla^\sigma G_{\mu\sigma;\mu'\nu'} - 2G_{\mu\nu;\mu'\nu'} + 2g_{\mu\nu} G^\sigma{}_{\sigma;\mu'\nu'} \end{aligned} \quad (6.25)$$

The solution to this equation is obtained by decomposing  $G$  onto a basis of 5 irreducible  $SO(1, d)$ -tensors, which may all be expressed in terms of the metric  $g_{\mu\nu}$  and the derivatives of the chordal distance  $\partial_\mu u$ ,  $\partial_\mu \partial_{\nu'} u$  etc. One finds that three linear combinations of these 5 tensors correspond to diffeomorphisms, so that we have

$$\begin{aligned} G_{\mu\nu;\mu'\nu'} = & (\partial_\mu \partial_{\mu'} u \partial_\nu \partial_{\nu'} u + \partial_\mu \partial_{\nu'} u \partial_\nu \partial_{\mu'} u) G(u) + g_{\mu\nu} g_{\mu'\nu'} H(u) \\ & + \nabla_{(\mu} S_{\nu);\mu'\nu'} + \nabla_{(\mu'} S_{\mu\nu);\nu'} \end{aligned} \quad (6.26)$$

The functions  $G$  precisely obeys the equation for a massless scalar propagator  $G_\Delta(u)$  with  $\Delta = d$ , so that  $G(u) = G_d(u)$ . The function  $H(u)$  is then given by

$$-(d-1)H(u) = 2(1+u)^2 G(u) + 2(d-2)(1+u) \int_\infty^u dv G(v) \quad (6.27)$$

which may also be expressed in terms of hypergeometric functions. The graviton propagator was derived using the above methods, or alternatively in De Donder gauge in [80]. Propagators for other fields, such as massive tensor and form fields were constructed in [81] and [82]; see also [83] and [84].

## 6.4 Conformal Structure of 1- 2- and 3- Point Functions

Conformal invariance is remarkably restrictive on correlation functions with 1, 2, and 3 conformal operators [90]. We illustrate this point for correlation functions of superconformal primary operators, which are all scalars.



The *1-point function* is given by

$$\langle \mathcal{O}_\Delta(x) \rangle = \delta_{\Delta,0} \quad (6.28)$$

Indeed, by translation invariance, this object must be independent of  $x$ , while by scaling invariance, an  $x$ -independent quantity can have dimension  $\Delta$  only when  $\Delta = 0$ , in which case we have the identity operator.

The *2-point function* is given by

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle = \frac{\delta_{\Delta_1, \Delta_2}}{|x_1 - x_2|^{2\Delta_1}} \quad (6.29)$$

Indeed, by Poincaré symmetry, this object only depends upon  $(x_1 - x_2)^2$ ; by inversion symmetry, it must vanish unless  $\Delta_1 = \Delta_2$ ; by scaling symmetry one fixes the exponent; and by properly normalizing the operators, the 2-point function may be put in diagonal form with unit coefficients.

The *3-point function* is given by

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \mathcal{O}_{\Delta_3}(x_3) \rangle = \frac{c_{\Delta_1 \Delta_2 \Delta_3}(g_s, N)}{|x_1 - x_2|^{\Delta - 2\Delta_3} |x_2 - x_3|^{\Delta - 2\Delta_1} |x_3 - x_1|^{\Delta - 2\Delta_2}} \quad (6.30)$$

where  $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ . The coefficient  $c_{\Delta_1 \Delta_2 \Delta_3}$  is independent of the  $x_i$  and will in general depend upon the coupling  $g_{YM}^2$  of the theory and on the Yang-Mills gauge group through  $N$ .

## 6.5 SYM Calculation of 2- and 3- Point Functions

All that is needed to compute the SYM correlation functions of the composite operators

$$\mathcal{O}_\Delta(x) \equiv \frac{1}{n_\Delta} \text{str} X^{i_1}(x) \cdots X^{i_\Delta}(x) \quad (6.31)$$

to Born level (order  $g_{YM}^0$ ) is the propagator of the scalar field

$$\langle X^{ic}(x_1) X^{jc'}(x_2) \rangle = \frac{\delta^{ij} \delta^{cc'}}{4\pi^2 (x_1 - x_2)^2} \quad (6.32)$$

where  $c$  is a color index running over the adjoint representation of  $SU(N)$  while  $i = 1, \dots, 6$  labels the fundamental representation of  $SO(6)$ . Clearly, the 2- and 3- point functions have the space-time behavior expected from the preceding discussion of conformal invariance. Normalizing the 2-point function as below, we have  $n_k^2 = \text{str}(T^{c_1} \cdots T^{c_k}) \text{str}(T^{c_1} \cdots T^{c_k})$ .

$$\begin{aligned} \langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle &= \frac{\delta_{\Delta_1, \Delta_2}}{(x_1 - x_2)^{2\Delta_1}} \\ \langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \mathcal{O}_{\Delta_3}(x_3) \rangle &\sim \frac{1}{(x_1 - x_2)^{\Delta_{12}} (x_2 - x_3)^{\Delta_{23}} (x_3 - x_1)^{\Delta_{31}}} \end{aligned} \quad (6.33)$$

Using the fact that the number  $\Delta_i$  of propagators emerging from operator  $\mathcal{O}_{\Delta_i}$  equals the sum  $\Delta_{ij} + \Delta_{ik}$ , we find  $2\Delta_{ij} = \Delta_i + \Delta_j - \Delta_k$ , in agreement with (6.30). The precise numerical coefficients may be worked out with the help of the contractions of color traces.

## 6.6 AdS Calculation of 2- and 3- Point Functions

On the AdS side, the 2-point function to lowest order is obtained by taking the boundary-to-bulk propagator  $K_\Delta(z, \vec{x})$  for a field with dimension  $\Delta$  and extracting the  $z_0^\Delta$  behavior as  $z_0 \rightarrow 0$ , which gives

$$\lim_{z_0 \rightarrow 0} z_0^{-\Delta} K_\Delta(z, \vec{x}) \sim \frac{1}{(\vec{z} - \vec{x})^{2\Delta}} \quad (6.34)$$

in agreement with the behavior predicted from conformal invariance [91].

The 3-point function involves an integral over the intermediate supergravity interaction point, and is given by

$$\mathcal{G}(\Delta_1, \Delta_2, \Delta_3) \int_{S^5} Y_{\Delta_1} Y_{\Delta_2} Y_{\Delta_3} \int_H \frac{d^5 z}{z_0^5} \prod_{i=1}^3 C_{\Delta_i} \left( \frac{z_0}{z_0^2 + (\vec{z} - \vec{x}_i)^2} \right)^{\Delta_i} \quad (6.35)$$

where  $\mathcal{G}(\Delta_1, \Delta_2, \Delta_3)$  stands for the supergravity 3-point coupling and the second factor is the integrals over the spherical harmonics of  $S^5$ . To carry out the integral over  $H$ , one proceeds in three steps. First, use a translation to set  $\vec{x}_3 = 0$ . Second, use an inversion about 0, given by  $z^\mu \rightarrow z^\mu/z^2$  to set  $\vec{x}'_3 = \infty$ . Third, having one point at  $\infty$ , one may now use translation invariance again, to obtain for the  $H$ -integral

$$\sim (x'_{13})^{2\Delta_1} (x'_{23})^{2\Delta_2} \int_H \frac{d^5 z}{z_0^5} \frac{z_0^{\Delta_1 + \Delta_2 + \Delta_3}}{z^{2\Delta_1} [z_0^2 + (\vec{z} - \vec{x}'_{13} - \vec{x}'_{23})^2]^{\Delta_2}} \quad (6.36)$$

Carrying out the  $\vec{z}$  integral using a Feynman parametrization of the integral and then carrying out the  $z_0$  integral, one recovers again the general space-time dependence of the 3-point function [91]. A more detailed account of the AdS calculations of the 2- and 3-point functions will be given in §8.4.

## 6.7 Non-Renormalization of 2- and 3- Point Functions

Upon proper normalization of the operators  $\mathcal{O}_\Delta$ , so that their 2-point function is canonically normalized, the three point couplings  $c_{\Delta_1, \Delta_2, \Delta_3}(g_{YM}^2, N)$  may be computed in a unique manner. On the SYM side, small coupling  $g_{YM}$  perturbation theory yields results for  $g_{YM} \ll 1$ , but all  $N$ . On the AdS side, the only calculation available in practice so far is at the level of classical supergravity, which means the large  $N$  limit (where quantum loops are being neglected), as well as large 't Hooft coupling  $\lambda = g_{YM}^2 N$  (where  $\alpha'$  string corrections to supergravity are being neglected). Therefore, a direct comparison between the two calculations cannot be made because the calculations hold in mutually exclusive regimes of validity.

Nonetheless, one may compare the results of the calculations in both regimes. This involves obtaining a complete normalization of the supergravity three-point couplings

$\mathcal{G}(\Delta_1, \Delta_2, \Delta_3)$ , which was worked out in [92]. It was found that

$$\lim_{N, \lambda = g_s N \rightarrow \infty} c_{\Delta_1, \Delta_2, \Delta_3}(g_s, N) \Big|_{AdS} = \lim_{N \rightarrow \infty} c_{\Delta_1, \Delta_2, \Delta_3}(0, N) \Big|_{SYM} \quad (6.37)$$

Given that this result holds irrespectively of the dimensions  $\Delta_i$ , it was conjectured in [92] that this result should be viewed as emerging from a *non-renormalization effect* for 2- and 3-point functions of 1/2 BPS operators. Consequently, it was conjectured that the equality should hold for all couplings, at large  $N$ ,

$$\lim_{N \rightarrow \infty} c_{\Delta_1, \Delta_2, \Delta_3}(g_s, N) \Big|_{AdS} = \lim_{N \rightarrow \infty} c_{\Delta_1, \Delta_2, \Delta_3}(g_{YM}^2, N) \Big|_{SYM} \quad (6.38)$$

and more precisely that  $c_{\Delta_1, \Delta_2, \Delta_3}(g_s, N)$  be independent of  $g_s$  in the  $N \rightarrow \infty$  limit.

Independence on  $g_{YM}$  of the three point coupling  $c_{\Delta_1, \Delta_2, \Delta_3}(g_{YM}^2, N)$  is now a problem purely in  $\mathcal{N} = 4$  SYM theory, and may be studied there in its own right. This issue has been pursued since by performing calculations of the same correlators to order  $g_{YM}^2$ . It was found that to this order, neither the 2- nor the 3-point functions receive any corrections [93]. Consequently, a stronger conjecture was proposed to hold for all  $N$ ,

$$c_{\Delta_1, \Delta_2, \Delta_3}(g_s, N) \Big|_{AdS} = c_{\Delta_1, \Delta_2, \Delta_3}(g_{YM}^2, N) \Big|_{SYM} \quad (6.39)$$

Further evidence that this relation holds has been obtained using  $\mathcal{N} = 1$  superfields [94, 95] and  $\mathcal{N} = 2$  off-shell analytic/harmonic superfield methods [108, 109]. The problem has also been investigated using  $\mathcal{N} = 4$  on-shell superspace methods [96, 97], via the study of nilpotent superconformal invariants, which had been introduced for  $OSp(1, N)$  in [98]. Similar non-renormalization effects may be derived for 1/4 BPS operators and their correlators as well [99]. Two and three point correlators have also been investigated for superconformal descendant fields; for the R-symmetry current in [91] and later in [101]; see also [100] and [102]. Additional references include [103] and [104]. A further test of the Maldacena conjecture involving the Weyl anomaly is in [105].

## 6.8 Extremal 3-Point Functions

We now wish to investigate the dependence of the 3-point function of 1/2 BPS single trace operators

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \mathcal{O}_{\Delta_3}(x_3) \rangle \quad (6.40)$$

on the dimensions  $\Delta_i$  a little more closely. Recall that these operators transform under the irreducible representations of  $SU(4)_R$  with Dynkin labels  $[0, \Delta_i, 0]$ . As a result, the correlators must vanish whenever  $\Delta_i > \Delta_j + \Delta_k$  for any one of the labels  $i \neq j, k$ , since in this case no  $SU(4)_R$  singlet exists. Whenever  $\Delta_i \leq \Delta_j + \Delta_k$ , for all  $i, j, k$ , the correlator is allowed by  $SU(4)_R$  symmetry.

These facts may also be seen at Born level in SYM perturbation theory by matching the number of  $X$  propagators connecting different operators. If  $\Delta_i > \Delta_j + \Delta_k$ , it will be impossible to match up the  $X$  propagator lines and the diagram will have to vanish.

The case where  $\Delta_i = \Delta_j + \Delta_k$  for one of the labels  $i$  is of special interest and is referred to as an *extremal correlator* [106]. Although allowed by  $SU(4)_R$  group theory, its Born graph effectively factorizes into two 2-point functions, because no  $X$  propagators directly connect the vertices operators  $j$  and  $k$ . Thus, the extremal 3-point function is non-zero. However, the supergravity coupling  $\mathcal{G}(\Delta_1, \Delta_2, \Delta_3) \sim \Delta_1 - \Delta_2 - \Delta_3$  vanishes in the extremal case as was shown in [92]. The reason that all these statements can be consistent with the AdS/CFT correspondence is because the AdS<sub>5</sub> integration actually has a pole at the extremal dimensions, as may indeed be seen by taking a closer look at the integrals,

$$\int_H \frac{d^5z}{z_0^5} \prod_{i=1}^3 \frac{z_0^{\Delta_i}}{(z_0^2 + (\vec{z} - \vec{x}_i)^2)^{\Delta_i}} \sim \frac{1}{\Delta_1 - \Delta_2 - \Delta_3} \quad (6.41)$$

Thus, the AdS/CFT correspondence for extremal 3-point functions holds because a zero in the supergravity coupling is compensated by a pole in the AdS<sub>5</sub> integrals.

Actually, the dimensions  $\Delta_i$  are really integers (which is why “pole” was put in quotation marks above) and direct analytic continuation in them is not really justified. It was shown in [106] that when keeping the dimensions  $\Delta_i$  integer, it is possible to study the supergravity integrands more carefully and to establish that while the bulk contribution vanishes, there remains a boundary contribution (which was immaterial for non-extremal correlators). A careful analysis of the boundary contribution allows one to recover agreement with the SYM calculation directly.

## 6.9 Non-Renormalization of General Extremal Correlators

Extremal correlators may be defined not just for 3-point functions, but for general  $(n+1)$ -point functions. Let  $\mathcal{O}_\Delta$  and  $\mathcal{O}_{\Delta_i}$  with  $i = 1, \dots, n$  be 1/2 BPS chiral primary operators obeying the relation  $\Delta = \Delta_1 + \dots + \Delta_n$ , which generalizes the extremality relation for the 3-point function. We have the *extremal correlation non-renormalization conjecture*, stating the form of the following correlator [106],

$$\langle \mathcal{O}_\Delta(x) \mathcal{O}_{\Delta_1}(x_1) \cdots \mathcal{O}_{\Delta_n}(x_n) \rangle = A(\Delta_i; N) \prod_{i=1}^n \frac{1}{(\vec{x} - \vec{x}_i)^{2\Delta_i}} \quad (6.42)$$

The conjecture furthermore states that the overall function  $A(\Delta_i; N)$  is independent of the points  $x_i$  and  $x$  and is also independent of the string coupling constant  $g_s = g_{YM}^2$ . *The conjecture also states that the associated supergravity bulk couplings  $\mathcal{G}(\Delta; \Delta_1, \dots, \Delta_n)$  must vanish* [106].

There is by now ample evidence for the conjecture and we shall briefly review it here. First, there is evidence from the SYM side. To Born level (order  $\mathcal{O}(g_{YM}^0)$ ), the factorization of the space-time dependence in a product of 2-point functions simply follows from the

fact that no  $X$ -propagator lines can connect different points  $x_i$ ; instead all  $X$ -propagator lines emanating from any vertex  $x_i$  flow into the point  $x$ . The absence of  $\mathcal{O}(g_{YM}^2)$  perturbative corrections was demonstrated in [110]. Off-shell  $\mathcal{N} = 2$  analytic/harmonic superspace methods have been used to show that  $g_{YM}$  corrections are absent to all orders of perturbation theory [108], [109].

On the AdS side, the simplest diagram that contributes to the extremal correlator is the contact graph, which is proportional to

$$\mathcal{G}(\Delta; \Delta_1, \dots, \Delta_n) \int_H \frac{d^5 z}{z_0^5} \frac{z_0^\Delta}{(z - \vec{x})^{2\Delta}} \prod_{i=1}^n \frac{z_0^{\Delta_i}}{(z - \vec{x}_i)^{2\Delta_i}} \quad (6.43)$$

In view of the relation  $\Delta = \Delta_1 + \dots + \Delta_n$ , the integration is convergent everywhere in  $H$ , except when  $\vec{z} \rightarrow \vec{x}$  and  $z_0 \rightarrow 0$ , where a simple pole arises in  $\Delta - \Delta_1 - \dots - \Delta_n$ . Finiteness of Type IIB superstring theory on  $\text{AdS}_5 \times S^5$  (which we take as an assumption here) guarantees that the full correlator must be convergent. Therefore, the associated supergravity bulk coupling must vanish,

$$\mathcal{G}(\Delta; \Delta_1, \dots, \Delta_n) \sim \Delta - \Delta_1 - \dots - \Delta_n \quad (6.44)$$

as indeed stated in the conjecture. Assuming that it makes sense to “analytically continue in the dimensions  $\Delta$ ”, one may proceed as follows. The pole of the  $z$ -integration and the zero of the supergravity coupling  $\mathcal{G}$  compensate one another and the contribution of the contact graph to the extremal correlator will be given by the residue of the pole, which is precisely of the form (6.42). It is also possible to carefully treat the boundary contributions generated by the supergravity action in the extremal case, to recover the same result [106].

The analysis of all other AdS graphs, which have at least one bulk-to-bulk exchange in them, was carried out in detail in [106]. For the exchange of chiral primaries in the graph, the extremality condition  $\Delta = \Delta_1 + \dots + \Delta_n$  implies that each of the exchange bulk vertices must be extremal as well. A non-zero contribution can then arise only if the associated integral is divergent, produces a pole in the dimensions, and makes the interaction point collapse onto the boundary  $\partial H$ . Dealing with all intermediate external vertices in this way, one recovers that all intermediate vertices have collapsed onto  $\vec{x}$ , thereby reproducing the space-time behavior of (6.42). The exchange of descendants may be dealt with in an analogous manner.

Assuming non-renormalization of 2- and 3-point functions for all (single and multiple trace)  $1/2$  BPS operators, and assuming the space-time form (6.42) of the extremal correlators, it is possible to *prove* that the overall factor  $A(\Delta_i; N)$  is independent of  $g_s = g_{YM}^2$ , as was done in [106] in a special case. We present only the simplest non-trivial case of  $n = 3$  and  $\Delta = 6$ ; the general case may be proved by induction. Assuming the space-time form, we have

$$\langle \mathcal{O}_6(x) \mathcal{O}_2(x_1) \mathcal{O}_2(x_2) \mathcal{O}_2(x_3) \rangle = A \prod_{i=1}^3 \frac{1}{(\vec{x} - \vec{x}_i)^4} \quad (6.45)$$

We begin with the OPE

$$\mathcal{O}_6(x)\mathcal{O}_2(x_1) \sim \frac{c\mathcal{O}_4(x) + c'[\mathcal{O}_2\mathcal{O}_2]_{\max}(x)}{(x-x_1)^4} + \text{less singular} \quad (6.46)$$

Using non-renormalization of the 3-point functions  $\langle \mathcal{O}_6\mathcal{O}_2\mathcal{O}_4 \rangle$  and  $\langle \mathcal{O}_6\mathcal{O}_2[\mathcal{O}_2\mathcal{O}_2]_{\max} \rangle$ , we find that  $c$  and  $c'$  are independent of the coupling  $g_{YM}$ . Now substitute the above OPE into the correlator (6.45), and use the fact that the 3-point functions  $\langle \mathcal{O}_4\mathcal{O}_2\mathcal{O}_2 \rangle$  and  $\langle \mathcal{O}_2\mathcal{O}_2[\mathcal{O}_2\mathcal{O}_2]_{\max} \rangle$  are not renormalized. It immediately follows that  $A$  in (6.45) is independent of the coupling.

## 6.10 Next-to-Extremal Correlators

The space-time dependence of extremal correlators was characterized by its factorization into a product of  $n$  2-point functions. The space-time dependence of *Next-to-extremal correlators*  $\langle \mathcal{O}_\Delta(x)\mathcal{O}_{\Delta_1}(x_1)\cdots\mathcal{O}_{\Delta_n}(x_n) \rangle$ , with the dimensions satisfying  $\Delta = \Delta_1 + \cdots + \Delta_n - 2$  is characterized by its factorization into a product of  $n-2$  two-point functions and one 3-point function. Therefore, the conjectured space-time dependence of *next-to-extremal correlators* is given by [109]

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_{\Delta_1}(x_1)\cdots\mathcal{O}_{\Delta_n}(x_n) \rangle = \frac{B(\Delta_i; N)}{x_{12}^2(x-x_1)^{2\Delta_1-2}(x-x_2)^{2\Delta_2-2}} \prod_{i=3}^n \frac{1}{(x-x_i)^{2\Delta_i}} \quad (6.47)$$

where the overall strength  $B(\Delta_i; N)$  is independent of  $g_{YM}$ . This form is readily checked at Born level and was verified at order  $\mathcal{O}(g_{YM}^2)$  by [107].

On the AdS side, the exchange diagrams, say with a single exchange, are such that one vertex is extremal while the other vertex is not extremal. A divergence arises when the extremal vertex is attached to the operator of maximal dimension  $\Delta$  and its collapse onto the point  $x$  now produces a 3-point correlator times  $n-2$  two-point correlators, thereby reproducing the space-time dependence of (6.47). Other exchange diagrams may be handled analogously. However, there is also a contact graph, whose AdS integration is now *convergent*. Since the space-time dependence of this contact term is qualitatively different from the factorized form of (6.47), the only manner in which (6.47) can hold true is if the supergravity bulk coupling associated with next-to-extremal couplings vanishes,

$$\mathcal{G}(\Delta; \Delta_1, \cdots, \Delta_n) = 0 \quad \text{whenever} \quad \Delta = \Delta_1 + \cdots + \Delta_n - 2 \quad (6.48)$$

which is to be included as part of the conjecture [111]. This type of cancellation has been checked to low order in [112].

## 6.11 Consistent Decoupling and Near-Extremal Correlators

The vanishing of the extremal and next-to-extremal supergravity couplings has a direct interpretation, at least in part, in supergravity. Recall that the operator  $\mathcal{O}_2$  and its descendants are dual to the 5-dimensional supergravity multiplet on AdS<sub>5</sub>, while the operators

$\mathcal{O}_\Delta$  with  $\Delta \geq 3$  and its descendants are dual to the Kaluza-Klein excitations on  $S^5$  of the 10-dimensional supergravity multiplet. Now, prior work on gauged supergravity [113, 114] has shown that the 5-dimensional gauged supergravity theory on  $\text{AdS}_5$  all by itself exists and is consistent. Thus, there must exist a *consistent truncation of the Kaluza-Klein modes of supergravity on  $\text{AdS}_5 \times S^5$  to only the supergravity on  $\text{AdS}_5$* ; see also [115]. In a perturbation expansion, this means that if only  $\text{AdS}_5$  supergravity modes are excited, then the Euler-Lagrange equations of the full  $\text{AdS}_5 \times S^5$  supergravity must close on these excitations alone without generating Kaluza-Klein excitation modes. This means that the one 1-point function of any Kaluza-Klein excitation operator in the presence of  $\text{AdS}_5$  supergravity alone must vanish, or

$$\mathcal{G}(\Delta, \Delta_1, \dots, \Delta_n) = 0, \quad \Delta_i = 2, \quad i = 1, \dots, n \quad \text{for all} \quad \Delta \geq 4 \quad (6.49)$$

When  $\Delta > 2n$ , the cancellation takes place by  $SU(4)_R$  group theory only. For  $\Delta = 2n$  and  $\Delta = 2n - 2$ , we have special cases of extremal and next-to-extremal correlators respectively, but for  $4 \leq \Delta \leq 2n - 4$ , they belong to a larger class. We refer to these as *near-extremal correlators* [111],

$$\langle \mathcal{O}_\Delta(x) \mathcal{O}_{\Delta_1}(x_1) \cdots \mathcal{O}_{\Delta_n}(x_n) \rangle \quad \Delta = \Delta_1 + \cdots + \Delta_n - 2m \quad 0 \leq m \leq n - 2 \quad (6.50)$$

The principal result on near-extremal correlators (but which are not of the extremal or next-to-extremal type) is that they do receive coupling dependent quantum corrections, but only through lower point functions [111]. Associated supergravity couplings must vanish,

$$\mathcal{G}(\Delta, \Delta_1, \dots, \Delta_n) = 0 \quad \Delta = \Delta_1 + \cdots + \Delta_n - 2m \quad 0 \leq m \leq n - 2 \quad (6.51)$$

Arguments in favor of this conjecture may be given based on the divergence structure of the AdS integrals and on perturbation calculations in SYM.

## 6.12 Problem Sets

(6.1) Using infinitesimal special conformal symmetry (or global inversion under which  $x^\mu \rightarrow x^\mu/x^2$ ) show that  $\langle \mathcal{O}_\Delta(x) \mathcal{O}_{\Delta'}(x') \rangle = 0$  unless  $\Delta' = \Delta$ .

(6.2) Gauge dependent correlators in gauge theories such as  $\mathcal{N} = 4$  SYM theory will, in general, depend upon a renormalization scale  $\mu$ . (a) Show that the general form of the scalar two point function to one loop order is given by

$$\langle X^{ic}(x) X^{j'c'}(y) \rangle = \frac{\delta^{cc'} \delta^{ij}}{(x-y)^2} \left( A + B \ln(x-y)^2 \mu^2 \right)$$

for some numerical constants  $A$  and  $B$ . (b) Show that the 2-pt function of the gauge invariant operator  $\mathcal{O}_2(x) \equiv \text{tr} X^i(x) X^j(x) - \frac{1}{6} \delta^{ij} \sum_k \text{tr} X^k(x) X^k(x)$  is  $\mu$ -independent. (c) Show that the 2-pt function of the gauge invariant operator  $\mathcal{O}_K(x) \equiv \text{tr} X^i(x) X^i(x)$  (the Konishi operator) is  $\mu$ -dependent. (d) Calculate the 1-loop anomalous dimensions of  $\mathcal{O}_2$  and  $\mathcal{O}_K$ .

(6.3) Consider the Laplace operator  $\Delta$  acting on scalar functions on the sphere  $S^d$  with round  $SO(d+1)$ -invariant metric and radius  $R$ . Compute the eigenvalues of  $\Delta$ . Suggestion :  $\Delta$  is related to the quadratic Casimir operator  $L^2 \equiv \sum_{i,j=1}^{d+1} L_{ij}^2$  where  $L_{ij}$  are the generators of  $d+1$ -dimensional angular momentum, i.e. generators of  $SO(d+1)$ ; thus the problem may be solved by pure group theory methods, analogous to those used for rotations on  $S^2$ .

(III.4) Continuing on the above problem, show that the eigenfunctions are of the form  $c_{i_1 \dots i_p} x^{i_1} \dots x^{i_p}$ , where we have now represented the sphere by the usual equation in  $\mathbf{R}^{d+1}$  :  $\sum_i (x^i)^2 = R^2$  and  $c$  is totally symmetric and traceless.



## 7 Structure of General Correlators

In the previous section, we have concentrated on matching between the SYM side and the AdS side of the Maldacena correspondence correlation functions that were not renormalized or were simply renormalized from their free form. This led us to uncover a certain number of important *non-renormalization effects*, most of which are at the level of conjecture.

However,  $\mathcal{N} = 4$  super-Yang-Mills theory is certainly not a free quantum field theory, and generic correlators will receive quantum corrections from their free field values, and therefore will acquire non-trivial coupling  $g_s = g_{YM}^2$  dependence. In this section, we analyze the behavior of such correlators. We shall specifically deal with the 4-point function. The relevant dynamical information available from correlators in conformal quantum field theory is contained in the *scaling dimensions of general operators*, in the *operator mixings* between general operators and in the values of the *operator product (OPE) coefficients*. As in the case of the 3-point function, a direct quantitative comparison between the results of weak coupling  $g_{YM}$  perturbation theory in SYM and the large  $N$ , large 't Hooft coupling  $\lambda = g_{YM}^2 N$  limit of supergravity cannot be made, because the domains of validity of the expansions do not overlap. Nonetheless, general properties lead to exciting and non-trivial comparisons, which we shall make here.

### 7.1 RG Equations for Correlators of General Operators

It is a general result of quantum field theory that all renormalizations of local operators are multiplicative. This is familiar for canonical fields; for example the bare field  $\phi_0(x)$  and the renormalized field  $\phi(x)$  in a scalar field theory are related by the field renormalization factor  $Z_\phi$  via the relation  $\phi_0(x) = Z_\phi \phi(x)$ . Composite operators often requires *additive renormalizations*; for example the proper definition of the operator  $\phi^2(x)$  requires the subtraction of a constant  $C$ . If this constant is viewed as multiplying the identity operator  $I$  in the theory, then renormalization may alternatively be viewed as multiplicative (by a matrix) on an array of two operators  $I$  and  $\phi^2(x)$  as follows,

$$\begin{pmatrix} I \\ \phi^2(x) \end{pmatrix}_0 = \begin{pmatrix} I & 0 \\ -C & Z_{\phi^2} \end{pmatrix} \begin{pmatrix} I \\ \phi^2(x) \end{pmatrix} \quad (7.1)$$

The general rule is that operators will renormalize with operators with the same quantum numbers but of lesser or equal dimension.

In more complicated theories such as  $\mathcal{N} = 4$  super-Yang-Mills theory, renormalization will continue to proceed in a multiplicative way. If we denote a basis of (local gauge invariant polynomial) operators by  $\mathcal{O}_I$ , and their bare counterparts by  $\mathcal{O}_{0I}$ , then we have the following multiplicative renormalization formula

$$\mathcal{O}_{0I}(x) = \sum_J Z_I^J \mathcal{O}_J(x) \quad (7.2)$$

The field renormalization matrix  $Z_I^J$  may be arranged in block lower triangular form, in ascending value of the operator dimensions, generalizing (7.1). Consider now a general

correlator of such operators

$$G_{I_1, \dots, I_n}(x_i; g, \mu) \equiv \langle \mathcal{O}_{I_1}(x_1) \cdots \mathcal{O}_{I_n}(x_n) \rangle \quad (7.3)$$

and its bare counterpart  $G_{0I_1, \dots, I_n}(x_i; g_0, \Lambda)$ , in a theory in which we schematically represent the dimensionless and dimensionful couplings by  $g$ , and their bare counterparts by  $g_0$ . The renormalization scale is  $\mu$  and the UV cutoff is  $\Lambda$ . Multiplicative renormalization implies the following relation between the renormalized and bare correlators

$$G_{0I_1, \dots, I_n}(x_i; g_0, \Lambda) = \sum_{J_1, \dots, J_n} Z_{I_1}^{J_1} \cdots Z_{I_n}^{J_n}(g, \mu, \Lambda) G_{J_1, \dots, J_n}(x_i; g, \mu) \quad (7.4)$$

Keeping the bare parameters  $g_0$  and  $\Lambda$  fixed and varying the renormalization scale  $\mu$ , we see that the lhs is independent of  $\mu$ . Differentiating both sides with respect to  $\mu$  is the standard way of deriving the renormalization group equations for the renormalized correlators, and we find

$$\left( \frac{\partial}{\partial \ln \mu} + \beta \frac{\partial}{\partial g} \right) G_{I_1, \dots, I_n}(x_i; g, \mu) - \sum_{j=1}^n \sum_J \gamma_{I_j}^J G_{I_1, \dots, I_{j-1}, J, I_{j+1}, \dots, I_n}(x_i; g, \mu) = 0 \quad (7.5)$$

where the RG  $\beta$ -function and anomalous dimension matrix  $\gamma_I^J$  are defined by

$$\beta(g) \equiv \left. \frac{\partial g}{\partial \ln \mu} \right|_{g_0, \Lambda} \quad \gamma_I^J(g) \equiv - \sum_K (Z^{-1})_I^K \left. \frac{\partial Z_K^J}{\partial \ln \mu} \right|_{g_0, \Lambda} \quad (7.6)$$

For each  $I$ , only a finite number of  $J$ 's are non-zero in the sum over  $J$ . The diagonal entries  $\gamma_I^I$  contribute to the anomalous dimension of the operator  $\mathcal{O}_I$ , while the off-diagonal entries are responsible for operator mixing. Operators that are eigenstates of the dimension operator  $D$  (at a given coupling  $g$ ) correspond to the eigenvectors of the matrix  $\gamma$ .

## 7.2 RG Equations for Scale Invariant Theories

Considerable simplifications occur in the RG equations for scale invariant quantum field theories. Scale invariance requires in particular that  $\beta(g_*) = 0$ , so that the theory is at a fixed point  $g_*$ . In rare cases, such as is in fact the case for  $\mathcal{N} = 4$  SYM, the theory is scale invariant for all couplings. If no dimensionful couplings occur in the Lagrangian, either from masses or from vacuum expectation values of dimensionful fields,  $\gamma_I^J$  is constant and the RG equation becomes a simple scaling equation

$$\frac{\partial}{\partial \ln \mu} G_{I_1, \dots, I_n}(x_i; g_*, \mu) - \sum_{j=1}^n \sum_J \gamma_{I_j}^J(g_*) G_{I_1, \dots, I_{j-1}, J, I_{j+1}, \dots, I_n}(x_i; g_*, \mu) = 0 \quad (7.7)$$

In conformal theories, the dilation generator may be viewed as a Hamiltonian of the system conjugate to radial evolution [116]. Therefore, in unitary scale invariant theories, the dilation generator should be self-adjoint, and hence the anomalous dimension matrix should

be Hermitian.<sup>vi</sup> As such,  $\gamma_I^J$  must be diagonalizable with real eigenvalues,  $\gamma_i$ . Standard scaling arguments then give the behavior of the correlators

$$G_{I_1, \dots, I_n}(\lambda x_i; g_*, \lambda^{-1} \mu) = \lambda^{-\Delta_1} \dots \lambda^{-\Delta_n} G_{I_1, \dots, I_n}(x_i; g_*, \mu) \quad (7.8)$$

where the full dimensions  $\Delta_i$  are given in terms of the canonical dimension  $\delta_i$  by  $\Delta_i = \gamma_i + \delta_i$ .

### 7.3 Structure of the OPE

One of the most useful tools of local quantum field theory is the Operator Product Expansion (OPE) which expresses the product of two local operators in terms of a sum over all local operators in the theory,

$$\mathcal{O}_I(x) \mathcal{O}_J(y) = \sum_K C_{IJ}^K(x-y; g, \mu) \mathcal{O}_K(y) \quad (7.9)$$

The OPE should be understood as a relations that holds when evaluated between states in the theory's Hilbert space or when inserted into correlators with other operators,

$$\langle \mathcal{O}_I(x) \mathcal{O}_J(y) \prod_L \mathcal{O}_L(z_L) \rangle = \sum_K C_{IJ}^K(x-y; g, \mu) \langle \mathcal{O}_K(y) \prod_L \mathcal{O}_L(z_L) \rangle \quad (7.10)$$

From the latter, together with the RG equations for the correlators, one deduces the RG equations for the OPE coefficient functions  $C_{IJ}^K$ ,

$$\left( \frac{\partial}{\partial \ln \mu} + \beta \frac{\partial}{\partial g} \right) C_{IJ}^K = \sum_L \left( \gamma_I^L C_{LJ}^K + \gamma_J^L C_{IL}^K - C_{IJ}^L \gamma_L^K \right) \quad (7.11)$$

In a scale invariant theory, we have  $\beta = 0$  and  $\gamma_I^J$  constant. Furthermore, if the theory is unitary,  $\gamma_I^J$  may be diagonalized in terms of operators  $\mathcal{O}_{\Delta_I}$  of definite dimension  $\Delta_I$ . The OPE then simplifies considerably and we have, [121],

$$\mathcal{O}_{\Delta_I}(x) \mathcal{O}_{\Delta_J}(y) = \sum_K \frac{c_{\Delta_I \Delta_J \Delta_K}}{(x-y)^{\Delta_I + \Delta_J - \Delta_K}} \mathcal{O}_{\Delta_K}(y) \quad (7.12)$$

The operator product coefficients  $c_{\Delta_I \Delta_J \Delta_K}$  are now independent of  $x$  and  $y$ , but they will depend upon the coupling constants and parameters of the theory, such as  $g_{YM}$  and  $N$ .

### 7.4 Perturbative Expansion of OPE in Small Parameter

Conformal field theories such as  $\mathcal{N} = 4$  SYM have coupling constants  $g_{YM}, \theta_I, N$  and the theory is (super)-conformal for any value of these parameters. In particular, the scaling dimensions are fixed but may depend upon these parameters in a non-trivial way,

$$\Delta_I = \Delta(g_{YM}, \theta_I, N) \quad (7.13)$$

The dependence of the composite operators on the canonical fields will in general also involve these coupling dependences.

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<sup>vi</sup>In non-unitary theories, the matrix  $\gamma_I^J$  may be put in Jordan diagonal form, and this form will produce dependence on  $\mu$  through  $\ln \mu$  terms. A fuller discussion is given in [117].

It is interesting to analyze the effects of small variations in any of these parameters on the structure of the OPE and correlation functions. Especially important is the fact that infinitesimal changes in  $\Delta_I$  produce *logarithmic dependences* in the OPE. To see this, assume that

$$\Delta_I = \Delta_I^0 + \delta_I \quad |\delta_I| \ll \Delta_I \quad (7.14)$$

and now observe that to first order in  $\delta_I$ , we have

$$\mathcal{O}_{\Delta_I}(x)\mathcal{O}_{\Delta_J}(y) = \sum_K \frac{c_{\Delta_I\Delta_J\Delta_K}\mathcal{O}_{\Delta_K}(y)}{(x-y)^{\Delta_I^0+\Delta_J^0-\Delta_K^0}} \left\{ 1 - (\delta_I + \delta_J - \delta_K) \ln|x-y|\mu \right\} \quad (7.15)$$

In the special case where the dimensions  $\Delta_I^0$  and  $\Delta_J^0$  are unchanged, because the operators are protected (e.g. BPS) then isolating the logarithmic dependence allows one to compute  $\delta_K$  and thus the correction to the dimension of operators occurring in the OPE. A useful reference on anomalous dimensions and the OPE, though not in conformal field theory, is in [122].

## 7.5 The 4-Point Function – The Double OPE

Recall that the AdS/CFT correspondence maps supergravity fields into single-trace 1/2 BPS operators on the SYM side. Thus, the only correlators that can be computed directly are the ones with one 1/2 BPS operator insertions. To explore even the simplest renormalization effects of non-BPS operators, such as their change in dimension, via the AdS/CFT correspondence, we need to go beyond the 3-point function. The simplest case is the 4-point function, which indeed can yield information on the anomalous dimensions of single and double trace operators.

Thanks to conformal symmetry, the 4-point function may be factorized into a factor capturing its overall non-trivial conformal dependence times a function  $F(s, t)$  that depends only upon 2 conformal invariants  $s, t$  of 4 points,

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4) \rangle = \frac{1}{|x_{13}|^{\Delta_1+\Delta_3}|x_{24}|^{\Delta_2+\Delta_4}} F(s, t) \quad (7.16)$$

The conformal invariants of the 4-point function  $s$  and  $t$  may be chosen as follows

$$s = \frac{1}{2} \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2} \quad t = \frac{x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2} \quad (7.17)$$

The fact that there are only 2 conformal invariants may be seen as follows. By a translation, take  $x_4 = 0$ ; under an inversion, we then have  $x'_4 = \infty$  and we may use translations again to choose  $x'_3 = 0$ . There remain 3 Lorentz invariants,  $x_1^2$ ,  $x_2^2$  and  $x_1 \cdot x_2$ , and thus 2 independent scale-invariant ratios. Note that 2 is also the number of Lorentz invariants of a massless 4-point function in momentum space.

A specific representation for the function  $F$  may be obtained by making use of the OPE twice in the 4-point function, one on the pair  $\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_3}(x_3)$  and once on the pair  $\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_4}(x_4)$ . One obtains the *double OPE*, first introduced in [123],

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4) \rangle = \sum_{\Delta\Delta'} \frac{c_{\Delta_1\Delta_3\Delta}}{|x_{13}|^{\Delta_1\Delta_3\Delta}} \frac{1}{|x_{12}|^{\Delta+\Delta'}} \frac{c_{\Delta_2+\Delta_4-\Delta'}}{|x_{24}|^{\Delta_2+\Delta_4-\Delta'}} \quad (7.18)$$

The OPE coefficients  $c_{\Delta_1\Delta_3\Delta}$  appeared in the simple OPE of the operators  $\mathcal{O}_{\Delta_1}$  and  $\mathcal{O}_{\Delta_3}$ . General properties of the OPE and double OPE have been studied recently in [124], [125], [126], [127] and [128] and from a perturbative point of view in [129]; see also [130].

## 7.6 4-pt Function of Dilaton/Axion System

The possible intermediary fields and operators are restricted by the  $SU(4)_R$  tensor product formula for external operators.<sup>vii</sup> Assuming external operators (such as the 1/2 BPS primaries) in representations  $[0, \Delta, 0]$  and  $[0, \Delta', 0]$ , their tensor product decomposes as

$$[0, \Delta, 0] \otimes [0, \Delta', 0] = \oplus_{\mu=0}^{\Delta'} \oplus_{\nu=0}^{\Delta'-\mu} [\nu, \Delta + \Delta' - 2\mu - 2\nu, \nu] \quad (7.19)$$

For example, the product of two AdS<sub>5</sub> supergravity primaries in the representation  $\mathbf{20}' = [0, 2, 0]$  is given by (the subscript  $A$  denotes antisymmetrization)

$$\mathbf{20}' \otimes \mathbf{20}' = \mathbf{1} \oplus \mathbf{15}_A \oplus \mathbf{20}' \oplus \mathbf{84} \oplus \mathbf{105} \oplus \mathbf{175}_A \quad (7.20)$$

Actually, the simplest group theoretical structure emerges when taking two  $SU(4)_R$  and Lorentz singlets which are  $SU(2, 2|4)$  descendants. We consider the system of dimension  $\Delta = 4$  half-BPS operators dual to the dilaton and axion fields in the bulk;

$$\mathcal{O}_\phi = \text{tr} F_{\mu\nu} F^{\mu\nu} + \dots \quad \mathcal{O}_C = \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} + \dots \quad (7.21)$$

The further advantage of this system is that the classical supergravity action is simple,

$$S[G, \Phi, C] = \frac{1}{2\kappa_5^2} \int_H \sqrt{G} \left[ -R_G + \Lambda + \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2} e^{2\Phi} \partial_\mu C \partial^\mu C \right] \quad (7.22)$$

In the AdS/CFT correspondence,  $\kappa_5^2$  may be related to  $N$  by  $\kappa_5^2 = 4\pi^2/N^2$ . This system was first examined in [131] and [132]. An investigation directly of the correlator of half-BPS chiral primaries may be found in [133].

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<sup>vii</sup>Actually, the possible intermediary fields and operators are restricted by the full  $SU(2, 2|4)$  superconformal algebra branching rules. Since no  $\mathcal{N} = 4$  off-shell superfield formulation is available, however, it appears very difficult to make direct use of this powerful fact.

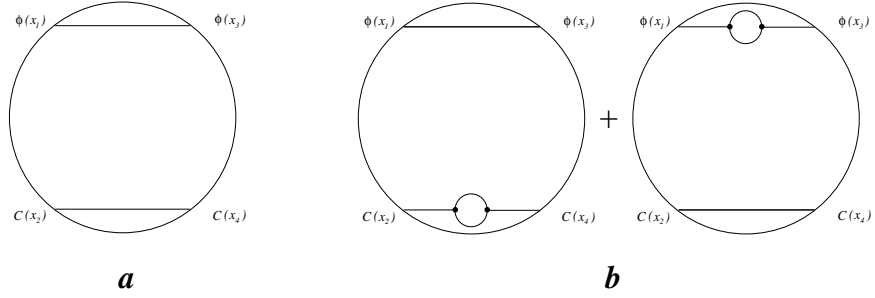


Figure 6: Disconnected contributions to the correlator  $\langle \mathcal{O}_\Phi \mathcal{O}_C \mathcal{O}_\Phi \mathcal{O}_C \rangle$  to order  $1/N^2$

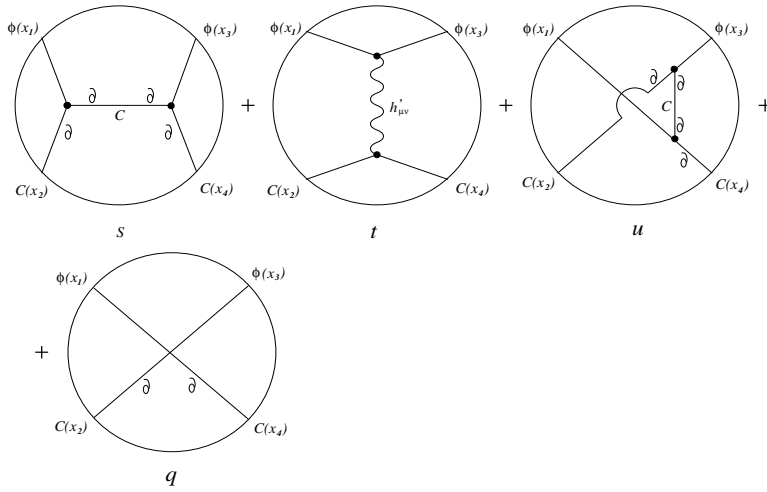


Figure 7: Connected contributions to the correlator  $\langle \mathcal{O}_\Phi \mathcal{O}_C \mathcal{O}_\Phi \mathcal{O}_C \rangle$  to order  $1/N^2$

## 7.7 Calculation of 4-point Contact Graph

The 4-point function receives contributions from the *contact graph* and from a number of *exchange graphs*, which we now discuss in turn. The most general 4-point contact term is given by the following integral,

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_i) \equiv \int_H \frac{d^{d+1}z}{z_0^{d+1}} \prod_{i=1}^4 \left( \frac{z_0}{z_0^2 + (z - \vec{x}_i)^2} \right)^{\Delta_i} \quad (7.23)$$

This integral is closely related to the momentum space integration of the box graph. In fact, we shall not need this object in all its generality, but may restrict to the case  $D_{\Delta\Delta\Delta'\Delta'}$ . The calculation in the general case is given in [134] and [135]; see also [136].

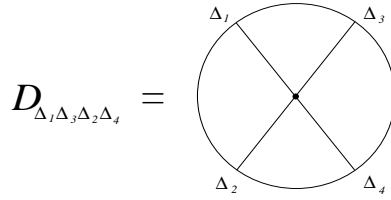


Figure 8: Definition of the contact graph function  $D$

To compute this object explicitly, it is convenient to factor out the overall non-trivial conformal dependence. This may be done by first translating  $x_1$  to 0, then performing an inversion and then translating also  $x'_3$  to 0. The result may be expressed in terms of

$$x \equiv x'_{13} - x'_{14} \quad y \equiv x'_{13} - x'_{12} \quad (7.24)$$

and is found to be

$$D_{\Delta\Delta\Delta'\Delta'}(x_i) = x_{12}^{2\Delta'} x_{13}^{2\Delta} x_{14}^{2\Delta'} \times \int_H \frac{d^{d+1}z}{z_0^{d+1}} \frac{z_0^{2\Delta+2\Delta'}}{z^{2\Delta} (z-x)^{2\Delta'} (z-y)^{2\Delta'}} \quad (7.25)$$

Introducing two Feynman parameters, and carrying out the  $z$ -integration, the integral may be re-expressed as

$$D_{\Delta\Delta\Delta'\Delta'}(x_i) = \frac{x_{12}^{2\Delta'} x_{13}^{2\Delta} x_{14}^{2\Delta'}}{(x^2 + y^2)^{\Delta'}} \frac{\pi^{d/2} \Gamma(\Delta + \Delta' - d/2)}{2^{\Delta'} \Gamma(\Delta) \Gamma(\Delta')} \times \int_0^\infty d\rho \int_{-1}^{+1} d\lambda \frac{\rho^{\Delta-1} (1-\lambda^2)^{\Delta-1}}{[1 + \rho(1-\lambda^2)]^\Delta} \frac{1}{(s + \rho + \rho\lambda t)^{\Delta'}} \quad (7.26)$$

Remarkably, for positive integers  $\Delta$  and  $\Delta'$ , the integral for any  $\Delta$ ,  $\Delta'$  and  $d$  may be re-expressed in terms of successive derivatives of a universal function  $I(s, t)$ ,

$$I(s, t) \equiv \int_{-1}^{+1} d\lambda \frac{1}{1 + \lambda t - s(1 - \lambda^2)} \ln \frac{1 + \lambda t}{s(1 - \lambda^2)} \quad (7.27)$$

in the following way,

$$D_{\Delta\Delta\Delta'\Delta'}(x_i) = (-)^{\Delta+\Delta'} \frac{x_{12}^{2\Delta'} x_{13}^{2\Delta} x_{14}^{2\Delta'}}{(x^2 + y^2)^{\Delta'}} \frac{\pi^{d/2} \Gamma(\Delta + \Delta' - d/2)}{\Gamma(\Delta)^2 \Gamma(\Delta')^2} \quad (7.28)$$

$$\times \left( \frac{\partial}{\partial s} \right)^{\Delta'-1} \left\{ s^{\Delta-1} \left( \frac{\partial}{\partial s} \right)^{\Delta-1} I(s, t) \right\}$$

While the function  $I(s, t)$  is not elementary, its asymptotic behavior is easily obtained.

In the *direct channel* or *t-channel*, we have  $|x_{13}| \ll |x_{12}|$  and  $|x_{24}| \ll |x_{12}|$ , so that we have both  $s, t \rightarrow 0$ . Of principal interest will be the contribution which contains logarithms of  $s$ , and this part is given by (for the full asymptotics, [135]); see also [139],

$$I^{\log}(s, t) = -\ln s \sum_{k=0}^{\infty} a_k(t) s^k \quad a_k(t) = \int_{-1}^{+1} d\lambda \frac{(1 - \lambda^2)^k}{(1 + \lambda t)^{k+1}} \quad (7.29)$$

In the two *crossed channels*, we have  $s \rightarrow 1/2$ : in the *s-channel*  $|x_{12}|, |x_{34}| \ll |x_{13}|$  for which  $t \rightarrow -1$ ; in the *u-channel*  $|x_{23}|, |x_{14}| \ll |x_{34}|$  for which  $t \rightarrow +1$ . Of principal interest will be the contribution which contains logarithms of  $(1 - t^2)$ , and this part is given by (for the full asymptotics, see [135]),

$$I^{\log}(s, t) = -\ln(1 - t^2) \sum_{k=0}^{\infty} (1 - 2s)^k \alpha_k(t) \quad \alpha_k(t) = \sum_{\ell=0}^{\infty} \frac{\Gamma(\ell + \frac{1}{2})(1 - t^2)^\ell}{\Gamma(\frac{1}{2})(2\ell + k + 1)\ell!} \quad (7.30)$$

## 7.8 Calculation of the 4-point Exchange Diagrams

A direct approach to the calculation of the exchange graphs for scalar and gravitons is to insert the scalar or graviton propagators computed previously and then to perform the integrals over the 3-point interaction vertices. This approach was followed in [79, 134, 135]. However, it is also possible to exploit the special space-time properties of conformal symmetry to take a more convenient approach discussed in [137]. This approach consists in first computing the 3-point interaction integral with two boundary-to-bulk propagators (say to vertices 1 and 3) with the bulk-to-bulk propagator between the same interaction vertex and an arbitrary bulk point. Conformal invariance and the assumption of integer dimension  $\Delta \geq d/2$  makes this into a very simple object. We shall follow the last method to evaluate the exchange graphs.

For the *scalar exchange diagram*, we need to compute the following integral<sup>viii</sup>

$$A(w, x_1, x_3) = \int_H \frac{d^{d+1}z}{z_0^{d+1}} G_\Delta(w, z) K_{\Delta_1}(z, x_1) K_{\Delta_3}(z, x_3) \quad (7.31)$$

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<sup>viii</sup>In this subsection, we shall not write explicitly the propagator normalization constants  $C_\Delta$ ; however, they will be properly restored in the next subsection.



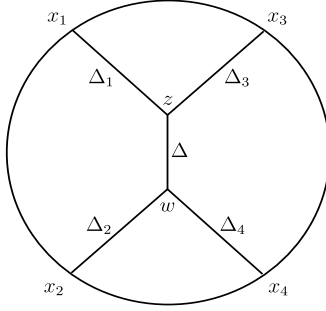


Figure 9: The  $t$ -channel exchange graph

As in the past, we simplify the integral by using translation invariance to translate  $x_1$  to 0, and then performing an inversion. As a result,

$$A(w, x_1, x_3) = |x_{13}|^{-2\Delta_3} I(w' - x'_{13}), \quad I(w) = \int_H \frac{d^5 z}{z_0^5} G_\Delta(w, z) \frac{z_0^{\Delta_1 + \Delta_3}}{z^{2\Delta_3}} \quad (7.32)$$

We now use the fact that  $G_\Delta$  is a Green function and satisfies  $(\square_w + \Delta(\Delta - d))G_\Delta(w, z) = \delta(w, z)$ , so that

$$(\square_w + \Delta(\Delta - d))I(w) = \frac{w_0^{\Delta_1 + \Delta_3}}{w^{2\Delta_3}} \quad (7.33)$$

In terms of the scale invariant combination  $\zeta = w_0^2/w^2$ , we have  $I(w) = w_0^{\Delta_{13}} f_S(\zeta)$ ,  $\Delta_{13} = \Delta_1 - \Delta_3$  and the function  $f_S$  now satisfies the following differential equation

$$4\zeta^2(\zeta - 1)f_S'' + 4\zeta[(\Delta_{13} + 1)\zeta - \Delta_{13} + d/2 - 1]f_S' + (\Delta - \Delta_{13})(\Delta + \Delta_{13} - d)f_S = \zeta^{\Delta_3} \quad (7.34)$$

Making the change of variables  $\sigma = 1/\zeta$ , we find that the new differential equation is manifestly of the hypergeometric type and is solved by

$$f_S(\zeta) = F\left(\frac{\Delta - \Delta_{13}}{2}, \frac{d - \Delta - \Delta_{13}}{2}; \frac{d}{2}; 1 - \frac{1}{\zeta}\right) \quad (7.35)$$

The other linearly independent solution to the hypergeometric equation is singular as  $\zeta \rightarrow 1$ , which is unacceptable since the original integral was perfectly regular in this limit (which corresponds to  $\vec{w} \rightarrow 0$ ).

It is easier, however, to find the solutions in terms of a power series,  $f_S(\zeta) = \sum_k f_{Sk} \zeta^k$ . Upon substitution into (7.34), we find solutions that truncate to a finite number of terms in  $\zeta$ , provided  $\Delta_1 + \Delta_3 - \Delta$  is a positive integer. Notice that  $k$  need not take integer values, rather  $k - \Delta_3$  must be integer. The series truncates from above at  $k_{\max} = \Delta_3 - 1$ , so that  $f_{Sk} = 0$  when  $k \geq \Delta_3$ , and

$$f_{Sk} = \frac{\Gamma(k)\Gamma(k + \Delta_{13})\Gamma(\frac{1}{2}\{\Delta_1 + \Delta_3 - \Delta\})\Gamma(\frac{1}{2}\{\Delta + \Delta_1 + \Delta_3 - d\})}{4\Gamma(\Delta_1)\Gamma(\Delta_3)\Gamma(k + 1 + \frac{1}{2}\{\Delta_{13} - \Delta\})\Gamma(k + 1 + \frac{1}{2}\{\Delta_{13} + \Delta - d\})} \quad (7.36)$$

Still under the assumption that  $\Delta_1 + \Delta_3 - \Delta$  is a positive integer, the series also truncates from below at  $k_{\min} = \frac{1}{2}(\Delta - \Delta_{13})$ .

It remains to complete the calculation and substitute the above partial result into the full exchange graphs. The required integral is

$$S(x_i) = \int_H dw \sqrt{g} K_{\Delta_2}(w, x_2) K_{\Delta_4}(w, x_4) A(w, x_1, x_3) \quad (7.37)$$

Remarkably, the expansion terms  $w_0^{\Delta_{13}} \zeta^k = w_0^{\Delta_1 + \Delta_3 + 2k} / w^{2k}$  are precisely of the form of the product of two boundary-to-bulk propagators, one with dimension  $k$ , the other with dimension  $\Delta_{13} + k$ . Thus, the scalar exchange diagram may be written as a sum over contact graphs in the following way,

$$S(x_i) = \sum_{k=k_{\min}}^{k_{\max}} f_{S_k} |x_{13}|^{-2\Delta_3 + 2k} D_{k \Delta_{13} + k \Delta_2 \Delta_4}(x_i) \quad (7.38)$$

The evaluation of the contact graphs was carried out in the preceding subsection for the special cases  $\Delta_1 = \Delta_3$  and  $\Delta_2 = \Delta_4$ .

For the *massless graviton exchange diagram*, we need to compute the integral,

$$A_{\mu\nu}(w, x_1, x_2) = \int_H \frac{d^{d+1}z}{z^{d+1}} G_{\mu\nu\mu'\nu'}(w, z) T^{\mu'\nu'}(z, x_1, x_3) \quad (7.39)$$

where the stress tensor is generated by two boundary-to-bulk scalar propagators which we assume both to be of dimension  $\Delta_1$ ,

$$\begin{aligned} T^{\mu'\nu'}(z, x_1, x_3) = & \nabla^{\mu'} K_{\Delta_1}(z, x_1) \nabla^{\nu'} K_{\Delta_1}(z, x_3) - \frac{1}{2} g^{\mu'\nu'} \left[ \nabla_{\rho'} K_{\Delta_1}(z, x_1) \nabla^{\rho'} K_{\Delta_1}(z, x_3) \right. \\ & \left. + \Delta_1(\Delta_1 - d) K_{\Delta_1}(z, x_1) \nabla^{\rho'} K_{\Delta_1}(z, x_3) \right] \end{aligned} \quad (7.40)$$

Under translation of  $x_1$  to 0 and inversion, then using the symmetries of rank 2 symmetric tensors on  $\text{AdS}_5$ , and finally using the operator  $W$  on both sides of (refgraveq), we find

$$\begin{aligned} A_{\mu\nu}(w, x_1, x_3) &= \frac{1}{w^4 |x_{13}|^{2\Delta_1}} J_{\mu\kappa}(w) J_{\nu\lambda}(w) I_{\kappa\lambda}(w' - x'_{13}) \\ I_{\kappa\lambda}(w) &= \left( \frac{\delta_{0\mu} \delta_{0\nu}}{w_0^2} - \frac{1}{d-1} g_{\mu\nu} \right) f_G(\zeta) + \nabla_{(\mu} v_{\nu)} \end{aligned} \quad (7.41)$$

where the field  $v_\mu$  represents an immaterial action of a diffeomorphism while the function  $f_G(\zeta)$  satisfies the first order differential equation

$$2\zeta(1-\zeta)f'_G(\zeta) - (d-2)f_G(\zeta) = \Delta_1 \zeta^{\Delta_1} \quad (7.42)$$

It is again possible to solve this equation via a power series  $f_G(\zeta) = \sum_k f_{Gk} \zeta^k$ . The range of  $k$  is found to be  $d/2 - 1 = k_{\min} \leq k \leq k_{\max} = \Delta_1 - 1$ , provided  $\Delta_1 - d/2$  is a non-negative

integer and  $d > 2$ . The coefficients are then given by

$$f_{Gk} = -\frac{\Delta_1 \Gamma(k) \Gamma(\Delta_1 + 1 - d/2)}{\Gamma(\Delta_1) \Gamma(k + 2 - d/2)} \quad (7.43)$$

The result is particularly simple for the case of interest here when  $d = 4$  and  $\Delta_1$  integer,

$$f_G(\zeta) = -\frac{\Delta_1}{2\Delta_1 - 2} (\zeta + \zeta^2 + \dots + \zeta^{\Delta_1 - 1}) \quad (7.44)$$

Again, this result may be substituted into the remaining integral in  $w$  versus the boundary-to-bulk propagators from the interaction point  $w$  to  $x_2$  and  $x_4$ , thereby yielding again contributions proportional to contact terms.

## 7.9 Structure of Amplitudes

The full calculations of the graviton exchange amplitudes are quite involved and will not be reproduced completely here [135]. Instead, we quote the contributions to the amplitudes from the correlator  $[\mathcal{O}_\phi(x_1) \mathcal{O}_C(x_2) \mathcal{O}_\phi(x_3) \mathcal{O}_C(x_4)]$ , where the graviton is exchanged in the  $t$ -channel only. The sum of the axion exchange graph  $I_s$  in the  $s$ -channel, of the axion exchange  $I_u$  in the  $u$ -channel and of the quartic contact graph  $I_q$  is listed separately from the graviton contribution  $I_g$ ,

$$\begin{aligned} I_s + I_u + I_q &= \frac{6^4}{\pi^8} \left[ 64x_{24}^2 D_{4455} - 32D_{4444} \right] \\ I_{\text{grav}} &= \frac{6^4}{\pi^8} \left[ 8\left(\frac{1}{s} - 2\right)x_{24}^2 D_{4455} + \frac{64x_{24}^2}{9s x_{13}^2} D_{3355} + \frac{16x_{24}^2}{3s x_{13}^4} D_{2255} \right. \\ &\quad \left. + 18D_{4444} - \frac{46}{9x_{13}^2} D_{3344} - \frac{40}{9x_{13}^4} D_{2244} - \frac{8}{3x_{13}^6} D_{1144} \right] \end{aligned} \quad (7.45)$$

The most interesting information is contained in the power singularity part of this amplitude as well as in the part containing logarithmic singularities. Both are obtained from the singular parts of the universal function  $I(s, t)$  in terms of which the contact functions  $D_{\Delta_1 \Delta_2 \Delta_4 \Delta_4}$  may be expressed.

## 7.10 Power Singularities

In the  $s$ -channel and  $u$ -channel, no power singularities occur in the supergravity result. This is consistent with the fact that there are no power singular terms in the OPE of  $\mathcal{O}_\phi$  with  $\mathcal{O}_C$ , since the resulting composite operator would have  $U(1)_Y$  hypercharge 4, and the lowest operator with those quantum numbers has dimension 8. (More details on this kind of argument will be given in §7.12.)

In the  $t$ -channel, where  $|x_{13}|, |x_{24}| \ll |x_{12}|$ , we have  $s, t \rightarrow 0$ , with  $s \sim t^2$ . The power singularities in this channel come entirely from the graviton exchange part, given by

$$I_{\text{grav}} \Big|_{\text{sing}} = \frac{2^{10}}{35\pi^6} \frac{1}{x_{13}^8 x_{24}^8} \left[ s(7t^2 + 6t^4) + s^2(-7 + 3t^2) - 8s^3 \right] \quad (7.46)$$

To compare this behavior with the singularities expected from the OPE, we derive first the behavior of the variables  $s$  and  $t$  in the  $t$ -channel limit,

$$s \sim \frac{x_{13}^2 x_{24}^2}{2x_{12}^4} \quad t \sim -\frac{x_{13} \cdot J(x_{12}) \cdot x_{24}}{x_{12}^2} \quad (7.47)$$

where  $J_{ij}(x) \equiv \delta_{ij} - 2x_i x_j / x^2$  is the conformal inversion Jacobian tensor. Therefore, the leading singularity in the graviton exchange contribution may be written as

$$I_{\text{grav}} \Big|_{\text{sing}} = \frac{2^6}{5\pi^6} \frac{1}{x_{13}^6 x_{24}^6} \frac{4(x_{13} \cdot J(x_{12}) \cdot x_{24})^2 - x_{13}^2 x_{24}^2}{x_{12}^8} \quad (7.48)$$

with further subleading terms suppressed by additional powers of  $x_{13}^2/x_{12}^2$  and  $x_{24}^2/x_{12}^2$ . The leading contribution above describes the exchanges of an operator of dimension 4, whose tensorial structure is that of the stress tensor.

Note that there is also a term corresponding to the exchange of the identity operator, with behavior  $x_{13}^{-8} x_{24}^{-8}$ , which derives from the disconnected contribution to the correlator in Fig 5 (a). Note that there is no contribution in the singular terms that corresponds to the exchange of an operator of dimension 2. One candidate would be  $\mathcal{O}_2$  which is a Lorentz scalar; however, it is a  $\mathbf{20}'$  under  $SU(4)_R$ , and therefore not allowed in the OPE of two singlets. The other candidate is the Konishi operator, which is both a Lorentz and  $SU(4)_R$  singlet. The fact that it is not seen here is consistent with the fact that its dimension becomes very large  $\sim \lambda^{1/4}$  in the limit  $\lambda \rightarrow \infty$  and is dual to a massive string excitation.

## 7.11 Logarithmic Singularities

The logarithmic singularities in the  $t$ -channel are produced by both the scalar exchange and contact graphs as well as by the graviton exchange graph [135]. They are given by

$$\begin{aligned} I_s + I_u + I_q \Big|_{\log} &= \frac{960}{\pi^6} \frac{\ln s}{x_{13}^8 x_{24}^8} \sum_{k=0}^{\infty} s^{k+4} (k+1)^2 (k+2)^2 (k+3)^2 (3k+4) a_{k+3}(t) \quad (7.49) \\ I_{\text{grav}} \Big|_{\log} &= \frac{24}{\pi^6} \frac{\ln s}{x_{13}^8 x_{24}^8} \sum_{k=0}^{\infty} s^{k+4} \frac{\Gamma(k+4)}{\Gamma(k+1)} \left\{ (k+4)^2 (15k^2 + 55k + 42) a_{k+4}(t) \right. \\ &\quad \left. - 2(5k^2 + 20k + 16)(3k^2 + 15k + 22) a_{k+3}(t) \right\} \end{aligned}$$

To leading order, these expressions simplify as follows,

$$\begin{aligned} I_s + I_u + I_q \Big|_{\log} &= + \frac{2^7 \cdot 3^3}{7\pi^6 x_{12}^{16}} \ln \frac{x_{13}^2 x_{24}^2}{x_{12}^4} \quad (7.50) \\ I_{\text{grav}} \Big|_{\log} &= - \frac{2^7 \cdot 3}{7\pi^6 x_{12}^{16}} \ln \frac{x_{13}^2 x_{24}^2}{x_{12}^4} \end{aligned}$$

Assembling all logarithmic contributions for the various correlators, we get, [117],

$$\begin{aligned}
\langle \mathcal{O}_\phi \mathcal{O}_\phi \mathcal{O}_\phi \mathcal{O}_\phi \rangle_{\log} &= -\frac{208}{21N^2} \frac{1}{x_{12}^{16}} \ln \frac{x_{13}^2 x_{24}^2}{x_{12}^4} && \text{t-channel} \\
\langle \mathcal{O}_C \mathcal{O}_C \mathcal{O}_C \mathcal{O}_C \rangle_{\log} &= -\frac{208}{21N^2} \frac{1}{x_{12}^{16}} \ln \frac{x_{13}^2 x_{24}^2}{x_{12}^4} && \text{t-channel} \\
\langle \mathcal{O}_\phi \mathcal{O}_C \mathcal{O}_\phi \mathcal{O}_C \rangle_{\log} &= +\frac{128}{21N^2} \frac{1}{x_{12}^{16}} \ln \frac{x_{13}^2 x_{24}^2}{x_{12}^4} && \text{t-channel} \\
\langle \mathcal{O}_\phi \mathcal{O}_C \mathcal{O}_\phi \mathcal{O}_C \rangle_{\log} &= -\frac{8}{N^2} \frac{1}{x_{13}^{16}} \ln \frac{x_{12}^2 x_{34}^2}{x_{13}^4} && \text{s-channel} \quad (7.51)
\end{aligned}$$

Here, the overall coupling constant factor of  $\kappa_5^2$  has been converted to a factor of  $1/N^2$  with the help of the relation  $\kappa_5^2 = 4\pi^2/N^2$ , a relation that will be explained and justified in (8.3). Further investigations of these log singularities may be found in [138].

## 7.12 Anomalous Dimension Calculations

We shall use the supergravity calculations of the 4-point functions for the operators  $\mathcal{O}_\phi$  and  $\mathcal{O}_C$  to extract anomalous dimensions of double-trace operators built out of linear combinations of  $[\mathcal{O}_\phi \mathcal{O}_\phi]$ ,  $[\mathcal{O}_C \mathcal{O}_C]$  and  $[\mathcal{O}_\phi \mathcal{O}_C]$ . This was done in [117] by taking the limits in various channels of the three 4-point functions  $\langle \mathcal{O}_\phi(x_1) \mathcal{O}_\phi(x_2) \mathcal{O}_\phi(x_3) \mathcal{O}_\phi(x_4) \rangle$ ,  $\langle \mathcal{O}_C(x_1) \mathcal{O}_C(x_2) \mathcal{O}_C(x_3) \mathcal{O}_C(x_4) \rangle$  and  $\langle \mathcal{O}_\phi(x_1) \mathcal{O}_C(x_2) \mathcal{O}_\phi(x_3) \mathcal{O}_C(x_4) \rangle$ . For example, we extract the following simple behavior from the *s-channel limit*  $x_{12}, x_{34} \rightarrow 0$  of the correlator  $\langle \mathcal{O}_\phi(x_1) \mathcal{O}_C(x_2) \mathcal{O}_\phi(x_3) \mathcal{O}_C(x_4) \rangle$ ,

$$\mathcal{O}_\phi(x_1) \mathcal{O}_C(x_2) = A_{\phi c}(x_{12}\mu) [\mathcal{O}_\phi \mathcal{O}_C]_\mu(x_2) + \dots \quad (7.52)$$

where  $\mu$  is an arbitrary renormalization scale for the composite operators and  $A_{\phi c}$  is the corresponding logarithmic coefficient function, whose precise value in the large  $N$ , large  $\lambda$  limit is available from the logarithmic singularities of the correlator, and is given by

$$A_{\phi c}(x_{12}\mu) = 1 - \frac{16}{N^2} \ln(x_{12}\mu) \quad (7.53)$$

This leading behavior receives further corrections both in inverse powers of  $N$  and  $\lambda$ .

From the *t-channel* and *u-channel* of the same correlators, we extract the leading terms in the OPE of two  $\mathcal{O}_\phi$ 's and of two  $\mathcal{O}_C$ 's as follows,

$$\begin{aligned}
\mathcal{O}_\phi(x_1) \mathcal{O}_\phi(x_3) &= S(x_1, x_3) + C_{\phi\phi} [\mathcal{O}_\phi \mathcal{O}_\phi]_\mu + C_{\phi c} [\mathcal{O}_C \mathcal{O}_C]_\mu + C_{\phi T} [TT]_\mu + \dots \\
\mathcal{O}_C(x_1) \mathcal{O}_C(x_3) &= S(x_1, x_3) + C_{c\phi} [\mathcal{O}_\phi \mathcal{O}_\phi]_\mu + C_{cc} [\mathcal{O}_C \mathcal{O}_C]_\mu + C_{cT} [TT]_\mu + \dots \quad (7.54)
\end{aligned}$$

where the term  $S(x_1, x_3)$  contains all the *power singular terms* in the expansion, and is given schematically by

$$S(x_1, x_3) = \frac{I}{x_{13}^8} + \frac{T(x_3)}{x_{13}^4} + \frac{\partial T(x_3)}{x_{13}^3} + \frac{\partial \partial T(x_3)}{x_{13}^2} + \frac{\partial \partial \partial T(x_3)}{x_{13}} \quad (7.55)$$

The coefficient functions may be extracted from the logarithmic behavior as before,

$$\begin{aligned} C_{\phi\phi} = C_{cc} &= 1 - \frac{208}{21N^2} \ln(x_{13}\mu) \\ C_{\phi c} = C_{c\phi} &= 1 + \frac{128}{21N^2} \ln(x_{13}\mu) \end{aligned} \quad (7.56)$$

Unfortunately, the coefficient functions  $C_{\phi T}$  and  $C_{cT}$  are not known at this time as their evaluation would involve the highly complicated calculation involving two external stress tensor insertions.

To make progress, we make use of a continuous symmetry of supergravity, namely  $U(1)_Y$  *hypercharge invariance*. Most important for us here is that the operator

$$\mathcal{O}_B \equiv \frac{1}{\sqrt{2}} \{ \mathcal{O}_\phi + i\mathcal{O}_C \} \quad (7.57)$$

has hypercharge  $Y = 2$ , which is the unique highest values attained amongst the canonical supergravity fields, as may be seen from the Table 7. We may now re-organize the OPE's of operators  $\mathcal{O}_\phi$  and  $\mathcal{O}_C$  in terms of  $\mathcal{O}_B$  and  $\mathcal{O}_B^*$ . The OPE of  $\mathcal{O}_B$  with  $\mathcal{O}_B^*$  contains the identity operator, the stress tensor and its derivatives and powers, as well as the  $Y = 0$  operator  $[\mathcal{O}_B\mathcal{O}_B^*]$ ,

$$\mathcal{O}_B(x_1)\mathcal{O}_B^*(x_2) = S(x_1, x_2) + C_{BT}[TT]_\mu + C_{BB^*}[\mathcal{O}_B\mathcal{O}_B^*]_\mu + \dots \quad (7.58)$$

while the  $Y = 4$  channel of the OPE is given by

$$\mathcal{O}_B(x_1)\mathcal{O}_B(x_2) = (C_{\phi\phi} - C_{\phi c})\text{Re}[\mathcal{O}_B\mathcal{O}_B]_\mu + iA_{\phi c}\text{Im}[\mathcal{O}_B\mathcal{O}_B]_\mu \quad (7.59)$$

Since the smallest dimensional operator of hypercharge  $Y = 4$  is the composite  $[\mathcal{O}_B\mathcal{O}_B]$ , we see that the power singularity terms  $S(x_1, x_3)$  indeed had to be the same for both OPE's in (7.54). By the same token, the rhs of (7.59) must be proportional to  $[\mathcal{O}_B\mathcal{O}_B]_\mu$ , so we must have  $C_{\phi\phi} - C_{\phi c} = A_{\phi c}$ , which is indeed borne out by the explicit calculational results of (7.53) and (7.56). In summary, we have a single simple OPE

$$\mathcal{O}_B(x_1)\mathcal{O}_B(x_2) = A_{\phi c}(x_{12}\mu)[\mathcal{O}_B\mathcal{O}_B]_\mu + \dots \quad (7.60)$$

from which the anomalous dimension may be found to be, [117],

$$\gamma_{[\mathcal{O}_B\mathcal{O}_B]} = \gamma_{[\mathcal{O}_B^*\mathcal{O}_B^*]} = -16/N^2 \quad (7.61)$$

There is another operator occurring in this OPE channel of which we know the anomalous dimension. Indeed, the double-trace operator  $[\mathcal{O}_2\mathcal{O}_2]_{105}$  is 1/2 BPS, and thus has vanishing anomalous dimension. Its maximal descendant  $Q^4\bar{Q}^4[\mathcal{O}_2\mathcal{O}_2]_{105}$  therefore has  $Y = 0$  and unrenormalized dimension 8. For more on the role of the  $U(1)_Y$  symmetry, see [118]. The study of the OPE via the 4-point function has also revealed some surprising non-renormalization effects, not directly related to the BPS nature of the intermediate operators. In the OPE of two half-BPS  $\mathbf{20}'$  operators, for example, the  $\mathbf{20}'$  intermediate state is *not* chiral. Yet, to lowest order at strong coupling, its dimension was found to be protected; see [119] and [120].

### 7.13 Check of N-dependence

The prediction for the anomalous dimension of the operator  $[\mathcal{O}_B\mathcal{O}_B]$  to order  $1/N^2$  obtained from supergravity calculations holds for infinitely large values of the 't Hooft coupling  $\lambda = g_{YM}^2 N$  on the SYM side. As the regimes of couplings for possible direct calculations do not overlap, we cannot directly compare this prediction with a calculation on the SYM side. However, it is very illuminating to reproduce the  $1/N^2$  dependence of the anomalous dimension from standard large  $N$  counting rules in SYM theory [117]. We proceed by expanding  $\mathcal{N} = 4$  SYM in  $1/N$ , while keeping the 't Hooft coupling fixed (and perturbatively small). The strategy will be to isolate the general structure of the expansion and then to seek the limit where  $\lambda \rightarrow \infty$ .

To be concrete, we study the correlator  $\langle \mathcal{O}_\phi \mathcal{O}_c \mathcal{O}_\phi \mathcal{O}_c \rangle$ , though our results will apply generally.

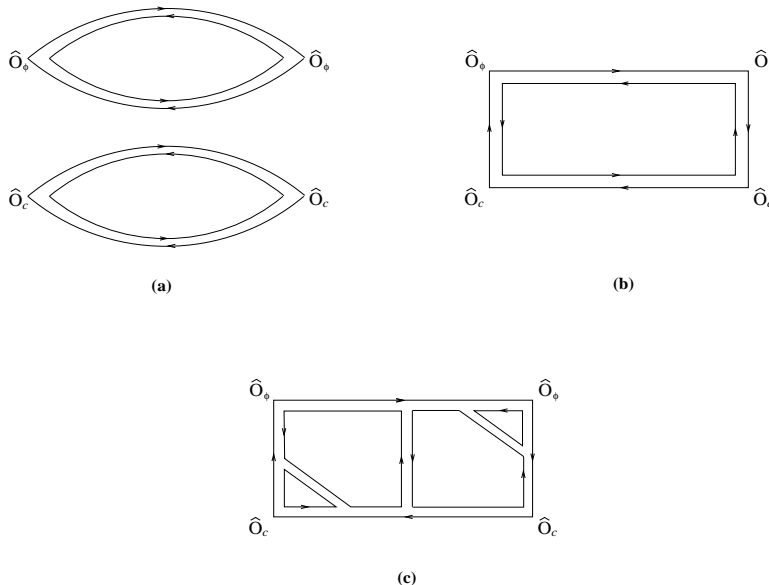


Figure 10: Large  $N$  counting for the 4-point function

First, we normalize the individual operators via their 2-point functions, which to leading order in large  $N$  requires

$$\mathcal{O}_c = \frac{1}{N} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} + \dots \quad \mathcal{O}_\phi = \frac{1}{N} \text{tr} F_{\mu\nu} F^{\mu\nu} + \dots \quad (7.62)$$

In computing the 4-point function  $\langle \mathcal{O}_\phi \mathcal{O}_c \mathcal{O}_\phi \mathcal{O}_c \rangle$ , there will first be a *disconnected contribution* of the form  $\langle \mathcal{O}_\phi \mathcal{O}_\phi \rangle \langle \mathcal{O}_c \mathcal{O}_c \rangle$ , which thus contributes precisely to order  $N^0$ . The simplest connected contribution is a Born graph with a single gluon loop; the operator normalizations contribute  $N^{-4}$ , while the two color loops contribute  $N^2$ , thereby suppressing the connected contribution by a factor of  $N^{-2}$  compared to the disconnected one. This Born graph has no logarithms because it is simply the product of 4 propagators. Perturbative corrections with internal interaction vertices will however generate logarithmic

corrections, and thus contributions to anomalous dimensions. In the large  $N$  limit, planar graphs will dominate and the only corrections are due to non-trivial  $\lambda$ -dependence with the same expansion order  $N^{-2}$  and the connected contribution will take the form

$$\langle \mathcal{O}_\phi \mathcal{O}_c \mathcal{O}_\phi \mathcal{O}_c \rangle_{\text{conn}} = \frac{1}{N^2} f(\lambda) + O\left(\frac{1}{N^4}\right) \quad (7.63)$$

For the anomalous dimensions, a similar expansion will hold,

$$\gamma(N, \lambda) = \frac{1}{N^2} \bar{\gamma}(\lambda) + O\left(\frac{1}{N^4}\right) \quad (7.64)$$

The above results were established perturbatively in the 't Hooft coupling. To compare with the supergravity results,  $f$  and  $\bar{\gamma}$  should admit well-behaved  $\lambda \rightarrow \infty$  limits. Our supergravity calculation in fact established that  $\bar{\gamma}_{[\mathcal{O}_B \mathcal{O}_B]}(\lambda = \infty) = -16$ , a result that could of course not have been gotten from Feynman diagrams in SYM theory.

The calculation of AdS four point functions in weak coupling perturbation theory was carried out in [140] and [141]; string corrections to 4-point functions were considered early on in [142] and [143]; further 4-point function calculations in the AdS setting may be found in [144], [145] and [146]. More general correlators of 4-point functions and higher corresponding to the insertion of currents and tensor forms may be found in [148], [149], [150] and [151]. Finally, an approach to correlation functions based on the existence of a higher spin field theory in Anti-de Sitter space-time may be found in a series of papers [152]; see also [153]. Finally, effects of instantons on SYM and AdS/CFT correlators were explored recently in [154], [156], [155], [159], [157], [158]. Possible constraints on correlators in AdS/CFT and  $\mathcal{N} = 4$  SYM from S-duality have been investigated by [160]. Finally, very recently, correlators have been evaluated exactly for strings propagating in AdS<sub>3</sub> in [161].



## 8 How to Calculate CFT<sub>d</sub> Correlation Functions from AdS<sub>d+1</sub> Gravity

The main purpose of this chapter is to discuss the techniques used to calculate correlation functions in  $\mathcal{N} = 4$ ,  $d = 4$  SYM field theory from Type IIB  $D = 10$  supergravity. We will begin with a quick summary of the basic ideas of the correspondence between the two theories. These were discussed in more detail in earlier sections, but we wish to make this chapter self-contained. Other reviews we recommend to readers are the broad treatment of [7] and the 1999 TASI lectures of Klebanov [162] in which the AdS/CFT correspondence is motivated from the viewpoints of  $D$ -brane and black hole solutions, entropy and absorption cross-sections.

The  $\mathcal{N} = 4$  SYM field theory is a 4-dimensional gauge theory with gauge group  $SU(N)$  and  $R$ -symmetry or global symmetry group  $SO(6) \sim SU(4)$ . Elementary fields are all in the adjoint representation of  $SU(N)$  and are represented by traceless Hermitean  $N \times N$  matrices. There are 6 elementary scalars  $X^i(x)$ , 4 fermions  $\psi^a(x)$ , and the gauge potential  $A_j(x)$ . The theory contains a unique coupling constant, the gauge coupling  $g_{YM}$ . It is known that the only divergences of elementary Green's functions are those of wave function renormalization which is unobservable and gauge-dependent. The  $\beta$ -function  $\beta(g_{YM})$  vanishes, so the theory is conformal invariant. The bosonic symmetry group of the theory is the direct product of the conformal group  $SO(2, 4) \sim SU(2, 2)$  and the  $R$ -symmetry  $SU(4)$ . These combine with 16 Poincaré and 16 conformal supercharges to give the superalgebra  $SU(2, 2|4)$  which is the over-arching symmetry of the theory.

Observables in a gauge theory must be gauge-invariant quantities, such as:

1. Correlation functions of gauge invariant local composite operators — the subject on which we focus,
2. Wilson loops — not to be discussed, See, for example, [163, 164, 165]

Our primary interest is in correlation functions of the chiral primary operators<sup>ix</sup>

$$\text{tr}X^k \equiv N^{\frac{1-k}{2}} \text{tr} \left( X^{\{i_1} X^{i_2} \dots X^{i_k\}} \right) - \text{traces} \quad (8.1)$$

These operators transform as rank  $k$  symmetric traceless  $SO(6)$  tensors – irreducible representations whose Dynkin designation is  $[0, k, 0]$ . For  $k = 2, 3, 4$  the dimensions of these representations are 20, 50, 105, respectively.

**Ex. 1:** *What is the dimension of the  $[0, 5, 0]$  representation?*

The  $\text{tr}X^k$  are lowest weight states of **short** representations of  $SU(2, 2|4)$ . The condition for a short representation is the relation  $\Delta_{\text{tr}X^k} = k$  between scale dimension  $\Delta$  and  $SO(6)$  rank. Since the latter must be an integer, the former is quantized. The scale dimension of chiral primary operators (and all descendents) is said to be “protected” It is given for all  $g_{YM}$  by its free-field value (i.e. the value at  $g_{YM} = 0$ ). This is to be contrasted with

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<sup>ix</sup>the normalization factor  $N^{\frac{1-k}{2}}$  is chosen so that all correlation functions of these operators are of order  $N^2$  for large  $N$ .

the many composite operators which belong to **long** representations of  $SU(2, 2|4)$ . For example, the Konishi operator  $K(x) = \text{tr}[X^i X^i]$  is the primary of a long representation. In the weak coupling limit, it is known [166] that  $\Delta_K = 2 + 3g_{YM}^2 N/4\pi^2 + \mathcal{O}(g_{YM}^4)$ . The existence of a gauge invariant operator with anomalous dimension is one sign that the field theory is non-trivial, not a cleverly disguised free theory.

In Sec. 3.4 it was discussed how  $SU(2, 2|4)$  representations are “filled out” with descendent states obtained by applying SUSY generators with  $\Delta = \frac{1}{2}$  to the primaries. Descendants can be important. For example, the descendants of the lowest chiral primary  $\text{tr}X^2$  include the 15  $SO(6)$  currents, the 4 supercurrents, and the stress tensor.

Some years ago ‘t Hooft taught us (for a review, see [167]) that it is useful to express amplitudes in an  $SU(N)$  gauge theory in terms of  $N$  and the ‘t Hooft coupling  $\lambda = g_{YM}^2 N$ . Any Feynman diagram can be redrawn as a sum of color-flow diagrams with definite Euler character  $\chi$  (in the sense of graph theory).  $n$ -point functions of the operators  $\text{tr}X^k$  are of the form

$$N^\chi F(\lambda, x_i) = N^\chi [f_0(x_i) + \lambda f_1(x_i) + \dots] \quad (8.2)$$

The right side shows the beginning of a weak coupling expansion. One can see that planar diagrams (those with  $\chi = 2$ ) dominate in the large  $N$ -limit.

The extremely remarkable fact of the AdS/CFT correspondence is that the planar contribution to  $n$ -point correlation functions of operators  $\text{tr}X^k$  and descendants can be calculated (in the limit  $N \rightarrow \infty$ ,  $\lambda \gg 1$ ) from **classical** supergravity, a strong coupling limit of a QFT<sub>4</sub> without gravity from classical calculations in a D=5 gravity theory. Results are interpreted as the sum of the series in (8.2). Information about operators in long representations can be obtained by including string scale effects. It is known that their scale dimensions are of order  $\lambda^{\frac{1}{4}}$  in the limit above. They decouple from supergravity correlators.

This claim brings us to the supergravity side of the duality, namely to type IIB,  $D = 10$  supergravity which has the product space-time  $\text{AdS}_5 \times S^5$  as a classical “vacuum solution”. The first hint of some relation to  $\mathcal{N} = 4$  SYM theory is the match of the isometry group  $SO(2, 4) \times SO(6)$  with the conformal and  $R$ -symmetry groups of the field theory. The vacuum solution is also invariant under 16+16 supercharges and thus has the same  $SU(2, 2|4)$  superalgebra as the field theory.

Type IIB supergravity is a complicated theory whose structure was discussed in Secs. 4.4 and 4.5. Here we describe only the essential points necessary to understand the correspondence with  $\mathcal{N} = 4$  SYM theory. Since the supergravity theory is the low energy limit of IIB string theory, the 10D gravitational coupling may be expressed in terms of the dimensionless string coupling  $g_s$  and the string scale  $\alpha'$  (of dimension  $l^2$ ). The relation is  $\kappa_{10}^2 = 8\pi G_{10} = 64\pi^7 g_s^2 \alpha'^4$ . The length scale of the  $\text{AdS}_5$  and  $S_5$  factors of the vacuum space-time is  $L$  with  $L^4 = 4\pi\alpha'^2 g_s N$ . The integer  $N$  is determined by the flux of the self-dual 5-form field strength on  $S_5$ . The volume of  $S_5$  is  $\pi^3 L^3$  so the effective 5D gravitational constant is

$$\frac{\kappa_5^2}{8\pi} = G_5 = \frac{G_{10}}{\text{Vol}(S_5)} = \frac{\pi L^3}{2N^2} \quad (8.3)$$

Among the bosonic fields of the theory, we single out the 10D metric  $g_{MN}$  and 5-form  $F_{MNPQR}$ , which participate in the vacuum solution, and the dilaton  $\phi$  and axion  $C$ . Other fields consistently decouple from these and the subsystem is governed by the truncated action (in Einstein frame)

$$S_{\text{IIB}} = \frac{1}{16\pi G_{10}} \int d^{10}z \sqrt{g_{10}} \left\{ R_{10} - \frac{1}{2 \cdot 5!} F_{MNPQR} F^{MNPQR} - \frac{1}{2} \partial_M \phi \partial^M \phi - \frac{1}{2} e^{2\phi} \partial_M C \partial^M C \right\} \quad (8.4)$$

Actually there is no covariant action which gives the self-dual relation  $F_5 = *F_5$  as an Euler-Lagrange equation, and the field equations from  $S_{\text{IIB}}$  must be supplemented by this extra condition.

Using  $x^i$ ,  $i = 0, 1, 2, 3$  as Cartesian coordinates of Minkowski space with metric  $\eta_{ij} = (-+++)$  and  $y^a$ ,  $a = 1, 2, 3, 4, 5, 6$  as coordinates of a flat transverse space, we write the following ansatz for the set of fields above:

$$\begin{aligned} ds_{10}^2 &= \frac{1}{\sqrt{H(y^a)}} \eta_{ij} dx^i dx^j + \sqrt{H(y^a)} \delta_{ab} dy^a dy^b \\ F &= dA + *dA & A &= \frac{1}{H(y^a)} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ \phi &= C \equiv 0 \end{aligned} \quad (8.5)$$

Remarkably the configuration above is a solution of the equations of motion provided that  $H(y^a)$  is a harmonic function of  $y^a$ , *i.e.*

$$\sum_{a=1}^6 \frac{\partial^2}{\partial y^a \partial y^a} H = 0 \quad (8.6)$$

**Ex. 2:** *Verify that the above is a solution. Compute the connection and curvature of the metric as an intermediate step. See the discussion of the Cartan structure equations in Section 9 for some guidance.*

The appearance of harmonic functions is typical of  $D$ -brane solutions to supergravity theories. The solutions (8.5) are  $\frac{1}{2}$ - $BPS$  solutions which support 16 conserved supercharges. This fact may be derived by studying the transformation rules of Type IIB supergravity to find the Killing spinors. A quite general harmonic function is given by

$$H = 1 + \sum_{I=1}^M \frac{L_I^4}{(y - y_I)^4} \quad L_I^4 = 4\pi\alpha'^2 g_s N_I \quad (8.7)$$

This describes a collection of  $M$  parallel stacks of  $D3$ -branes, with  $N_I$  branes located at position  $y^a = y_I^a$  in the transverse space. This “multi-center” solution of IIB supergravity defines a 10-dimensional manifold with  $M$  infinitely long throats as  $y \rightarrow y_I$  and which is asymptotically flat as  $y \rightarrow \infty$ . The curvature invariants are non-singular as  $y \rightarrow y_I$ , and these loci are simply degenerate horizons. The solution has an AdS/CFT interpretation as the dual of a Higgs branch vacuum state of  $\mathcal{N} = 4$  SYM theory, a vacuum in which conformal symmetry is spontaneously broken. However, we are jumping too far ahead.

Let's consider the simplest case of a single stack of  $N$  D3-branes at  $y_I = 0$ . We replace the  $y^a$  by a radial coordinate  $r = \sqrt{y^a y^a}$  plus 5 angular coordinates  $y^\alpha$  on an  $S_5$ . At the same time we take the near-horizon limit. The physical and mathematical arguments for this limit are rather complex and discussed in Sec 5.2 above, in [7] and elsewhere. We simply state that it is the throat region of the geometry that determines the physics of AdS/CFT. We therefore restrict to the throat simply by dropping the 1 in the harmonic function  $H(r)$ . Thus we have the metric

$$ds_{10}^2 = \frac{r^2}{L^2} \eta_{ij} dx^i dx^j + \frac{L^2 dr^2}{r^2} + L^2 d\Omega_5^2 \quad (8.8)$$

where  $d\Omega_5^2$  is the  $SO(6)$  invariant metric on the unit  $S_5$ . The metric describes the product space  $\text{AdS}_5 \times S^5$ . The coordinates  $(x^i, r)$  are collectively called  $z_\mu$  below. These coordinates give the Poincaré patch of the induced metric on the hyperboloid embedded in 6-dimensions [7].

$$Y_0^2 + Y_5^2 - Y_1^2 - Y_2^2 - Y_3^2 - Y_4^2 = L^2 \quad (8.9)$$

**Ex. 3:** Show that the curvature tensor in the  $z_\mu$  directions has the maximal symmetric form  $R_{\mu\nu\rho\lambda} = -\frac{1}{L^2}(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho})$ .

The bulk theory may now be viewed as a supergravity theory in the  $\text{AdS}_5$  space-time with an infinite number of 5D fields obtained by Kaluza-Klein analysis on the internal space  $S_5$ . We will discuss the KK decomposition process and the properties of the 5D fields obtained from it. The main point is to emphasize the 1:1 correspondence between these bulk fields and the composite operators of the  $\mathcal{N} = 4$  SYM theory discussed above.

The linearized field equations of fluctuations about the background (8.8) were analyzed in [51]. All fields of the  $D = 10$  theory are expressed as series expansions in appropriate spherical harmonics on  $S_5$ . Typically the independent 5D fields are mixtures of KK modes from different 10D fields. For example the scalar fields which correspond to the chiral primary operators are superpositions of the trace  $h_\alpha^\alpha$  of metric fluctuations on  $S_5$  with the  $S_5$  components of the 4-form potential  $A_{\alpha\beta\gamma\delta}$ . The independent 5D fields transform in representations of the isometry group  $SU(4) \sim SO(6)$  of  $S_5$  which are determined by the spherical harmonics.

The analysis of [51] leads to a graviton multiplet plus an infinite set of KK excitations. We list the fields of the graviton multiplet, together with the dimensionalities of the corresponding  $SO(6)$  representations: graviton  $h_{\mu\nu}$ ,  $\mathbf{1}$ , gravitini  $\psi_\mu$ ,  $\mathbf{4} \oplus \mathbf{4}^*$ , 2-form potentials  $A_{\mu\nu}$ ,  $\mathbf{6}_c$ , gauge potentials  $A_\mu$ ,  $\mathbf{15}$ , spinors  $\lambda$ ,  $\mathbf{4} \oplus \mathbf{4}^* \oplus \mathbf{20} \oplus \mathbf{20}^* = 48$ , and finally scalars  $\phi$ ,  $\mathbf{20}' \oplus \mathbf{10} \oplus \mathbf{10}^* \oplus \mathbf{1}_c = 42$ . In this notation  $\mathbf{10}^*$  denotes the conjugate of the complex irrep  $\mathbf{10}$ , while  $\mathbf{6}_c$  denotes a doubling of the real 6-dimensional (defining) representation of  $SO(6)$ .

Each of these fields is the base of a KK tower. For the scalar primaries one effectively has the following expansion, after mixing is implemented,

$$\phi(z, y) = \sum_{k=2}^{\infty} \phi_k(z) Y^k(y) \quad (8.10)$$

Here  $Y^k(y)$  denotes a spherical harmonic of rank  $k$ , so that  $\phi_k(z)$  is a scalar field on  $\text{AdS}_5$  which transforms<sup>x</sup> in the  $[0, k, 0]$  irrep of  $SO(6)$ . In the same way that every scalar field on Minkowski space contains an infinite number of momentum modes, each  $\phi_k$  contains an infinite number of modes classified in a unitary irreducible representation of the  $\text{AdS}_5$  isometry group  $SO(2, 4)$ . We will describe these irreps briefly. For more information, see [168, 169, 24, 170]. The group has maximal compact subgroup  $SO(2) \times SO(4)$  and irreps are denoted by  $(\Delta, s, s')$ . The generator of the  $SO(2)$  factor is identified with the energy in the physical setting, and  $\Delta$  is the lowest energy eigenvalue that occurs in the representation. The quantum numbers  $s, s'$  designate the irrep of  $SO(4)$  in which the lowest energy components transform. Unitarity requires the bounds

$$\Delta \geq 2 + s + s' \quad \text{if } ss' > 0 \quad \Delta \geq 1 + s \quad \text{if } s' = 0. \quad (8.11)$$

In general  $\Delta$  need not be integer, but our KK scalars  $\phi_k$  transform in the irrep  $[0, \Delta = k, 0]$  in which the energy and internal symmetry eigenvalues are locked, a condition which gives a short representation of  $SU(2, 2|4)$ .

Each  $\phi_k(z)$  satisfies an equation of motion of the form

$$(\square_{\text{AdS}} - M^2)\phi_k = \text{nonlinear interaction terms} \quad (8.12)$$

The symbol  $\square$  is the invariant Laplacian on  $\text{AdS}_5$ ,

**Ex. 4:** *Obtain its explicit form from the metric in (8.8).*

Each KK mode has a definite mass  $M^2 = m^2/L^2$  and the dimensionless  $m^2$  is essentially determined by  $SO(6)$  group theory<sup>xi</sup> to be  $m^2 = k(k - 4)$ . Formulas of this type are important in the AdS/CFT correspondence, because the energy quantum number,  $\Delta = k$  in this case, is identified with the scale dimension of the dual operator in the  $\mathcal{N} = 4$  SYM theory. Later we will see how this occurs.

Since the superalgebra  $SU(2, 2|4)$  operates in the dimensionally reduced bulk theory all KK modes obtained in the decomposition process can be classified in representations of  $SU(2, 2|4)$ . It turns out that one gets exactly the set of short representation discussed above for the composite operators of the field theory. There is thus a 1:1 correspondence between the KK fields of Type IIB  $D = 10$  supergravity and the composite operators (in short representations) of  $\mathcal{N} = 4$  SYM theory. The  $\phi_k$  we have been discussing are dual to the chiral primary operators  $\text{tr}X^k$ . Within the lowest  $k = 2$  multiplet, the 15 bulk gauge fields  $A_\mu$  are dual to the conserved currents  $\mathcal{J}_i$  of the  $SO(6)$  R-symmetry group, and the  $\text{AdS}_5$  metric fluctuation  $h_{\mu\nu}$  is dual to the field theory stress tensor  $T_{ij}$ .

Critics of the AdS/CFT correspondence legitimately ask whether results are due to dynamics or simply to symmetries. It thus must be admitted that the operator duality just

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<sup>x</sup>indices for components of this irrep are omitted on both  $\phi_k$  and  $Y^k$ .

<sup>xi</sup>In the simplest case of the dilaton field, whose linearized 10D field equation is uncoupled, the masses in the KK decomposition are simply given by the eigenvalues of the Laplacian on  $S_5$ , namely  $m^2 = k(k + 4)$ . The mass formula which follows differs because of the mixing discussed above.

discussed was essentially ensured by symmetry. The superalgebra representations which can occur in the KK reduction of a gravity theory whose “highest spin” field is the metric tensor  $g_{MN}$  are strongly constrained. In the present case of  $SU(2,2|4)$  there was no choice but to obtain the series of short representations which were found. So what we have uncovered so far is just the working of the same symmetry algebra in two different physical settings, a field theory without gravity in 4 dimensions and a gravity theory in 5 dimensions. The more dynamical aspects of the correspondence involve the interactions of the dimensionally reduced bulk theory, *e.g.* the nonlinear terms in (8.12). It is notoriously difficult to find these terms,<sup>xiii</sup> but fortunately enough information has been obtained to give highly non-trivial tests of AdS/CFT, some of which are discussed later.

## 8.1 AdS<sub>d+1</sub> Basics—Geometry and Isometries

We now begin our discussion of how to obtain information on correlation functions in conformal field theory from classical gravity. For applications to the “realistic” case of  $\mathcal{N} = 4$  SYM theory, we will need details of Type IIB supergravity, but we can learn a lot from toy models of the bulk dynamics. In most cases we will use Euclidean signature models in order to simplify the discussions and calculations.

Consider the Euclidean signature gravitational action in  $d + 1$  dimensions

$$S = \frac{-1}{16\pi G} \int d^{d+1}z \sqrt{g}(R - \Lambda) \quad (8.13)$$

with  $\Lambda = -d(d - 1)/L^2$ . The maximally symmetric solution is Euclidean AdS<sub>d+1</sub> which should be more properly called the hyperbolic space  $H_{d+1}$ . The metric can be presented in various coordinate systems, each of which brings out different features. For now we will use the upper half-space description

$$\begin{aligned} ds^2 &= \frac{L^2}{z_0^2} (dz_0^2 + \sum_{i=1}^d dz_i^2) \\ &= \bar{g}_{\mu\nu} dz^\mu dz^\nu \end{aligned} \quad (8.14)$$

**Ex. 5:** Calculate the curvatures  $R_{\mu\nu} = \frac{-d}{L^2} \bar{g}_{\mu\nu}$ ,  $R = \frac{-d(d+1)}{L^2}$ .

The space is conformally flat and one may think of the coordinates as a  $(d+1)$ -dimensional Cartesian vector which we will variously denote as  $z_\mu = (z_0, z_i) = (z_0, \vec{z})$ , with  $z_0 > 0$ . Scalar products  $z \cdot w$  and invariant squares  $z^2$  involve a sum over all  $d + 1$  components, *e.g.*  $z \cdot w = \delta^{\mu\nu} z_\mu w_\nu$ .

The plane  $z_0 = 0$  is at infinite geodesic distance from any interior point. Yet it is technically a boundary. Data must be specified there to obtain unique solutions of wave equations on the spacetime, as we will see later. We will usually set the scale  $L = 1$ . Equivalently, all dimensionful quantities are measured in units set by  $L$ .

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<sup>xiii</sup>Except in subsectors such as that of the  $15 A_\mu$  where non-abelian gauge invariance in 5 dimensions governs the situation.

The continuous isometry group of Euclidean  $\text{AdS}_{d+1}$  is  $SO(d+1, 1)$ . This consists of rotations and translations of the  $z_i$  with  $\frac{1}{2}d(d-1) + d$  parameters, scale transformations  $z_\mu \rightarrow \lambda z_\mu$  with 1 parameter, and special conformal transformations whose infinitesimal form is  $\delta z_\mu = 2c \cdot z z_\mu - z^2 c_\mu$ , with  $c_\mu = (0, c_i)$  and thus  $d$  parameters. The total number of parameters is  $(d+2)(d+1)/2$  which is the dimension of the group  $SO(d+1, 1)$ .

**Ex. 6:** *Verify explicitly the Killing condition  $D_\mu K_\nu + D_\nu K_\mu = 0$  for all infinitesimal transformations. The covariant derivative  $D_\mu$  includes the Christoffel connection for the metric (8.14).*

**Ex. 7:** *(Extra credit !)* *Since  $\text{AdS}_{d+1}$  is conformally flat, it has the same conformal group  $SO(d+2, 1)$  as flat  $(d+1)$ -dimensional Euclidean space. There are  $d+2$  additional conformal Killing vectors  $\bar{K}_\mu$  which satisfy  $D_\mu \bar{K}_\nu + D_\nu \bar{K}_\mu - \frac{2}{d+1} \bar{g}_{\mu\nu} D^\rho \bar{K}_\rho = 0$ . Find them!*

The  $\text{AdS}_{d+1}$  space also has the important **discrete** isometry of **inversion**. We will discuss this in some detail because it has applications to the computation of AdS/CFT correlation functions and in conformal field theory itself. Under inversion the coordinates  $z_\mu$  transform to new coordinates  $z'_\mu$  by  $z_\mu = z'_\mu / z'^2$ , and it is not hard to show that the line element (8.14) is invariant under this transformation.

**Ex. 8:** *Show this explicitly.*

Inversion is also a discrete conformal isometry of flat Euclidean space.

The Jacobian of the transformation is also useful,

$$\begin{aligned} \frac{\partial z_\mu}{\partial z'_\nu} &= \frac{1}{z'^2} J_{\mu\nu}(z) \\ J_{\mu\nu}(z) &= J_{\mu\nu}(z') = \delta_{\mu\nu} - \frac{2z_\mu z_\nu}{z^2} \end{aligned} \quad (8.15)$$

The Jacobian tells us how a tangent vector of the manifold transforms under inversion.

**Ex. 9:** *View  $J_{\mu\nu}(z)$  as a matrix. Show that it satisfies  $J_{\mu\rho}(z) J_{\rho\nu}(z) = \delta_{\mu\nu}$  and has  $d$  eigenvalues  $+1$  and 1 eigenvalue  $-1$ .*

$J_{\mu\nu}$  is thus a matrix of the group  $O(d+1)$  which is not in the proper subgroup  $SO(d+1)$ . As an isometry, inversion is an improper reflection which cannot be continuously connected to the identity in  $SO(d+1, 1)$ .

**Ex. 10:** *(Important but tedious !)* *Let  $z_\mu, w_\mu$  denote two vectors with  $z'_\mu, w'_\mu$  their images under inversion. Show that*

$$\frac{1}{(z-w)^2} = \frac{(z')^2 (w')^2}{(z'-w')^2} \quad (8.16)$$

$$J_{\mu\nu}(z-w) = J_{\mu\mu'}(z') J_{\mu'\nu'}(z'-w') J_{\nu'\nu}(w') \quad (8.17)$$

## 8.2 Inversion and CFT Correlation Functions

Although we have derived the properties of inversion in the context of  $\text{AdS}_{d+1}$ , the manipulations are essentially the same for flat  $d$ -dimensional Euclidean space. We simply replace  $z_\mu, w_\mu$  by  $d$ -vectors  $x_i, y_i$  and take  $x_i = \frac{x'_i}{x'^2}$ , etc. Inversion is now a conformal isometry and in most cases <sup>xiii</sup> a symmetry of  $\text{CFT}_d$ . Under the inversion  $x_i \rightarrow x'_i$ , a scalar operator of scale dimension  $\Delta$  is transformed as  $\mathcal{O}_\Delta(x) \rightarrow \mathcal{O}'_\Delta(x) = x'^{2\Delta} \mathcal{O}_\Delta(x')$ . Correlation functions then transform covariantly under inversion, viz.

$$\begin{aligned} \langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \cdots \mathcal{O}_{\Delta_n}(x_n) \rangle \\ = (x'_1)^{2\Delta_1} (x'_2)^{2\Delta_2} \cdots (x'_n)^{2\Delta_n} \langle \mathcal{O}_{\Delta_1}(x'_1) \mathcal{O}_{\Delta_2}(x'_2) \cdots \mathcal{O}_{\Delta_n}(x'_n) \rangle \end{aligned} \quad (8.18)$$

It is well known that the spacetime forms of 2- and 3-point functions are unique in any  $\text{CFT}_d$ , a fact which can be established using the transformation law under inversion. These forms are

$$\langle \mathcal{O}_\Delta(x) \mathcal{O}_{\Delta'}(y) \rangle = \frac{c \delta_{\Delta\Delta'}}{(x-y)^{2\Delta}} \quad (8.19)$$

$$\begin{aligned} \langle \mathcal{O}_{\Delta_1}(x) \mathcal{O}_{\Delta_2}(y) \mathcal{O}_{\Delta_3}(z) \rangle &= \frac{\tilde{c}}{(x-y)^{\Delta_{12}} (y-z)^{\Delta_{23}} (z-x)^{\Delta_{31}}} \\ \Delta_{12} &= \Delta_1 + \Delta_2 - \Delta_3, \text{ and cyclic permutations} \end{aligned} \quad (8.20)$$

It follows immediately from the exercise above that they do transform correctly.

Operators such as conserved currents  $\mathcal{J}_i$  and the conserved traceless stress tensor  $T_{ij}$  are important in a  $\text{CFT}_d$ . Under inversion  $\mathcal{J}_i(x) \rightarrow J_{ij}(x') x'^{2(d-1)} \mathcal{J}_j(x')$  with an analogous rule for  $T_{ij}$ . The 2-point function of a conserved current takes the form

$$\begin{aligned} \langle \mathcal{J}_i(x) \mathcal{J}_j(y) \rangle &\approx (\partial_i \partial_j - \square \delta_{ij}) \frac{1}{(x-y)^{(2d-4)}} \\ &\sim \frac{J_{ij}(x-y)}{(x-y)^{(2d-2)}} \end{aligned} \quad (8.21)$$

The exercise above can be used to show this tensor does transform correctly. Here are some new exercises.

**Ex. 11:** *show that the second line in (8.21) follows from the manifestly conserved first form and obtain the missing coefficient.*

**Ex. 12:** *Use the projection operator  $\pi_{ij} = \partial_i \partial_j - \square \delta_{ij}$  to write the 2-point correlator of the stress tensor and then convert to a form with manifestly correct inversion properties,*

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<sup>xiii</sup>Inversion is an improper reflection similar to parity and is not always a symmetry of a field theory action containing fermions.



$$\begin{aligned} \langle T_{ij}(x)T_{kl}(y) \rangle &= [2\pi_{ij}\pi_{kl} - 3(\pi_{ik}\pi_{jl} + \pi_{il}\pi_{jk})] \frac{c}{(x-y)^{(2d-4)}} \\ &\sim \frac{J_{ik}(x-y)J_{jl}(x-y) + k \leftrightarrow l - \frac{2}{d}\delta_{ij}\delta_{kl}}{(x-y)^{2d}} \end{aligned} \quad (8.22)$$

This form is unique. For  $d \geq 4$  there are two independent tensor structures for a 3-point function of conserved currents and three structures for the 3-point function of  $T_{ij}$ . For more information on the tensor structure of conformal amplitudes, see the work of Osborn and collaborators, for example [171, 172].

It is useful to mention that any finite special conformal transformation can be expressed as a product of (inversion)(translation)(inversion).

**Ex. 13:** Show that the finite transformation is  $x_i \rightarrow (x_i + x^2 a_i)/(1 + 2a \cdot x + a^2 x^2)$ . Show that the flat Euclidean line element transforms with a conformal factor under this transformation. Show that the commutator of an infinitesimal special conformal transformation and a translation involves a rotation plus scale transformation.

The behavior of amplitudes under rotations and translations is rather trivial to test. Special conformal symmetry is more difficult, but it can be reduced to inversion. Thus the behavior under inversion essentially establishes covariance under the full conformal group.

We will soon put the inversion to good use in our study of the AdS/CFT correspondence, but we first need to discuss how the dynamics of the correspondence works.

### 8.3 AdS/CFT Amplitudes in a Toy Model

Let us consider a toy model of a scalar field  $\phi(z)$  in an  $\text{AdS}_{d+1}$  Euclidean signature background. The action is

$$S = \frac{1}{8\pi G} \int d^{d+1}z \sqrt{g} \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{3} b \phi^3 + \dots \right) \quad (8.23)$$

We will outline the general prescription for correlation functions due to Witten [3] and then give further details. The first step is to solve the non-linear classical field equations

$$\frac{\delta S}{\delta \phi} = (-\square + m^2)\phi + b\phi^2 + \dots = 0 \quad (8.24)$$

with the boundary condition

$$\begin{aligned} \phi(z_0, \vec{z}) &\xrightarrow{z_0 \rightarrow 0} z_0^{d-\Delta} \bar{\phi}(\vec{z}) \\ \Delta &= \frac{d}{2} + \frac{1}{2} \sqrt{d^2 + 4m^2} \end{aligned} \quad (8.25)$$

This is a modified Dirichlet boundary value problem with boundary data  $\bar{\phi}(\vec{z})$ . The scaling rate  $z_0^{d-\Delta}$  is that of the leading Frobenius solution of the linearized version<sup>xiv</sup> of (8.24)

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<sup>xiv</sup>To simplify the discussion we restrict throughout to the range  $m^2 > -\frac{d^2}{4}$  and consider  $\Delta > \frac{1}{2}d$ . See [173] for an extension to the region  $\frac{1}{2}d \geq \Delta \geq \frac{1}{2}(d-2)$  close to the unitarity bound; see also [174].

Exact solutions of the non-linear equation (8.24) with general boundary data are beyond present ability, so we work with the iterative solution

$$\phi_0(z) = \int d^d \vec{x} K_\Delta(z_0, \vec{z} - \vec{x}) \bar{\phi}(\vec{x}) \quad (8.26)$$

$$\phi(z) = \phi_0(z) + b \int d^{d+1} w \sqrt{g} G(z, w) \phi_0^2(w) + \dots \quad (8.27)$$

The linear solution  $\phi_0$  involves the bulk-to-boundary propagator

$$K_\Delta(z_0, \vec{z}) = C_\Delta \left( \frac{z_0}{z_0^2 + \vec{z}^2} \right)^\Delta \quad C_\Delta = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}} \Gamma(\Delta - \frac{d}{2})}, \quad (8.28)$$

which satisfies  $(\square + m^2)K_\Delta(z_0, \vec{z}) = 0$ . Interaction terms require the bulk-to-bulk propagator  $G(z, w)$  which satisfies  $(-\square_z + m^2)G(z, w) = \delta(z, w)/\sqrt{g}$  and is given by the hypergeometric function

$$\begin{aligned} G_\Delta(u) &= \tilde{C}_\Delta (2u^{-1})^\Delta F\left(\Delta, \Delta - d + \frac{1}{2}; 2\Delta - d + 1; -2u^{-1}\right) \\ \tilde{C}_\Delta &= \frac{\Gamma(\Delta)\Gamma(\Delta - \frac{d}{2} + \frac{1}{2})}{(4\pi)^{(d+1)/2}\Gamma(2\Delta - d + 1)} \\ u &= \frac{(z - w)^2}{2z_0 w_0}. \end{aligned} \quad (8.29)$$

This differs from the form given in Sec. 6.3 by a quadratic hypergeometric transformation, see [134].

For several purposes in dealing with the AdS/CFT correspondence it is appropriate to insert a cutoff at  $z_0 = \epsilon$  in the bulk geometry and consider a true Dirichlet problem at this boundary. This is the situation of 19th century boundary value problems where Green's formula gives a well known relation between  $G$  and  $K$ . Essentially  $K$  is the normal derivative at the boundary of  $G$ . The cutoff region has less symmetry than full AdS. Exact expressions for  $G$  and  $K$  in terms of Bessel functions in the  $\vec{p}$ -space conjugate to  $\vec{z}$  are straightforward to obtain, but the Fourier transform back to  $z_0, \vec{z}$  is unknown. See Sec. 8.5 below.

The next step is to substitute the solution  $\phi(z)$  into the action (8.23) to obtain the on-shell action  $S[\bar{\phi}]$  which is a functional of the boundary data. The key dynamical statement of the AdS/CFT correspondence is that  $S[\bar{\phi}]$  is the generating functional for correlation functions of the dual operator  $\mathcal{O}(\vec{x})$  in the boundary field theory, so that

$$\langle \mathcal{O}(\vec{x}_1) \cdots \mathcal{O}(\vec{x}_n) \rangle = (-)^{n-1} \frac{\delta}{\delta \phi(\vec{x}_1)} \cdots \frac{\delta}{\delta \phi(\vec{x}_n)} S[\bar{\phi}] \Big|_{\bar{\phi}=0} \quad (8.30)$$

Another way to state things is that the boundary data for bulk fields play the role of sources for dual field theory operators. The integrals in the on-shell action diverge at the boundary and must be cut off either as discussed above or by a related method [175, 176]. However we will proceed formally here.

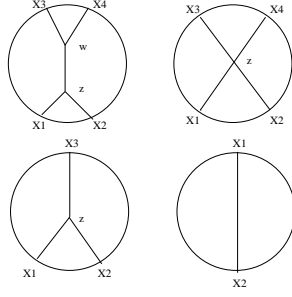


Figure 11: Some Witten Diagrams

From the expansion of  $S[\bar{\phi}]$  in powers of  $\bar{\phi}$ , one obtains a diagrammatic algorithm (in terms of Witten diagrams) for the correlation functions. Some examples are given in Figure 11. In these diagrams the interior and boundary of each disc denote the interior and boundary of the AdS geometry. The rules for interpretation and computation associated with the diagrams are as follows:

- boundary points  $\vec{x}_i$  are points of flat Euclidean<sub>d</sub> space where field theory operators are inserted.
- bulk points  $z, w \in AdS_{d+1}$  and are integrated as  $\int d^{d+1}z \sqrt{\bar{g}(z)}$
- Each bulk-to-boundary line carries a factor of  $K_\Delta$  and each bulk-to-bulk line a factor of  $G(z, w)$
- An  $n$ -point vertex carries a coupling factor from the interaction terms of the bulk Lagrangian, e.g.  $\mathcal{L} = \frac{1}{3}b\phi^3 + \frac{1}{4}c\phi^4 + \dots$  with the same combinatoric weights as for Feynman-Wick diagrams. This is most clearly derived using the cutoff discussed above.

Let us examine this construction more closely beginning with the linear solution for bulk fields.

**Ex. 14:** Show that the linearized field equation can be written as

$$(z_0^2 \partial_0^2 - (d-1)z_0 \partial_0 + z_0^2 \nabla^2 - m^2)\phi = 0 \quad (8.31)$$

and that  $K(z_0, \vec{z})$  given above is a solution. Plot  $K(z_0, \vec{z})$  as a function of  $|\vec{z}|$  for several fixed values of  $z_0$ . Note that it becomes more and more like  $\delta(\vec{z})$  as  $z_0 \rightarrow 0$ .

The exercise shows that  $\phi_0(z)$  in (8.26) is indeed a solution of (8.31) and suggests that it satisfies the right boundary condition. Let's verify that it has the correct normalization at the boundary. Because of translation symmetry there is no loss of generality in taking  $\vec{z} = 0$ . We then have

$$\begin{aligned} \phi(z_0, 0) &= C_\Delta \int d^d \vec{x} \left( \frac{z_0}{z_0^2 + \vec{x}^2} \right)^\Delta \bar{\phi}(\vec{x}) \\ &= C_\Delta z_0^{d-\Delta} \int d^d \vec{y} \left( \frac{1}{1+\vec{y}^2} \right)^\Delta \bar{\phi}(z_0 \vec{y}) \\ &\xrightarrow{z_0 \rightarrow 0} C_\Delta z_0^{d-\Delta} I_\Delta \bar{\phi}(0) \\ I_\Delta &= \int \frac{d^d \vec{y}}{(1+\vec{y}^2)^\Delta} \end{aligned} \quad (8.32)$$

Thus we do satisfy the boundary condition (8.25) provided that  $C_\Delta = \frac{1}{I_\Delta}$  and the integral does indeed give the value of  $C_\Delta$  in (8.28).

## 8.4 How to calculate 3-point correlation functions

Two-point correlations do not contain a bulk integral and turn out to require a careful cut-off procedure which we discuss later. For these reasons 3-point functions are the prototype case, and we now discuss them in some detail. The basic integral to be done is:

$$A(\vec{x}, \vec{y}, \vec{z}) = \int \frac{dw_0 d^d \vec{w}}{w_0^{d+1}} \left( \frac{w_0}{(w-\vec{x})^2} \right)^{\Delta_1} \left( \frac{w_0}{(w-\vec{y})^2} \right)^{\Delta_2} \left( \frac{w_0}{(w-\vec{z})^2} \right)^{\Delta_3} \quad (8.33)$$

$$(w - \vec{x})^2 \equiv w_0^2 + (\vec{w} - \vec{x})^2$$

Let us first illustrate the use of the method of inversion. We change integration variable by  $w_\mu = w'_\mu/w'^2$  and at the same time refer boundary points to their inverses, *i.e.*  $\vec{x} = \vec{x}'/(\vec{x}')^2$  and the same for  $\vec{y}, \vec{z}$ . The bulk-to-boundary propagator transform very simply

$$K_\Delta(w, \vec{x}) = |\vec{x}'|^{2\Delta} K_\Delta(w', \vec{x}') \quad (8.34)$$

with the prefactor associated with a field theory operator  $\mathcal{O}_\Delta(\vec{x})$  clearly in evidence. The AdS volume element is invariant, *i.e.*  $d^{d+1}w/w_0^{d+1} = d^{d+1}w'/w_0'^{d+1}$  since inversion is an isometry.

**Ex. 15:** *Use results of previous exercises to prove these important facts.*

We then find that

$$A(\vec{x}, \vec{y}, \vec{z}) = |\vec{x}'|^{2\Delta_1} |\vec{y}'|^{2\Delta_2} |\vec{z}'|^{2\Delta_3} A(\vec{x}', \vec{y}', \vec{z}') \quad (8.35)$$

Thus the AdS/CFT procedure produces a 3-point function which transforms correctly under inversion. See (8.18).

This is a very general property which holds for **all** AdS/CFT correlators. Suppose you wish to calculate  $\langle \mathcal{J}_i^a \mathcal{J}_j^b \mathcal{J}_k^c \rangle$ . The Witten amplitude is the product (see [91]) of 3 vector bulk-to-boundary propagators, each given by

$$G_{\mu i}(w, \vec{x}) = \frac{1}{2} c_d \frac{w_0^{d-1}}{(w - \vec{x})^{d-1}} J_{\mu i}(w - \vec{x}), \quad (8.36)$$

in which the Jacobian (8.15) appears. The bulk indices are contracted with a vertex rule from the Yang-Mills interaction  $f^{abc} A_\mu^a A_\nu^b \partial_\mu A_\nu^c$ . If you try to do the change of variable in detail, you get a mess. But the process is guaranteed to produce the correct inversion factors for the conserved currents, namely  $|\vec{x}'|^{2(d-1)} J_{ii}(\vec{x}')$ , etc, because inversion is an isometry of  $\text{AdS}_{d+1}$  and all pieces of the amplitude conspire to preserve this symmetry.

**Ex. 16:** *Show that  $G_{\mu i}(w, \vec{x})$  satisfies the bulk Maxwell equation*

$$\partial_\mu \sqrt{\bar{g}} \bar{g}^{\mu\nu} (\partial_\nu G_{\rho i}(w, \vec{x}) - \partial_\rho G_{\nu i}(w, \vec{x})) = 0 \quad (8.37)$$

where  $\partial_\mu = \partial/\partial w_\mu$ . Express  $G_{\mu i}(w, \vec{x})$  in terms of the inverted  $G_{\mu' i'}(w', \vec{x}')$ .

We can conclude that all AdS/CFT amplitudes are conformal covariant! A transformation of the  $SO(d+1, 1)$  isometry group of the bulk is dual to an  $SO(d+1, 1)$  conformal transformation on the boundary. Since there is a unique covariant form for scalar 3-point functions, given in (8.18), the AdS/CFT integral  $A(\vec{x}, \vec{y}, \vec{z})$  is necessarily a constant multiple of this form. Our exercise also shows conclusively that a scalar field of AdS mass  $m^2$  is dual to an operator  $\mathcal{O}_\Delta(\vec{x})$  of dimension  $\Delta$  given by (8.25).

We still need to **do** the bulk integral to obtain the constant  $\tilde{c}$ . It is hard to do the integral in the original form (8.33) because it contains 3 denominators and the restriction  $w_0 > 0$ . But we can simplify it by using inversion in a somewhat different way. We use translation symmetry to move the point  $\vec{z} \rightarrow 0$ , *i.e.*  $A(\vec{x}, \vec{y}, \vec{z}) = A(\vec{x} - \vec{z}, \vec{y} - \vec{z}, 0) \equiv A(\vec{u}, \vec{v}, 0)$ . The integral for  $A(\vec{u}, \vec{v}, 0)$  is similar to (8.33) except that the third propagator is simplified,

$$\left(\frac{w_0}{(w-\vec{z})^2}\right)^{\Delta_3} \longrightarrow \left(\frac{w_0}{w^2}\right)^{\Delta_3} = (w'_0)^{\Delta_3}. \quad (8.38)$$

There is no denominator in the inverted frame since  $\vec{z} = 0 \rightarrow \vec{z}' = \infty$ . After inversion the integral is

$$A(\vec{u}, \vec{v}, 0) = \frac{1}{|\vec{u}|^{2\Delta_1} |\vec{v}|^{2\Delta_2}} \int \frac{d^{d+1}w'}{(w')^{d+1}} \left(\frac{w'_0}{(w'-\vec{u})^2}\right)^{\Delta_1} \left(\frac{w'_0}{(w'-\vec{v})^2}\right)^{\Delta_2} (w'_0)^{\Delta_3} \quad (8.39)$$

The integral can now be done by conventional Feynman parameter methods, which give

$$A(\vec{u}, \vec{v}, 0) = \frac{1}{|\vec{u}|^{2\Delta_1} |\vec{v}|^{2\Delta_2}} \frac{a}{|\vec{u}-\vec{v}|^{\Delta_1+\Delta_2-\Delta_3}}$$

$$a = \frac{\pi^{d/2}}{2} \frac{\Gamma(\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3))\Gamma(\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1))\Gamma(\frac{1}{2}(\Delta_3+\Delta_1-\Delta_2))}{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)} \Gamma[\frac{1}{2}(\Delta_1 + \Delta_2 + \Delta_3 - d)] \quad (8.40)$$

**Ex. 17:** *Repristiniate the original variables  $\vec{x}, \vec{y}, \vec{z}$  to obtain the form (8.20) with  $\tilde{c} = a$ .*

The major application of this result was already discussed in Sec. 6.7. A Princeton group [92] obtained the cubic couplings  $b_{klm}$  of the Type IIB supergravity modes on  $\text{AdS}_5 \times S^5$  which are dual to the chiral primary operators  $\text{tr}X^k$ , etc. of  $\mathcal{N} = 4$  SYM theory. They combined these couplings with the Witten integral above and observed that the AdS/CFT prediction

$$\langle \text{tr}X^k(\vec{x}) \text{tr}X^l(\vec{y}) \text{tr}X^m(\vec{z}) \rangle = b_{klm} c_k c_l c_m A(\vec{x}, \vec{y}, \vec{z}) \quad (8.41)$$

for the large  $N$ , large  $\lambda$  supergravity limit agreed with the **free field** Feynman amplitude for these correlators. They conjectured a broader non-renormalization property. It was subsequently confirmed in weak coupling studies in the field theory that order  $g^2, g^4$  and non-perturbative instanton contributions to these correlations vanished for all  $N$  and all gauge groups. General all orders arguments for non-renormalization have also been developed. The non-renormalization of 3-point functions of chiral primaries (and their descendants) was a surprise and the first major new result about  $\mathcal{N} = 4$  SYM obtained from AdS/CFT. (See the references cited in Sec 6.7.)

## 8.5 2-point functions

This is an important case, but more delicate, since a cutoff procedure is required to obtain a concrete result from the formal integral expression. Since 3-point functions do not require a cutoff, one way to bypass this problem is to study the 3-point function  $\langle \mathcal{J}_i(z) \mathcal{O}_\Delta(x) \mathcal{O}_\Delta^*(y) \rangle$  of a conserved current and a scalar operator  $\mathcal{O}_\Delta(x)$  assumed to carry one unit of  $U(1)$  charge.<sup>xv</sup> The Ward identity relates  $\langle \mathcal{J}_i \mathcal{O}_\Delta \mathcal{O}_\Delta^* \rangle$  to  $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta^* \rangle$ . There is a unique conformal tensor for  $\langle \mathcal{J}_i \mathcal{O}_\Delta \mathcal{O}_\Delta^* \rangle$  in any  $\text{CFT}_d$ , namely

$$\langle J_i(z) \mathcal{O}_\Delta(x) \mathcal{O}_\Delta^*(y) \rangle = -i\xi \frac{1}{(x-y)^{2\Delta-d+2}} \frac{1}{(x-z)^{d-2}(y-z)^{d-2}} \left[ \frac{(x-z)i}{(x-z)^2} - \frac{(y-z)i}{(y-z)^2} \right] \quad (8.42)$$

and the Ward identity is

$$\begin{aligned} \frac{\partial}{\partial z_i} \langle J_i(z) \mathcal{O}_\Delta(x) \mathcal{O}_\Delta^*(y) \rangle &= i[\delta(x-z) - \delta(y-z)] \langle \mathcal{O}_\Delta(x) \mathcal{O}_\Delta^*(y) \rangle \\ &= i[\delta(x-z) - \delta(y-z)] \frac{2\pi^{d/2}}{\Gamma(d/2)} \xi \frac{1}{(x-y)^{2\Delta}} \end{aligned} \quad (8.43)$$

**Ex. 18:** Derive (8.43) from (8.42).

To implement the gravity calculation of  $\langle \mathcal{J}_i \mathcal{O}_\Delta \mathcal{O}_\Delta^* \rangle$  we extend the bulk toy model (8.23) to include a  $U(1)$  gauge coupling

$$L = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{g}^{\mu\nu} (\partial_\mu + iA_\mu) \phi^* (\partial_\nu - iA_\nu) \phi \quad (8.44)$$

In application to the duality between Type IIB sugra and  $\mathcal{N} = 4$  SYM, the  $U(1)$  would be interpreted as a subgroup of the  $SO(6)$  R-symmetry group. The cubic vertex leads to the AdS integral

$$\langle \mathcal{J}_i(z) \mathcal{O}_\Delta(x) \mathcal{O}_\Delta^*(y) \rangle = -i \int \frac{d^{d+1}w}{w_0^{d+1}} G_{\mu i}(w, \vec{z}) w_0^2 K_\Delta(w, \vec{x}) \overleftrightarrow{\frac{\partial}{\partial w_\mu}} K_\Delta(w, \vec{y}). \quad (8.45)$$

**Ex. 19:** The integral can be done by the inversion technique, please do it.

The result is the tensor form (8.42) with coefficient

$$\xi = \frac{(\Delta-d/2)\Gamma(\frac{d}{2})\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta-d/2)} \quad (8.46)$$

Using (8.43) we thus obtain the 2-point function

$$\langle \mathcal{O}_\Delta(x) \mathcal{O}_\Delta^*(y) \rangle = \frac{(2\Delta-d)\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta-d/2)} \frac{1}{(x-y)^{2\Delta}} \quad (8.47)$$

We now discuss a more direct computation [2, 91] of 2-point correlators from a Dirichlet boundary value problem in the AdS bulk geometry with cutoff at  $z_0 = \epsilon$ . This method

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<sup>xv</sup>When no ambiguity arises we will denote boundary points by  $x, y, z$  etc. rather than  $\vec{x}, \vec{y}, \vec{z}$ .

illustrates the use of a systematic cutoff, and it may be applied to (some) 2-point functions in holographic RG flows for which the 3-point function  $\langle \mathcal{J}_i \mathcal{O}_\Delta \mathcal{O}_\Delta^* \rangle$  cannot readily be calculated.

The goal is to obtain a solution of the linear problem

$$\begin{aligned} (\square - m^2)\phi(z_0, \vec{z}) &= 0 \\ \phi(\epsilon, \vec{z}) &= \bar{\phi}(\vec{z}) \end{aligned} \tag{8.48}$$

The result will be substituted in the bilinear part of the toy model action to obtain the on-shell action. After partial integration we obtain the boundary integral

$$S[\bar{\phi}] = \frac{1}{2\epsilon^{d-1}} \int d^d \vec{z} \bar{\phi}(\vec{z}) \partial_0 \phi(\epsilon, \vec{z}) \tag{8.49}$$

Since the cutoff region  $z_0 \geq \epsilon$  does not have the full symmetry of AdS, an exact solution of the Dirichlet problem is impossible in  $x$ -space, so we work in  $p$ -space. Using the Fourier transform

$$\phi(z_0, \vec{z}) = \int d^d \vec{p} e^{i\vec{p}\cdot\vec{z}} \phi(z_0, \vec{p}) \tag{8.50}$$

we find the boundary value problem

$$\begin{aligned} [z_0^2 \partial_0^2 - (d-1)z_0 \partial_0 - (p^2 z_0^2 + m^2)]\phi(z_0, \vec{p}) &= 0 \\ \phi(\epsilon, \vec{p}) &= \bar{\phi}(\vec{p}) \end{aligned} \tag{8.51}$$

where  $\bar{\phi}(\vec{p})$  is the transform of the boundary data. The differential equation is essentially Bessel's equation, and we choose the solution involving the function  $z_0^{d/2} K_\nu(pz_0)$ , where  $\nu = \Delta - d/2$ ,  $p = |\vec{p}|$ , which is exponentially damped as  $z_0 \rightarrow \infty$  and behaves as  $z_0^{d-\Delta}$  as  $z_0 \rightarrow 0$ . The second solution  $z_0^{d/2} I_\nu(pz_0)$  is rejected because it increases exponentially in the deep interior. The normalized solution of the boundary value problem is then

$$\phi(z_0, \vec{p}) = \frac{z_0^{d/2} K_\nu(pz_0)}{\epsilon^{d/2} K_\nu(p\epsilon)} \bar{\phi}(\vec{p}), \tag{8.52}$$

The on-shell action in  $p$ -space is

$$S[\bar{\phi}] = \frac{1}{2\epsilon^{d-1}} \int d^d p d^d q (2\pi)^d \delta(\vec{p} + \vec{q}) \phi(\epsilon, \vec{p}) \partial_0 \phi(\epsilon, \vec{q}) \tag{8.53}$$

which leads to the cutoff correlation function

$$\begin{aligned} \langle \mathcal{O}_\Delta(\vec{p}) \mathcal{O}_\Delta(\vec{q}) \rangle_\epsilon &= -\frac{\delta^2 S}{\delta \bar{\phi}(\vec{p}) \delta \bar{\phi}(\vec{q})} \\ &= -\frac{(2\pi)^d \delta(\vec{p} + \vec{q})}{\epsilon^{d-1}} \frac{d}{d\epsilon} \ln(\epsilon^{d/2} K_\nu(p\epsilon)) \end{aligned} \tag{8.54}$$

To extract a physical result, we need the boundary asymptotics of the Bessel function  $K_\nu(p\epsilon)$ . The values of  $\nu = \Delta - d/2$  which occur in most applications of AdS/CFT are

integer. The asymptotics were worked out for continuous  $\nu$  in the Appendix of [91] with an analytic continuation to the final answer. Here we assume integer  $\nu$ , although an analytic continuation will be necessary to define Fourier transform to  $x$ -space. The behavior of  $K_\nu(u)$  near  $u = 0$  can be obtained from a standard compendium on special functions such as [177]. For integer  $\nu$ , the result can be written schematically as

$$K_\nu(u) = u^{-\nu}(a_0 + a_1u^2 + a_2u^4 + \dots) + u^\nu \ln(u) (b_0 + b_1u^2 + b_2u^4 + \dots) \quad (8.55)$$

where the  $a_i, b_i$  are functions of  $\nu$  given in [177]. This expansion may be used to compute the right side of (8.54) leading to

$$\langle \mathcal{O}_\Delta(\vec{p}) \mathcal{O}_\Delta(\vec{q}) \rangle_\epsilon = \frac{(2\pi)^d \delta(\vec{p} + \vec{q})}{\epsilon^d} \left[ -\frac{d}{2} + \nu(1 + c_2\epsilon^2 p^2 + c_4\epsilon^4 p^4 + \dots) - \frac{2\nu b_0}{a_0} \epsilon^{2\nu} p^{2\nu} \ln(p\epsilon)(1 + d_2\epsilon^2 p^2 + \dots) \right] \quad (8.56)$$

where the new constants  $c_i, d_i$  are simply related to  $a_i, b_i$ . From [177] we obtain the ratio

$$\frac{2\nu b_0}{a_0} = \frac{(-)^{(\nu-1)}}{2^{(2\nu-2)} \Gamma(\nu)^2} \quad (8.57)$$

which is the only information explicitly needed.

This formula is quite important for applications of AdS/CFT ideas to both conformal field theories and RG flows where similar formulas appear. The physics is obtained in the limit as  $\epsilon \rightarrow 0$ , and we scale out the factor  $\epsilon^{2(\Delta-d)}$  which corresponds to the change from the true Dirichlet boundary condition to the modified form (8.25) for the full AdS space. We also drop the conventional momentum conservation factor  $(2\pi)^d \delta(\vec{p} + \vec{q})$  and study

$$\langle \mathcal{O}_\Delta(p) \mathcal{O}_\Delta(-p) \rangle = \frac{\beta_0 + \beta_1 \epsilon^2 p^2 + \dots + \beta_\nu (\epsilon p)^{2(\nu-1)}}{\epsilon^{2\Delta-d}} - \frac{2\nu b_0}{a_0} p^{2\nu} \ln(p\epsilon) + \mathcal{O}(\epsilon^2) \quad (8.58)$$

The first part of this expression is a sum of non-negative integer powers  $p^{2m}$  with singular coefficients in  $\epsilon$ . The Fourier transform of  $p^{2m}$  is  $\square^m \delta(\vec{x} - \vec{y})$ , a pure contact term in the  $\vec{x}$ -space correlation. Such terms are usually physically uninteresting and scheme dependent in quantum field theory. Indeed it is easy to see that the singular powers  $\epsilon^{2(m-\Delta)+d}$  carried by the terms corresponds to their dependence on the ultraviolet cutoff  $\Lambda^{2(\Delta-m)-d}$  in a field theory calculation. This gives rise to the important observation that the  $\epsilon$ -cutoff in AdS space which cuts off long distance effects in the bulk corresponds to an ultraviolet cutoff in field theory. Henceforth we drop the polynomial contact terms in (8.58).

The physical  $p$ -space correlator is then given by

$$\langle \mathcal{O}_\Delta(p) \mathcal{O}_\Delta(-p) \rangle = -\frac{2\nu b_0}{a_0} p^{2\nu} \ln p. \quad (8.59)$$

This has an absorptive part which is determined by unitarity in field theory. Its Fourier transform is proportional to  $1/(x-y)^{2\Delta}$  which is the correct CFT behavior for  $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta \rangle$ . The precise constant can be obtained using differential regularization [178] or by analytic continuation in  $\nu$  from the region where the Fourier transform is defined.

The result agrees exactly with the 2-point function calculated from the Ward identity in (8.47).



## 8.6 Key AdS/CFT results for $\mathcal{N}=4$ SYM and $\text{CFT}_d$ correlators.

We can now summarize the important results discussed in this chapter and earlier ones for  $\text{CFT}_d$  correlation functions from the AdS/CFT correspondence.

- i. the non-renormalization of  $\langle \text{tr}X^k \text{tr}X^l \text{tr}X^m \rangle$  in  $\mathcal{N}=4$  SYM theory
- ii. 4-point functions are less constrained than 2- and 3- point functions in any CFT. In general they contain arbitrary functions  $F(\xi, \eta)$  of two invariant variables, the cross ratios

$$\xi = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} \quad \eta = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2} \quad x_{ij} = x_i - x_j \quad (8.60)$$

One way to extract the physics of 4-point functions is to use the operator product expansion. This is written

$$\mathcal{O}_\Delta(x) \mathcal{O}_{\Delta'}(y) \xrightarrow{x \rightarrow y} \sum_p \frac{a_{\Delta \Delta' \Delta_p}}{(x-y)^{\Delta + \Delta' - \Delta_p}} \mathcal{O}_{\Delta_p}(y) \quad (8.61)$$

which is interpreted to mean that at short distance inside any correlation function, the product of two operators acts as a sum of other local operators with power coefficients. For simplicity we have indicated only the contributions of primary operators. Thus, in the limit where  $|x_{12}|, |x_{34}| \ll |x_{13}|$ , a 4-point function must factor as

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \mathcal{O}_{\Delta_3}(x_3) \mathcal{O}_{\Delta_4}(x_4) \rangle \approx \sum_p \frac{a_{12p}}{(x_{12})^{\Delta_1 + \Delta_2 - \Delta_p}} \frac{c_p}{(x_{13})^{2\Delta_p}} \frac{a_{34p}}{(x_{34})^{\Delta_3 + \Delta_4 - \Delta_p}} \quad (8.62)$$

One must expect that AdS/CFT amplitudes satisfy this property and indeed they do in a remarkably simple way. The amplitude of a Witten diagram of the bulk field  $\phi_p(z)$  dual to  $\mathcal{O}_{\Delta_p}(\vec{z})$  factors with the correct coefficients  $c_p, a_{12p}, a_{34p}$  determined from 2- and 3-point functions. This holds for singular powers, *e.g.*  $\Delta_1 + \Delta_2 - \Delta_p > 0$ .

The AdS/CFT amplitude also contains a  $\ln(\xi)$  term in its short distance asymptotics. This is the level of the *OPE* at which  $\mathcal{O}_p =: \mathcal{O}_{\Delta_1}(\vec{x}) \mathcal{O}_{\Delta_2}(\vec{y})$  : contributes. In  $\mathcal{N}=4$  SYM theory the normal product is a double trace operator, *e.g.*  $: \text{tr}X^k(y) \text{tr}X^l(y) :$ , which has components in irreps of  $SO(6)$  contained in the direct product  $(0, k, 0) \otimes (0, l, 0)$ . The irreducible components are generically primaries of long representations of  $SU(2, 2|4)$ . Their scale dimensions are not fixed, and have a large  $N$  expansion of the form  $\Delta_{kl} = k + l + \gamma_{kl}/N^2 + \dots$ . The contribution  $\Delta_{\gamma_{kl}}$  can be read from the  $\ln(\xi)$  term of the 4-point function. It is a strong coupling prediction of AdS/CFT, which cannot yet be checked by field theoretic methods.

iii. Another surprising fact about  $\mathcal{N}=4$  SYM correlators suggested by the AdS/CFT correspondence is that **extremal**  $n$ -point functions are not renormalized. The extremal condition for 4-point functions is  $\Delta_1 = \Delta_2 + \Delta_3 + \Delta_4$ . The name extremal comes from the fact that the correlator vanishes by  $SO(6)$  symmetry for any larger value of  $\Delta_1$ . As discussed in detail in Secs. 6.8 and 6.9, the absence of radiative corrections was suggested by the form of the supergravity couplings and Witten integrals. This prediction was confirmed by weak coupling calculation and general arguments in field theory. Field theory then suggested that next-to-external correlators ( $\Delta_1 = \Delta_2 + \Delta_3 + \Delta_4 - 2$ ) were also not renormalized, and this was subsequently verified by AdS/CFT methods.

It is clear that the AdS/CFT correspondence is a new principle which stimulated an interplay of work involving both supergravity and field theory methods. As a result we have much new information about the  $N=4$  SYM theory. It confirms that AdS/CFT has quantitative predictive power, so we can go ahead and apply it in other settings.

## 9 Holographic Renormalization Group Flows

We have already seen that AdS/CFT has taught us a great deal of useful information about  $\mathcal{N} = 4$  SYM theory as a  $\text{CFT}_4$ . But years of elegant work in  $\text{CFT}_2$  has taught us to consider both the pure conformal theory and its deformation by relevant operators. The deformed theory exhibits RG flows in the space of coupling constants of the relevant deformations. For general dimension  $d$  we can also consider the  $\text{CFT}_d$  perturbed by relevant operators. For  $\mathcal{N} = 4$  SYM theory, the perturbed Lagrangian would take the form

$$\mathcal{L} = \mathcal{L}_{\mathcal{N}=4} + \frac{1}{2}m_{ij}^2 \text{tr} X^i X^j + \frac{1}{2}M_{ab} \text{tr} \psi^a \psi^b + b_{ijk} \text{tr} X^i X^j X^k. \quad (9.1)$$

For  $d > 2$  there is the additional option of Coulomb and Higgs phases in which gauge symmetry is spontaneously broken. The Lagrangian is not changed, but certain operators acquire vacuum expectation values, e.g.  $\langle X^i \rangle \neq 0$  in  $\mathcal{N} = 4$  SYM. In all these cases conformal symmetry is broken because a scale is introduced. The resulting theories have the symmetry of the Poincaré group in  $d$  dimensions which is smaller than the conformal group  $SO(1, d+1)$ . Our purpose in this chapter is to explore the description of such theories using  $D = d + 1$  dimensional gravity. We will focus on relevant operator deformations.

### 9.1 Basics of RG flows in a toy model

The basic ideas for the holographic description of field theories with RG flow were presented in [179, 180]. We will discuss these ideas in a simple model in which Euclidean  $(d + 1)$ -dimensional gravity interacts with a single bulk scalar field with action

$$S = \frac{1}{4\pi G} \int d^{d+1} \sqrt{g} \left[ -\frac{1}{4}R + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + V(\phi) \right] \quad (9.2)$$

We henceforth choose units in which  $4\pi G = 1$ . In these units  $\phi$  is dimensionless and all terms in the Lagrangian have dimension 2. We envisage a potential  $V(\phi)$  which has one or more critical points, *i.e.*  $V'(\phi_i) = 0$ , at which  $V(\phi_i) < 0$ . We consider both maxima and minima. See Figure 12.

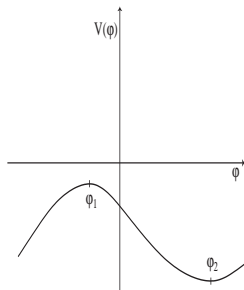


Figure 12: Potential  $V(\phi)$

The Euler-Lagrange equations of motion of our system are

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) - V'(\phi) = 0 \quad (9.3)$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 2\left[\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}(\partial\phi)^2 + V(\phi)\right)\right] = 2T_{\mu\nu} \quad (9.4)$$

For each critical point  $\phi_i$  there is a trivial solution of the scalar equation, namely  $\phi(z) \equiv \phi_i$ . The Einstein equation then reduces to

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -2g_{\mu\nu}V(\phi_i). \quad (9.5)$$

This is equivalent to the Einstein equation of the action (8.13) if we identify  $\Lambda_i = 4V(\phi_i) = -d(d-1)/L_i^2$ . Thus constant scalar fields with  $\text{AdS}_{d+1}$  geometries of scale  $L_i$  are solutions of our model. We refer to them as critical solutions.

However, more general solutions in which the scalar field is not constant are needed to describe the gravity duals of RG flows in field theory. Since the symmetries must match on both sides of the duality, we look for solutions of the  $D = d+1$ -dimensional bulk equations with  $d$ -dimensional Poincaré symmetry. The most general such configuration is

$$\begin{aligned} ds^2 &= e^{2A(r)}\delta_{ij}dx^i dx^j + dr^2 \\ \phi &= \phi(r) \end{aligned} \quad (9.6)$$

This is known as the domain wall ansatz. The coordinates separate into a radial coordinate  $r$  plus  $d$  transverse coordinates  $x^i$  with manifest Poincaré symmetry. Several equivalent forms which differ only by change of radial coordinate also appear in the literature.

Domain wall metrics have several modern applications, and it is worth outlining a method to compute the connection and curvature. Symbolic manipulation programs are very useful for this purpose, but analytic methods can also be useful, and we discuss a method which uses the Cartan structure equations. A similar method works quite well for brane metrics such as (8.5). One proceeds as follows using the notation of differential forms:

1. The first step is to choose a basis of frame 1-forms  $e^a = e^a_\mu dx^\mu$  such that the metric is given by the inner product  $ds^2 = e^a \delta_{ab} e^b$ .
2. The torsion-free connection 1-form is then defined by  $de^a + \omega^{ab} \wedge e^b = 0$  with the condition  $\omega^{ab} = -\omega^{ba}$ . The connection is valued in the Lie algebra of  $SO(d+1)$ .
3. The curvature 2-form is

$$R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega^{cb} = \frac{1}{2}R^{ab}_{cd} e^c \wedge e^d. \quad (9.7)$$

The general formulas for  $\omega^{ab}_\mu$  and  $R^{ab}_{\mu\nu}$  which appear in textbooks can be deduced from these definitions. However, for a reasonably simple metric ansatz and suitable choice of frame, it is frequently more convenient to use the definitions and compute directly. It takes

some experience to learn to use the  $d$  and  $\wedge$  operations efficiently. One must also remember to convert from frame to coordinate components of the curvature as needed.

For the domain wall metric a convenient frame is given by the transverse forms  $e^{\hat{i}} = e^{A(r)} dx^i$ ,  $i = 1 \cdots d$ , and the radial form  $e^D = dr$ .

**Ex. 20:** Use the Cartan structure equations with the frame 1-forms above to obtain the domain wall connection forms:

$$\omega^{\hat{i}\hat{j}} = 0 \quad \omega^{D\hat{i}} = A'(r)e^{\hat{i}} \quad (9.8)$$

Find next the curvature 2-forms:

$$\begin{aligned} R^{\hat{i}\hat{j}} &= -A'^2 e^{\hat{i}} \wedge e^{\hat{j}} \\ R^{\hat{i}D} &= -(A'' + A'^2) e^{\hat{i}} \wedge e^D \end{aligned} \quad (9.9)$$

Next obtain the curvature tensor (with coordinate indices)

$$\begin{aligned} R_{kl}^{ij} &= -A'^2 (\delta_k^i \delta_l^j - \delta_l^i \delta_k^j) \\ R_{jD}^{iD} &= -(A'' + A'^2) \delta_j^i \\ R_{kD}^{ij} &= 0 \end{aligned} \quad (9.10)$$

The final task is to find the Ricci tensor components

$$\begin{aligned} R_{ij} &= -e^{2A} (A'' + dA'^2) \delta_{ij} \\ R_{DD} &= -d(A'' + A'^2) \\ R_{iD} &= 0 \end{aligned} \quad (9.11)$$

**Ex. 21:** If you still have some energy compute the non-vanishing components of the Christoffel connection, namely

$$\mathcal{G}_{ij}^D = -e^{2A} A' \delta_{ij} \quad \mathcal{G}_{jD}^i = A' \delta_j^i \quad (9.12)$$

The fact that certain connection and curvature components vanish could have been seen in advance, since there are no possible Poincaré invariant tensors with the appropriate symmetries. We can introduce a new radial coordinate  $z$ , defined by  $\frac{dz}{dr} = e^{-A(r)}$ . This brings the domain wall metric to conformally flat form. Its Weyl tensor thus vanishes.

We now ask readers to manipulate the Einstein equation  $G_\nu^\mu \equiv R_\nu^\mu - \frac{1}{2} \delta_\nu^\mu R = 2T_\nu^\mu$  for the domain wall and deduce a simple condition on the scale factor  $A(r)$ .

**Ex. 22:** Deduce that

$$\begin{aligned} G_D^D &= \frac{d(d-1)}{2} A'^2 = 2T_D^D \\ G_j^i &= \delta_j^i (d-1) \left( A'' + \frac{1}{2} dA'^2 \right) = 2T_j^i \end{aligned} \quad (9.13)$$

Compute  $G_i^i - G_D^D$  for any fixed diagonal component (no sum on  $i$ ) and deduce that

$$A'' = \frac{2}{d-1} (T_i^i - T_D^D) = -\frac{2}{d-1} \phi'^2 \quad (9.14)$$

Thus we certainly have  $A'' < 0$  in the dynamics of the toy model. However there is a much more general result, namely  $T_i^i - T_D^D < 0$  for any Poincaré invariant matter configuration in all conventional models for the bulk dynamics, for example, several scalars with non-linear  $\sigma$ -model kinetic term. In Lorentzian signature, the condition above is one of the standard energy conditions of general relativity. Later we will see the significance of the fact that  $A''(r) < 0$ .

**Ex. 23:** Complete the analysis of the Einstein and scalar equations of motion for the domain wall and obtain the equations

$$\begin{aligned} A'^2 &= \frac{2}{d(d-1)}[\phi'^2 - 2V(\phi)] \\ \phi'' + dA'\phi' &= \frac{dV(\phi)}{d\phi} \end{aligned} \tag{9.15}$$

It is frequently the case that the set of equations obtained from a given ansatz for a gravity-matter system is not independent because of the Bianchi identity. Indeed in our system the derivative of the  $A'^2$  equation combines simply with the the scalar equation to give (9.14). We can thus view the system (9.15) as independent.

It is easy to see how the previously discussed critical solutions fit into the domain wall framework. At each critical point  $\phi_i$  of the potential, the scalar equation is satisfied by  $\phi(r) \equiv \phi_i$ . The  $A'^2$  equation then gives  $A(r) = \pm \frac{r}{L_i} + a_0$ . The integration constant  $a_0$  has no significance since it can be eliminated by scaling the coordinates  $x^i$  in (9.6). The sign above is a matter of convention and we choose the positive sign. The metric (9.6) is then equivalent to our previous description of  $\text{AdS}_{d+1}$  with the change of radial coordinate  $z_0 = L_i e^{-\frac{r}{L_i}}$ . With this sign convention we find that  $r \rightarrow +\infty$  is the boundary region and  $r \rightarrow -\infty$  is the deep interior.

Our main goal now is to discuss more general solutions of the system (9.15) in a potential of the type shown in Figure 12. We are interested in solutions which interpolate between two critical points, producing a domain wall geometry which approaches the boundary region of an AdS space with scale  $L_1$  as  $r \rightarrow +\infty$  and the deep interior of another AdS with scale  $L_2$  as  $r \rightarrow -\infty$ . Such geometries are dual to field theories with RG flow.

To develop this interpretation let's first look at the quadratic approximation to the potential near a critical point,

$$V(\phi) \approx V(\phi_i) + \frac{1}{2} \frac{m_i^2}{L_i^2} h^2, \tag{9.16}$$

where we use the fluctuation  $h = \phi - \phi_i$  and the scaled mass  $m_i^2 = L_i^2 V''(\phi_i)$  with  $V(\phi_i) = -d(d-1)/4L_i^2$ . Let's recall the basic AdS/CFT idea that the boundary data for a bulk scalar field is the source for an operator in quantum field theory. We apply this to the fluctuation  $h(r, \vec{x})$  which will be interpreted as the bulk dual of an operator  $\mathcal{O}_\Delta(\vec{x})$  whose scale dimension is related to the mass  $m_i^2$  by (8.25). Given the discussion of Sec. 8.3 it is reasonable to suppose that a general solution of the non-linear scalar equation of motion

(9.3) will approach the critical point with the following boundary asymptotics for the fluctuation,

$$h(r, \vec{x}) \underset{r \rightarrow \infty}{\longrightarrow} e^{(\Delta-d)r} \tilde{h}(\vec{x}) = e^{(\Delta-d)r} (\bar{\phi} + \bar{h}(\vec{x})). \quad (9.17)$$

in which  $\tilde{h}(\vec{x})$  contains  $\bar{\phi}$ , describing the boundary behavior of the domain wall profile plus a remainder  $\bar{h}(\vec{x})$ . We can form the on-shell action  $S[\bar{\phi} + \bar{h}]$  which is a functional of this boundary data.<sup>xvi</sup>

A neat way to package the statement that the bulk on-shell action generates correlation functions in the boundary field theory is through the generating functional relation

$$\langle e^{-[S_{\text{CFT}} + \int d^d \vec{x} \mathcal{O}_\Delta(\vec{x})(\bar{\phi} + \bar{h}(\vec{x}))]} \rangle = e^{-S[\bar{\phi} + \bar{h}]} \quad (9.18)$$

in which  $\langle \dots \rangle$  on the left side indicates a path integral in the field theory. This is a simple generalization of a formula which we have implicitly used in Sec. 8.3 for CFT correlators, and  $S_{\text{CFT}}$  must still appear. The natural procedure in the present case is to define correlation functions by

$$\frac{(-)^{n-1} \delta^n}{\delta \bar{h}(\vec{x}_1) \dots \delta \bar{h}(\vec{x}_n)} S[\bar{\phi} + \bar{h}] \Big|_{\bar{h}=0}. \quad (9.19)$$

The term  $\Delta S \equiv \int d^d \vec{x} \mathcal{O}_\Delta(\vec{x}) \bar{\phi}$  then remains in the QFT Lagrangian and describes an operator deformation of the CFT with coupling constant  $\bar{\phi}$ . If  $0 > m^2 > -\frac{d^2}{4} < 0$ , that is if the critical point  $\phi_i$  is a local maximum which is not too steep, then  $d > \Delta > \frac{1}{2}d$ , and we are describing a **relevant deformation** of a  $\text{CFT}_{UV}$ , one which will give a new long distance realization of the field theory. It is worth remarking that the lower bound agrees exactly with the stability criterion [183, 184] for field theory in Lorentzian  $\text{AdS}_{d+1}$ . It is the lower mass limit for which the energy of normalized scalar field configurations is conserved and positive.

If the critical point is a local minimum, then  $m_i^2 > 0$ , and the dual operator has dimension  $\Delta > d$ . We thus have the deformation of the CFT by an **irrelevant** operator, exactly as describes the approach of an RG flow to a  $\text{CFT}_{IR}$  at long distance. We thus see the beginnings of a gravitational description of RG flows in quantum field theory!

## 9.2 Interpolating Flows, I

Interpolating flows are solutions of the domain wall equations (9.15) in which the scalar field  $\phi(r)$  approaches the maximum  $\phi_1$  of  $V(\phi)$  in Fig. 12 as  $r \rightarrow +\infty$  and the minimum  $\phi_2$ , as  $r \rightarrow -\infty$ . The associated metric approaches an AdS geometry in these limits as discussed in the previous section. Exact solutions of the second order non-linear system (9.15) are difficult (although we discuss an interesting method in the next section). However, we can learn a lot by linearizing about each critical point.

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<sup>xvi</sup>A complete discussion should include the bulk metric which is coupled to  $\phi(r, \vec{x})$ . We have omitted this for simplicity. See [181, 182] for a recent general treatment.

We thus set  $\phi(r) = \phi_i + h(r)$  and  $A' = \frac{1}{L_i} + a'(r)$  and work with the quadratic approximate potential in (9.16). (See Footnote xiv.) The linearized scalar equation of motion and its general solution are

$$h'' + \frac{d}{L_i}h' - \frac{m_i^2}{L_i^2}h = 0 \quad (9.20)$$

$$h(r) = Be^{(\Delta_i-d)r/L_i} + Ce^{-\Delta_i r/L} \quad (9.21)$$

$$\Delta_i = \left( d + \sqrt{d^2 + 4m_i^2} \right) / 2 \quad (9.22)$$

One may then linearize the scale factor equation in (9.15) to find  $a' = \mathcal{O}(h^2)$  so that the scale factor  $A(r)$  is not modified to linear order

**Ex. 24:** *Verify the statements above.*

The basic idea of linearization theory is that there is an exact solution of the nonlinear equations of motion that is well approximated by a linear solution near a critical point. Thus as  $r \rightarrow +\infty$ , we assume that the exact solution behaves as

$$\phi(r) \underset{r \gg 0}{\approx} \phi_1 + B_1 e^{(\Delta_1-d)r/L_1} + C_1 e^{-\Delta_1 r/L_1}. \quad (9.23)$$

The fluctuation must disappear as  $r \rightarrow +\infty$ . For a generic situation in which the dominant  $B$  term is present, this requires  $\frac{d}{2} < \Delta_1 < d$  or  $m_1^2 < 0$ . Hence the critical point associated with the boundary region of the domain wall must be a local maximum, and everything is consistent with an interpretation as the dual of a  $\text{QFT}_d$  which is a relevant deformation of an ultraviolet  $\text{CFT}_d$ .

Near the critical point  $\phi_2$ , which is a minimum, we have  $m_2^2 > 0$  so  $\Delta_2 > d$ . This critical point must be approached at large negative  $r$ , where the exact solution is approximated by

$$\phi(r) \underset{r \ll 0}{\approx} \phi_2 + B_2 e^{(\Delta_2-d)r/L_2} + C_2 e^{-\Delta_2 r/L_2}. \quad (9.24)$$

The second term diverges, so we must choose the solution with  $C_2 = 0$ . Thus the domain wall approaches the deep interior region with the scaling rate of an irrelevant operator of scale dimension  $\Delta_2 > d$  exactly as required for infrared fixed points by RG ideas on field theory.

The non-linear equation of motion for  $\phi(r)$  has two integration constants. We must fix one of them to ensure  $C = 0$  as  $r \rightarrow -\infty$ . The remaining freedom is just the shift  $r \rightarrow r + r_0$  and has no effect on the physical picture. A generic solution with  $C = 0$  in the IR would be expected to approach the UV critical point at the dominant rate  $Be^{(\Delta_1-d)r/L_1}$  which we have seen to be dual to a relevant operator deformation of the  $\text{CFT}_{UV}$ . It is possible (but exceptional) that the  $C = 0$  solution in the IR would have vanishing  $B$  term in the UV and approach the boundary as  $C_1 e^{-\Delta_1 r/L_1}$ . In this case the physical interpretation is that of the deformation of the  $\text{CFT}_{UV}$  by a vacuum expectation value,  $\langle \mathcal{O}_{\Delta_1} \rangle \sim C_1 \neq 0$ . See [185, 186, 187, 173].



The domain wall flow “sees” the  $\text{AdS}_{IR}$  geometry only in the deep interior limit. To discuss the  $\text{CFT}_{IR}$  and its operator perturbations in themselves, we must think of extending this interior region out to a complete  $\text{AdS}_{d+1}$  geometry with scale  $L_{IR} = L_2$ .

The interpolating solution we are discussing is plotted in Figure 13. The scale factor  $A(r)$  is concave downward since  $A''(r) < 0$  from (9.14). This means that slopes of the linear regions in the deep interior and near boundary are related by  $1/L_{IR} > 1/L_{UV}$  (where we have set  $L_{UV} = L_1$ ). Hence,

$$V_{IR} = \frac{-d(d-1)}{4L_{IR}^2} < V_{UV} = \frac{-d(d-1)}{4L_{UV}^2}. \quad (9.25)$$

Thus the flow from the boundary to the interior necessarily goes to a deeper critical point of  $V(\phi)$ . Recall that the condition  $A''(r) < 0$  is very general and holds in any physically reasonable bulk theory, *e.g.* a system of many scalars  $\phi^I$  and potential  $V(\phi^I)$ . Thus any Poincaré invariant domain wall interpolating between AdS geometries is **irreversible**.

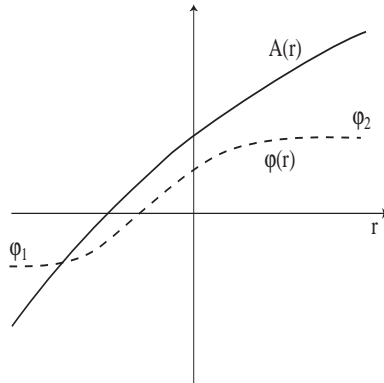


Figure 13: Profile of the scale factor  $A(r)$

The philosophy of the AdS/CFT correspondence suggests that any conspicuous feature of the bulk dynamics should be dual to a conspicuous property of quantum field theory. The irreversibility property reminds us of Zamolodchikov’s  $c$ -theorem [188] which implies that RG flow in  $\text{QFT}_2$  is irreversible. We will discuss the  $c$ -theorem and its holographic counterpart later. Our immediate goals are to present a very interesting technique for exact solutions of the non-linear flow equations (9.15) and to discuss a “realistic” application of supergravity domain walls to deformations of  $\mathcal{N}=4$  SYM theory.

### 9.3 Interpolating Flows, II

The domain wall equations

$$\phi'' + dA'\phi' = \frac{dV(\phi)}{d\phi} \quad (9.26)$$

$$A'^2 = \frac{2}{d(d-1)} (\phi'^2 - 2V(\phi)) \quad (9.27)$$

constitute a non-linear second order system with no apparent method of analytic solution. Nevertheless, a very interesting procedure which does give exact solutions in a number of examples has emerged from the literature [189, 190, 191, 192].

Given the potential  $V(\phi)$ , suppose we could solve the following differential equation in field space and obtain an auxiliary quantity, the superpotential  $W(\phi)$ :

$$\frac{1}{2} \left( \frac{dW}{d\phi} \right)^2 - \frac{d}{d-1} W^2 = V(\phi) \quad (9.28)$$

We then consider the following set of first order equations

$$\frac{d\phi}{dr} = \frac{dW}{d\phi} \quad (9.29)$$

$$\frac{dA}{dr} = -\frac{2}{d-1} W(\phi(r)) \quad (9.30)$$

These decoupled equations have a trivial structure and can be solved sequentially, the first by separation of variables, and the second by direct integration. (We assume that the two required integrals are tractable.) It is then easy to show that **any solution of the first order system (9.28, 9.29, 9.30) is also a solution of the original second order system (9.26, 9.27)**.

**Ex. 25:** *Prove this!*

It is also elementary to see that any critical point of  $W(\phi)$  is also a critical point of  $V(\phi)$  but not conversely.

**Ex. 26:** *Suppose that  $W(\phi)$  takes the form  $W \approx -\frac{1}{L_i}(\lambda + \frac{1}{2}\mu h^2)$  near a critical point. Show that  $\lambda, \mu$  are related to the parameters of the approximate potential in (9.16) by  $\lambda = \frac{1}{2}(d-1)$  and  $m^2 = \mu(\mu-d)$ . Show that the solution to the flow equation (9.29) approaches the critical point at the rate  $h \sim e^{-\mu r/L}$  with  $\mu = \Delta$ , the vev rate, or  $\mu = d - \Delta$  the operator deformation rate.*

This apparently miraculous structure generalizes to bulk theories with several scalars  $\phi^I$  and Lagrangian

$$L = -\frac{1}{4}R + \frac{1}{2}\partial_\mu \phi^I \partial^\mu \phi^I + V(\phi^I) \quad (9.31)$$

The superpotential  $W(\phi^I)$  is defined to satisfy the partial differential equation

$$\frac{1}{2} \sum_I \left( \frac{\partial W}{\partial \phi^I} \right)^2 - \frac{d}{d-1} W^2 = V \quad (9.32)$$

The first order flow equations

$$\frac{d\phi^I}{dr} = \frac{\partial W}{\partial \phi^I} \quad (9.33)$$

$$\frac{dA}{dr} = -\frac{2}{d-1} W \quad (9.34)$$

automatically give a solution of the second order Euler-Lagrange equations of (9.31) for Poincaré invariant domain walls.

**Ex. 27:** Prove this and derive first order flow equations with the same property for the non-linear  $\sigma$ -model (in which the kinetic term of (9.31) is replaced by  $\frac{1}{2}G_{IJ}(\phi^K)\partial_\mu\phi^I\partial^\mu\phi^J$ ).

The equations (9.33) are conventional gradient flow equations. The solutions are paths of steepest descent for  $W(\phi^I)$ , everywhere perpendicular to the contours  $W(\phi^I) = \text{const.}$  In applications to RG flows, the  $\phi^I(r)$  represent scale dependent couplings of relevant operators in a QFT Lagrangian, so we are talking about gradient flow in the space of couplings—an idea which is frequently discussed in the RG literature!

There are two interesting reasons why there are first order flow equations which reproduce the dynamics of the second order system (9.26, 9.27).

1. They emerge as *BPS* conditions for supersymmetric domain walls in supergravity theories. For a review, see [193]. The superpotential  $W(\phi^I)$  emerges by algebraic analysis of the quantum transformation rule. Bulk solutions have Killing spinors, and bulk supersymmetry is matched in the boundary field theory which describes a supersymmetric deformation of an SCFT.
2. They are the Hamilton-Jacobi equations for the dynamical system of gravity and scalars [194]. The superpotential  $W(\phi^I)$  is the classical Hamilton-Jacobi function, and one must solve (9.28) or (9.32) to obtain it from the potential  $V(\phi^I)$ . This is very interesting theoretically but rather impractical because it is rare that one can actually use the Hamilton-Jacobi formulation to solve a dynamical system explicitly. Numerical and approximate studies have been instructive [192, 195]. However, most applications involve superpotentials from *BPS* conditions in gauged supergravity. One may also employ a toy model viewpoint in which  $W(\phi^I)$  is postulated with potential  $V(\phi^I)$  defined through (9.28) or (9.32).

## 9.4 Domain Walls in $D = 5$ , $\mathcal{N} = 8$ , Gauged Supergravity

The framework of toy models is useful to illustrate the correspondence between domain walls in  $(d + 1)$ -dimensional gravity and RG flows in  $\text{QFT}_d$ . However it is highly desirable to have “realistic examples” which describe deformations of  $\mathcal{N} = 4$  SYM in the strong coupling limit of the AdS/CFT correspondence. There are two reasons to think first about supersymmetric deformations. As just discussed, the bulk dynamics is then governed by a superpotential  $W(\phi^I)$  with first order flow equations. Further the methods of Seiberg dynamics can give control of the long distance non-perturbative behavior of the field theory, so that features of the supergravity description can be checked.

We can only give a brief discussion here. We begin by discussing the relation between  $D = 10$  Type IIB sugra dimensionally reduced on  $\text{AdS}_5 \times \text{S}^5$  of [51] and the  $D = 5$ ,  $\mathcal{N} = 8$  supergravity theory with gauge group  $SO(6)$  first completely constructed in [196]. As discussed in Section 9.3 above, the spectrum of the first theory consists of the graviton multiplet, whose fields are dual due to all relevant and marginal operators of  $\mathcal{N} = 4$  SYM plus Kaluza-Klein towers of fields dual to operators of increasing  $\Delta$ . On the other hand gauged  $\mathcal{N} = 8$  supergravity is a theory formulated in 5 dimensions with only the fields of

the graviton multiplet above, namely

$$\begin{array}{cccccc}
g_{\mu\nu} & \Psi_\mu^a & A_\mu^A & B_{\mu\nu} & X^{abc} & \phi^I \\
1 & 8 & 15 & 12 & 48 & 42
\end{array} \tag{9.35}$$

It is a complicated theory in which the scalar dynamics is that of a nonlinear  $\sigma$ -model on the coset  $E(6,6)/USp(8)$  with a complicated potential  $V(\phi^I)$ .

Gauged  $\mathcal{N} = 8$  supergravity has a maximally symmetric ground state in which the metric is that of  $AdS_5$ . The global symmetry is  $SO(6)$  with 32 supercharges, so that the superalgebra is  $SU(2,2|4)$ . Symmetries then match the vacuum configuration of Type IIB sugra on  $AdS_5 \times S^5$ . Indeed  $D = 5$ ,  $\mathcal{N} = 8$  sugra is believed to be the **consistent truncation** of  $D = 10$  Type IIB sugra to the fields of its graviton multiplet. This means that **every** classical solution of  $D = 5$ ,  $\mathcal{N} = 8$  sugra can be “lifted” to a solution of  $D = 10$  Type IIB sugra. For example the  $SO(6)$  invariant  $AdS_5$  ground state solution lifts to the  $AdS_5 \times S^5$  geometry of (8.8) (with other fields either vanishing or maximally symmetric). There is not yet a general proof of consistent truncation, but explicit lifts of nontrivial domain wall solutions have been given [197, 198, 199, 200]. Consistent truncation has been established in other similar theories [201, 202].

In the search for classical solutions with field theory duals it is more elegant, more geometric, and more “braney” to work at the level of  $D = 10$  Type IIB sugra. There are indeed very interesting examples of Polchinski and Strassler [203] and Klebanov and Strassler [204]. Another example is the multi-center  $D3$ -brane solution of (8.5,8.7) which is dual to a Higgs deformation of  $\mathcal{N} = 4$  SYM in which the  $SU(N)$  gauge symmetry is broken spontaneously to  $SU(N_1) \otimes SU(N_2) \otimes \dots \otimes SU(N_M)$ . In these examples the connection with field theory is somewhat different from the emphasis in the present notes. For this reason we confine our discussion to domain wall solutions of  $D = 5$   $\mathcal{N} = 8$  sugra. This is a realistic framework since the  $D = 5$  theory contains all relevant deformations of  $\mathcal{N} = 4$  SYM, and experience indicates that  $5D$  domain wall solutions can be lifted to solutions of  $D = 10$  Type IIB sugra.

For domain walls, we can restrict to the metric and scalars of the theory which are governed by the action

$$S = \int d^5z \sqrt{g} \left[ -\frac{1}{4}R + \frac{1}{2}G_{IJ}(\phi^K) \partial_\mu \phi^I \partial^\mu \phi^J + V(\phi^k) \right] \tag{9.36}$$

The 42 scalars sit in the 27-bein matrix  $V_{AB}^{ab}(\phi^K)$  of  $E(6,6)/USp(8)$ . The indices  $a, b$  and  $AB$  have 8 values. They are anti-symmetrized (with symplectic trace removed) in most expressions we write. The coset metric  $G_{IJ}$ , the potential  $V$  and other quantities in the theory are constructed from  $V_{AB}^{ab}$ . Symmetries govern the construction, but the nested structure of symmetries makes things very complicated.

A simpler question than domain walls is that of critical points of  $V(\phi^k)$ . The AdS/CFT correspondence requires that every stable critical point with  $V < 0$  corresponds to a  $CFT_4$ . Stability means simply that mass eigenvalues of fluctuations satisfy  $m^2 > -4$  so that

bulk fields transform in unitary, positive energy representations of  $SO(2, 4)$  (for Lorentz signature).

Even the task of extremizing  $V(\phi^k)$  is essentially impossible in a space of 40 variables, ( $V$  does not depend on the dilaton and axion fields), so one uses the following simple but practically important trick, [205]:

- a. Select a subgroup  $H$  of the invariance group  $SO(6)$  of  $V(\phi^K)$ .
- b. The  $42\phi^K$  may be grouped into fields  $\phi$  which are singlets of  $H$  and others  $\xi$  which transform in non-trivial representations of  $H$ .
- c. It follows from naive group theory that the expansion of  $V$  takes the form  $V(\phi, \xi) = V_0(\phi) + V_2(\phi)\xi^2 + \mathcal{O}(\xi^3)$  with no linear term.
- d. Thus, if  $\hat{\phi}$  is a stationary point of  $V_0(\phi)$ , then  $\hat{\phi}, \xi = 0$  is a stationary point of  $V(\phi, \xi)$ . The problem is then reduced to minimization in a much smaller space.

The same method applies to all solutions of the equations of motion  $\frac{\partial S}{\partial \phi^K} = 0$ , and to the Killing spinor problem since that gives a solution to the equations of motion. The general principle is that if  $S$  is invariant under  $G$ , in this case  $G = SO(6)$ , and  $H \subset G$  is a subgroup, then a consistent  $H$ -invariant solution to the dynamics can be obtained by restricting, ab initio, to singlets of  $H$ .

All critical points with preserved symmetry  $H \supseteq SU(2)$  are known [206]. There are 5 critical points of which 3 are non-supersymmetric and unstable [196, 207]. There are two  $SUSY$  critical points of concern to us. The first with  $H = SO(6)$  and full  $\mathcal{N} = 8$   $SUSY$  is the maximally symmetric state discussed above, and the second has  $H = SU(2) \otimes U(1)$  and  $\mathcal{N} = 2$   $SUSY$ . The associated critical bulk solutions are dual to the undeformed  $\mathcal{N} = 4$  SYM and the critical IR limit of a particular deformation of  $\mathcal{N} = 4$  SYM.

The search for supersymmetric domain walls in  $\mathcal{N} = 8$  gauged supergravity begins with the fermionic transformation rules<sup>xvii</sup> which have the form:

$$\delta\psi_\mu^a = D_\mu\epsilon^a - \frac{1}{3}W_b^a\gamma_\mu\epsilon^b \quad (9.37)$$

$$\delta\chi^A = \left(\gamma^\mu P_{aI}^A \partial\phi^I - Q_a^A(\varphi)\right)\epsilon^a \quad (9.38)$$

where  $A$  is an index for the 48 spinor fields  $\chi^{abc}$ . The  $\epsilon^a$  are 4-component symplectic Majorana spinors [190] (with spinor indices suppressed and  $a = 1, \dots, 8$ ). The matrices  $W_b^a$ ,  $P_{aI}^A$  and  $Q_a^A$  are functions of the scalars  $\varphi^I$  which are part of the specification of the classical supergravity theory. Killing spinors  $\epsilon^a(\vec{x}, r)$  are spinor configurations which satisfy  $\delta\psi_\mu^a = 0$  and  $\delta\chi^A = 0$ . The process of solving these equations leads both to the  $\epsilon^a(\vec{x}, r)$  and to conditions which determine the domain wall geometry which supports them. These conditions, in this case the first order field equations (9.33, 9.34), imply that the bosonic equations of motion of the theory are satisfied.

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<sup>xvii</sup>Conventions for spinors and  $\gamma$ -matrices are those of [190] with spacetime signature  $+- - - -$ .

**Ex. 28:** For a generic *SUSY* or *sugra* theory, show that if there are Killing spinors for a given configuration of bosonic fields, then that configuration satisfies the equations of motion. *Hint:*

$$\delta S = \int \left[ \frac{\delta S}{\delta B} \delta B + \frac{\delta S}{\delta \psi} \delta \psi \right] \equiv 0, \quad (9.39)$$

where  $B$  and  $\psi$  denote the boson and fermion fields of the theory and  $\delta B$  and  $\delta \psi$  their transformation rules.

We now discuss the Killing spinor analysis to outline how the first order flow equations arise.

**Ex. 29:** Using the spin connection of Ex: 20, show that the condition  $\delta \psi_j^a = 0$  can be written out in detail as

$$\delta \psi_j^a = \partial_j \epsilon - \frac{1}{2} A'(r) \gamma_j \gamma_5 \epsilon^a - \frac{1}{3} W_b^a \gamma_j \epsilon^b = 0 \quad (9.40)$$

We can drop the first term because the Killing spinor must be translation invariant. What remains is a purely algebraic condition, and we can see that the flow equation (9.30) for the scale factor directly emerges with superpotential  $W(\phi)$  identified as one of the eigenvalues of the tensor  $W_b^a$ . In detail one actually has a symplectic eigenvalue problem, with 4 generically distinct  $W$ 's as solutions. Each of these is a candidate superpotential. One must then examine the 48 conditions

$$\delta \chi^A = (\gamma^5 P_{aI}^A \partial_r \phi^I - Q_a^A) \epsilon^a = 0 \quad (9.41)$$

to see if *SUSY* is supported on any of the eigenspaces. One can see how the gradient flow equation (9.33) can emerge. Success is not guaranteed, but when it occurs, it generically occurs on one of the four (symplectic) eigenspaces. The 5D Killing spinor solution satisfies a  $\gamma^5$  condition effectively yielding a 4d Weyl spinor, giving  $\mathcal{N} = 1$  *SUSY* in the dual field theory. Extended  $\mathcal{N} > 1$  *SUSY* requires further degeneracy of the eigenvalues. (The  $\delta \Psi_r^a = 0$  condition which has not yet been mentioned gives a differential equation for the  $r$ -dependence of  $\epsilon^a(r)$ )

**Ex. 30:** It is a useful exercise to consider a simplified version of the Killing spinor problem involving one complex (Dirac) spinor with superpotential  $W(\phi)$  with one scalar field. The equations are

$$\begin{aligned} (D_\mu - \frac{1}{3} i W \gamma_\mu) \epsilon &= 0 \\ (-i \gamma^\mu \partial_\mu \phi - \frac{dW}{d\phi}) \epsilon &= 0 \end{aligned} \quad (9.42)$$

Show that the solution of this problem yields the flow equations (9.29,9.30) and

$$\epsilon = e^{\frac{A}{2}} \eta \quad (9.43)$$

where  $\eta$  is a constant eigenspinor of  $\gamma^5$ . Show that at a critical point of  $W$ , there is a second Killing spinor (which depends on the transverse coordinates  $x^i$ ). See [70]. This appears because of the the doubling of supercharges in superconformal *SUSY*.

Needless to say the analysis is impossible on the full space of 42 scalars. Nor do we expect a solution in general, since many domain walls are dual to non-supersymmetric deformations and cannot have Killing spinors. In [190] a symmetry reduction to singlets of an  $SU(2)$  subgroup of  $SO(6)$  was used. After further simplification it was found that  $\mathcal{N} = 1$   $SUSY$  with  $SU(2) \times U(1)$  global symmetry was supported for flows involving two scalar fields,  $\phi_2$  a field with  $\Delta = 2$  in the  $20'$  of  $SO(6)$  in the full theory, and  $\phi_3$  a field with  $\Delta = 3$  in the  $10 + \overline{10}$  representation. (In [190], these fields were called  $\phi_3, \phi_1$  respectively.) The fields  $\phi_2, \phi_3$  have canonical kinetic terms as in (9.31). Using  $\rho = e^{\frac{\phi_2}{\sqrt{6}}}$ , the superpotential is

$$W(\phi_2, \phi_3) = \frac{1}{4L\rho^2} [\cosh(2\phi_3)(\rho^6 - 2) - 3\rho^6 - 2] \quad (9.44)$$

**Ex. 31:** Show that  $W(\phi_2, \phi_3)$  has the following critical points:

- i. a maximum at  $\phi_2 = 0, \phi_3 = 0$ , at which  $W = -\frac{3}{2L}$
- ii. a saddle point at  $\phi_2 = \frac{1}{\sqrt{6}} \ln 2, \phi_3 = \pm \frac{1}{2} \ln 3$  at which  $W = -\frac{2^{2/3}}{L}$ . (The two solutions are related by a  $Z_2$  symmetry and are equivalent).

Thus there is a possible domain wall flow interpolating between these two critical points. The flow equation (9.33) cannot yet be solved analytically for  $W$  of (9.44), but a numerical solution and its asymptotic properties were discussed in [190]. See Figure 14.

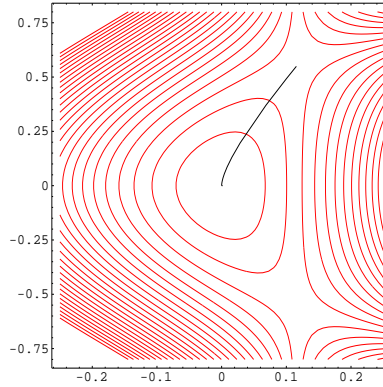


Figure 14: Contour plot of  $W(\phi_2, \phi_3)$

In accord with the general discussion of Section 9.2, the solution should be dual to a relevant deformation of  $\mathcal{N} = 4$   $SYM$  theory which breaks  $SUSY$  to  $\mathcal{N} = 1$  and flows to an  $SCFT_4$  at long distance. In the next section we discuss this field theory and the evidence that the supergravity description is correct.

In the space of the two bulk fields  $\phi_2, \phi_3$  there is a continuously infinite set of gradient flow trajectories emerging from the  $\mathcal{N} = 4$  critical point. One must tune the initial direction to find the one which terminates at the  $\mathcal{N} = 1$  point. All other trajectories approach infinite

values in field space, and the associated geometries, obtained from the flow equation (9.34) for  $A(r)$  have curvature singularities. There are analytic domain wall flows with  $\phi_3 \equiv 0$  [208, 209] and in other sectors [210] of the space of scalars of  $\mathcal{N} = 8$  sugra, and a number of 2-point correlation functions have been computed [211, 212, 213, 214, 181, 182]. Nevertheless the curvature singularities are at least a conceptual problem for the AdS/CFT correspondence. In any case we do not discuss singular flows here.

## 9.5 SUSY Deformations of $\mathcal{N} = 4$ SYM Theory

It is useful for several purposes to describe the  $\mathcal{N} = 4$  SYM theory in terms of  $\mathcal{N} = 1$  superfields. The 4 spinor fields  $\lambda^\alpha$  are regrouped, and  $\lambda^4$  is paired with gauge potential  $A_j$  in a gauge vector superfield  $V$ . The remaining  $\lambda^{1,2,3}$  may be renamed  $\psi^{1,2,3}$  and paired with complex scalars  $z^1 = X^1 + iX^4$ ,  $z^2 = X^2 + iX^5$ ,  $z^3 = X^3 + iX^6$  to form 3 chiral superfields  $\Phi^i$ . In the notation of Section 2.5, the Lagrangian consists of a gauge kinetic term plus matter terms

$$\mathcal{L} = \int d^4\theta \text{tr}(\bar{\Phi}^i e^{gV} \Phi^i) + \int d^2\theta g \text{tr} \Phi^3 [\Phi^1, \Phi^2] + h.c. \quad (9.45)$$

The manifest supersymmetry is  $\mathcal{N} = 1$  with  $R$ -symmetry  $SU(3) \otimes U(1)$ . Full symmetry is regained after re-expression in components because the Yukawa coupling  $g$  is locked to the  $SU(N)$  gauge coupling. This formulation is commonly used to explore perturbative issues since the  $\mathcal{N} = 1$  supergraph formalism (first reference in [13]) is quite efficient. This formulation suits our main purpose which is to discuss  $SUSY$  deformations of the theory.

A general relevant  $\mathcal{N} = 1$  perturbation of  $\mathcal{N} = 4$  SYM is obtained by considering the modified superpotential

$$U = g \text{tr} \Phi^3 [\Phi^1, \Phi^2] + \frac{1}{2} M_{\alpha\beta} \text{tr} \Phi^\alpha \Phi^\beta \quad (9.46)$$

This framework is called the  $\mathcal{N} = 1^*$  theory. The moduli space of vacua,  $\frac{\partial U}{\partial \Phi^\alpha} = 0$ , has been studied [215, 216, 203] and describes a rich panoply of dynamical realizations of gauge theories, confinement and Higgs-Coulomb phases, and as we shall see, a superconformal phase.

We discuss here the particular deformation with a mass term for one chiral superfield only

$$U = g \text{tr} \Phi^3 [\Phi^1, \Phi^2] + \frac{1}{2} m \text{tr} (\Phi^3)^2 \quad (9.47)$$

The  $R$ -symmetry is now the direct product of  $SU(2)$  acting on  $\Phi^{1,2}$  and  $U(1)_R$  with charges  $(\frac{1}{2}, \frac{1}{2}, 1)$  for  $\Phi^{1,2,3}$ . The massive field  $\Phi^3$  drops out of the long distance dynamics, leaving the massless fields  $\Phi^{1,2}$ . We thus find symmetries which match those of the supergravity flow of the last section. However to establish the duality, it needs to be shown that long distance dynamics is conformal. We will briefly discuss the pretty arguments of Leigh and Strassler [217] that show this is the case.



The key condition for conformal symmetry is the vanishing of  $\beta$ -functions for the various couplings in the Lagrangian. In a general  $\mathcal{N} = 1$  theory with gauge group  $G$  and chiral superfields  $\Phi_\alpha$  in representations  $R_\alpha$  of  $G$ , the exact  $NSVZ$  gauge  $\beta$ -function is

$$\beta(g) = -\frac{g^3}{8\pi^2} \frac{3T(G) - \sum_\alpha T(R_\alpha)(1-2\gamma_\alpha)}{1 - \frac{g^2 T(G)}{8\pi^2}} \quad (9.48)$$

where  $\gamma_\alpha$  is the anomalous dimension of  $\Phi^\alpha$  and  $T(R_\alpha)$  is the Dynkin index of the representation. (If  $T^a$  are the generators in the representation  $R_\alpha$ , then  $\text{tr} T^a T^b \equiv T(R_\alpha) \delta^{ab}$ ). In the present case, in which  $G = SU(N)$  and all fields are in the adjoint, we have  $T(R_\alpha) = T(G) = N$ , and

$$\beta(g) \sim 2N(\gamma_1 + \gamma_2 + \gamma_3) \quad (9.49)$$

In addition we need the  $\beta$ -function for various invariant field monomials  $\text{Tr}(\Phi^1)^{n_1}(\Phi^2)^{n_2}(\Phi^3)^{n_3}$ ,

$$\beta_{n_1, n_2, n_3} = 3 - \sum_{\alpha=1}^3 n_\alpha - \sum_{\alpha=1}^3 n_\alpha \gamma_\alpha \quad (9.50)$$

This form is a consequence of the non-renormalization theorem for superpotentials in  $\mathcal{N} = 1$   $SUSY$ . The first two terms are fixed by classical dimensions and the last is due to wave function renormalization. For the two couplings in the superpotential (9.47) we have

$$\beta_{1,1,1} = \gamma_1 + \gamma_2 + \gamma_3 \quad (9.51)$$

$$\beta_{0,0,2} = 1 - 2\gamma_3. \quad (9.52)$$

One should view the  $\gamma_\alpha(g, m)$  as functions of the two couplings. The conditions for the vanishing of the 3  $\beta$ -functions have the unique  $SU(2)$  invariant solution

$$\gamma_1 = \gamma_2 = -\frac{1}{2}\gamma_3 = -\frac{1}{4} \quad (9.53)$$

which imposes one relation between  $g, m$ , suggesting that the theory has a fixed line of couplings. The  $\beta = 0$  conditions are necessary conditions for a superconformal realization in the infrared, and Leigh and Strassler give additional arguments that the conformal phase is realized.

$\mathcal{N} = 1$  superconformal symmetry in 4 dimensions is governed by the superalgebra  $SU(2, 2, |1)$ . This superalgebra has several types of short representations. (See Appendix of [190]). For example, chiral superfields, either elementary or composite, are short multiplets in which scale dimensions and  $U(1)_R$  charge are related by  $\Delta = \frac{3}{2}r$ . For elementary fields  $\Delta_\alpha = 1 + \gamma_\alpha$ , and one can see that the  $\gamma_\alpha$  values in (9.53) are correctly related to the  $U(1)_R$  charges of the  $\Phi^\alpha$ .

The observables in the  $SCFT_{IR}$  are the correlation functions of gauge invariant composites of the light superfields<sup>xviii</sup>  $W_\alpha, \Phi^1, \Phi^2$ . We list several short multiplets together with

<sup>xviii</sup>the index of the field strength superfield  $W_a$  is that of a Lorentz group spinor, while that of  $\Phi^\alpha$  is that of  $SU(2)$  flavor.

the scale dimensions of their primary components

$$\begin{array}{cccccc}
\mathcal{O} & \text{tr}\Phi^\alpha\Phi^\beta & \text{tr}W_\alpha\Phi^\beta & \text{tr}W_\alpha W^\alpha & \text{tr}\Phi^+T^A\Phi & \text{tr}(W_\alpha W_\beta + \dots) \\
\Delta & \frac{3}{2} & \frac{9}{4} & 3 & 2 & 3
\end{array} \tag{9.54}$$

The first 3 operators are chiral, the next is the multiplet containing the  $SU(2)$  current, and the last is the multiplet containing  $U(1)_R$  current, supercurrent, and stress tensor. Each multiplet has several components.

## 9.6 AdS/CFT Duality for the Leigh-Strassler Deformation

We now discuss the evidence that the domain wall of  $\mathcal{N} = 8$  gauged supergravity of Section 9.4 is the dual of the mass deformation of  $\mathcal{N} = 4$  SYM of Section 9.5. There are two types of evidence, the match of dimensions of operators, discussed here, and the match of conformal anomalies discussed in the next chapter. Critics may argue that much of the detailed evidence is a consequence of symmetries rather than dynamics. But it is dynamically significant that the potential  $V(\Phi^k)$  contains an IR critical point with the correct symmetries and the correct ratio  $V_{IR}/V_{UV}$  to describe the IR fixed point of the Leigh-Strassler theory. The AdS/CFT correspondence would be incomplete if  $D = 5$   $\mathcal{N} = 8$  sugra did not contain this SCFT<sub>4</sub>.

Whether due to symmetries or dynamics, much of the initial enthusiasm for AdS/CFT came from the 1 : 1 map between bulk fields of Type IIB sugra and composite operators of  $\mathcal{N} = 4$  SYM. The map was established using the relationship between the AdS masses of fluctuations about the  $\text{AdS}_5 \times S^5$  solution and scale dimensions of operators. The same idea may be applied to fluctuations about the IR critical point of the flow of Section 9.4. One can check the holographic description of the dynamics by computing the mass eigenvalues of all fields in the theory, namely all fields of the graviton multiplet listed in Section 9.4. This task is complicated because the Higgs mechanism acts in several sectors. Scale dimensions are then assigned using the formula in (8.25) for scalars and its generalizations to other spins. The next step is to assemble component fields into multiplets of the  $SU(2, 2|1)$  superalgebra. One finds exactly the 5 short multiplets listed at the end of Section 9.5 together with 4 long representations. The detailed match of short multiplets confirms the supergravity description, while the scale dimensions of operators in long representations are non-perturbative predictions of the supergravity description.

It would be highly desirable to study correlation functions of operators in the Leigh-Strassler flow, but this requires an analytic solution for the domain wall, which is so far unavailable.

## 9.7 Scale Dimension and AdS Mass

For completeness we now list the relation between  $\Delta$  and the mass for the various bulk fields which occur in a supergravity theory. For  $d = 4$  some results were given in [170]. For the general case of for  $\text{AdS}_{d+1}$ , the relations are given below with references. There are exceptional cases in which the lower root of  $\pm$  is appropriate.

1. scalars [3]:  $\Delta_{\pm} = \frac{1}{2}(d \pm \sqrt{d^2 + 4m^2})$ ,
2. spinors [218]:  $\Delta = \frac{1}{2}(d + 2|m|)$ ,
3. vectors  $\Delta_{\pm} = \frac{1}{2}(d \pm \sqrt{(d-2)^2 + 4m^2})$ ,
4.  $p$ -forms [148]:  $\Delta = \frac{1}{2}(d \pm \sqrt{(d-2p)^2 + 4m^2})$ ,
5. first-order  $(d/2)$ -forms ( $d$  even):  $\Delta = \frac{1}{2}(d + 2|m|)$ ,
6. spin-3/2 [219, 220]:  $\Delta = \frac{1}{2}(d + 2|m|)$ ,
7. massless spin-2 [221]:  $\Delta = \frac{1}{2}(d + \sqrt{d^2 + 4m^2})$ .

## 10 The $c$ -theorem and Conformal Anomalies

In this chapter we develop a theme introduced in Sec. 9.2, the irreversibility of domain walls in supergravity and the suggested connection with the  $c$ -theorem for RG flows in field theory. The  $c$ -theorem is related to the conformal anomaly. We discuss this anomaly for 4d field theory and the elegant way it is treated in the AdS/CFT correspondence. This suggests a simple form for a holographic  $c$ -function, and monotonicity follows from the equation  $A''(r) < 0$ . It follows that any RG flow which can be described by the AdS/CFT correspondence satisfies the  $c$ -theorem. The holographic computation of anomalies agrees with field theory for both the undeformed  $\mathcal{N} = 4$  SYM theory and the  $\mathcal{N} = 1$  Leigh-Strassler deformation.

### 10.1 The $c$ -theorem in Field Theory

We briefly summarize the essential content of Zamolodchikov's  $c$ -theorem [188] which proves that RG flows in QFT<sub>2</sub> are irreversible. We consider the correlator  $\langle T_{zz}(z, \bar{z})T_{zz}(0) \rangle$  in a flow from a CFT<sub>UV</sub> to a CFT<sub>IR</sub>. It has the form

$$\langle T_{zz}(z, \bar{z})T_{zz}(0) \rangle = \frac{c(M^2 z \bar{z})}{z^4} \quad (10.1)$$

where  $M^2$  is a scale that is present since conformal symmetry is broken. The function  $c(M^2 z \bar{z})$  has the properties:

1.  $c(M^2 z \bar{z}) \rightarrow c_{UV}$  as  $|z| \rightarrow 0$  and  $c(M^2 z \bar{z}) \rightarrow c_{IR}$  as  $|z| \rightarrow \infty$  where  $c_{UV}$  and  $c_{IR}$  are central charges of the critical theories CFT<sub>UV</sub> and CFT<sub>IR</sub>.
2.  $c(M^2 z \bar{z})$  is not necessarily monotonic, but there are other (non-unique)  $c$ -functions which decrease monotonically toward the infrared and agree with  $c(M^2 z \bar{z})$  at fixed points. Hence  $c_{UV} > c_{IR}$  which proves irreversibility of the flow!
3. the central charges are also measured by the curved space Weyl anomaly in which the field theory is coupled to a fixed external metric  $g_{ij}$  and one has

$$\langle \theta \rangle = -\frac{c}{12} R \quad (10.2)$$

for both CFT<sub>UV</sub> and CFT<sub>IR</sub>.

The intuition for the  $c$ -theorem comes from the ideas of Wilsonian renormalization and the decoupling of heavy particles at low energy. Since  $T_{ij}$  couples to all the degrees of freedom of a theory, the  $c$ -function measures the effective number of degrees of freedom at scale  $x = \sqrt{z\bar{z}}$ . This number decreases monotonically as we proceed toward longer distance and more and more heavy particles decouple from the low energy dynamics. These are fundamental ideas and we should see if and how they are realized in QFT<sub>4</sub> and AdS<sub>5</sub>/CFT<sub>4</sub>.

First we define two projection operators constructed from the basic  $\pi_{ij} = \partial_i \partial_j - \delta_{ij} \square$ ,

$$\begin{aligned} \Pi_{ijkl}^{(0)} &= \pi_{ij} \pi_{kl} \\ \partial_{ijkl}^{(2)} &= 2\pi_{ij} \pi_{kl} - 3(\pi_{ik} \Pi_{jl} + \pi_{il} \pi_{jk}) \end{aligned} \quad (10.3)$$

In any QFT<sub>4</sub> the  $\langle TT \rangle$  correlator then takes the form

$$\langle T_{ij}(x) T_{kl}(0) \rangle = \frac{-1}{48\pi^4} P_{ijkl}^{(2)} \frac{c(m^2 x^2)}{x^4} + P_{ijkl}^{(0)} \frac{f(M^2 x^2)}{x^4} \quad (10.4)$$

In a flow between two CFT's, the central function [222]  $c(m^2 x^2)$  approaches central charges  $c_{UV}, c_{IR}$  in the appropriate limits, but  $f(M^2 x^2) \rightarrow 0$  in the UV and IR since effects of the trace  $T_i^i$  must vanish in conformal limits.

The correlators of  $T_{ij}$  can be obtained from a generating functional formally constructed by coupling the flat space theory covariantly to a non-dynamical background metric  $g_{ij}(x)$ . For example, in a gauge theory one would take

$$S[g_{ij}, A_k] \equiv \frac{1}{4} \int d^4 x \sqrt{g} g^{ik} g^{jl} F_{ij} F_{kl} \quad (10.5)$$

The effective action is then defined as the path integral over elementary fields, *e.g.*

$$e^{-S_{eff}[g]} \equiv \int [dA_i] e^{-S[g, A]} \quad (10.6)$$

Correlation functions are obtained by functional differentiation, viz.

$$\langle T_{i_1 j_1}(x_1) \cdots T_{i_n j_n}(x_n) \rangle = \frac{(-)^{n-1} 2^n}{\sqrt{g(x_1)} \cdots \sqrt{g(x_n)}} \frac{\delta^n}{\delta g^{i_1 j_1}(x_1) \cdots \delta g^{i_n j_n}(x_n)} S_{eff}[g] \quad (10.7)$$

with  $g_{ij} \rightarrow \delta_{ij}$ .

Consider two background metrics related by a Weyl transformation  $g'_{ij}(x) = e^{2\sigma(x)} g_{ij}(x)$ . Since the trace of  $T_{ij}$  vanishes in a CFT and  $\langle T_i^i \rangle = -\delta S / \delta \sigma$ , one might expect that  $S_{eff}[g] = S_{eff}[g']$ . However,  $S_{eff}[g]$  is divergent and must be regulated. This must be done even for a free theory (such as the pure  $U(1)$  Maxwell theory). In a free theory the correlators of composite operators such as  $T_{ij}$  are well defined for separated points but must be regulated since they are too singular at short distance to have a well defined Fourier transform. Regularization introduces a scale and leads to the Weyl anomaly, which is expressed as

$$\langle T_i^i \rangle = \frac{c}{16\pi^2} W_{ijkl}^2 - \frac{a}{16\pi^2} \tilde{R}_{ijkl}^2 + \alpha \square R + \beta R^2 \quad (10.8)$$

where the Weyl tensor and Euler densities are

$$W_{ijkl}^2 = R_{ijkl}^2 - 2R_{ij}^2 + \frac{1}{3}R^2 \quad (10.9)$$

$$\left(\frac{1}{2}\epsilon_{ij}{}^{mn} R_{mnlk}\right)^2 = R_{ijkl}^2 - 4R_{ij}^2 + R^2$$

The anomaly must be local since it comes from ultraviolet divergences, and we have written all possible local terms of dimension 4 above. One can show that  $\beta R^2$  violates the Wess-Zumino consistency condition while  $\square R$  is the variation of the local term  $\int d^4 x \sqrt{g} R^2$

in  $S_{eff}[g]$ . Finite local counter terms in an effective action depend on the regularization scheme and are usually considered not to carry dynamical information. (But see [223] for a proposed  $c$ -theorem based on this term. See [224] for a more extensive discussion of the Weyl anomaly.)

For the reasons above attention is usually restricted to the first two terms in (10.8). The scheme-independent coefficients  $c, a$  are central charges which characterize a  $\text{CFT}_4$ . One can show by a difficult argument [225, 226] that  $c$  for the critical theories  $\text{CFT}_{UV}$ ,  $\text{CFT}_{IR}$  agrees with the fixed point limits  $c_{UV}, c_{IR}$  of  $c(M^2x^2)$  in (10.4). The central charge  $a$  is not measured in  $\langle TT \rangle$  but agrees with constants  $a_{UV}, a_{IR}$  obtained in short and long distance limits of the 3-point function  $\langle TTT \rangle$ , see [172, 227]

What can be said about monotonicity? One might expect  $c_{UV} > c_{IR}$ , since the Weyl central charge is related to  $\langle TT \rangle$  and thus closer to the notion of unitarity which was important in Zamolodchikov's proof. However this inequality fails in some field theory models. Cardy [228] conjectured that the inequality  $a_{UV} > a_{IR}$  is the expression of the  $c$ -theorem in  $\text{QFT}_4$ . This is plausible since  $a$  is related to the topological Euler invariant in common with  $c$  for  $\text{QFT}_2$ , and Cardy showed that the inequality is satisfied in several models. Despite much effort (see [229] and references therein), there is no generally accepted proof of the  $c$ -theorem in  $\text{QFT}_4$ .

The values of  $c, a$  for free fields have been known for years. They were initially calculated by heat kernel methods, as described in [224]. The free field values agree with  $c_{UV}, a_{UV}$  in any asymptotically free gauge theory, since the interactions vanish at short distance. For a theory of  $N_0$  real scalars,  $N_{\frac{1}{2}}$  Dirac fermions, and  $N_1$  gauge bosons, the results are

$$\begin{aligned} c_{UV} &= \frac{1}{120}[N_0 + 5N_{\frac{1}{2}} + 12N_1] \\ a_{UV} &= \frac{1}{360}[N_0 + 11N_{\frac{1}{2}} + 62N_1] \end{aligned} \tag{10.10}$$

In a  $SUSY$  gauge theory, component fields assemble into chiral multiplets (2 real scalars plus 1 Majorana (or Weyl) spinor) and vector multiplets (1 gauge boson plus 1 Majorana spinor). For a theory with  $N_\chi$  chiral and  $N_V$  vector multiplets, the numbers above give

$$\begin{aligned} c_{UV} &= \frac{1}{24}[N_\chi + 3N_V] \\ a_{UV} &= \frac{1}{48}[N_\chi + 9N_V]. \end{aligned} \tag{10.11}$$

It is worthwhile to present some simple ways to calculate these central charges which are directly accessible to field theorists. Because of the relation to  $\langle T_{ij}T_{kl} \rangle$  detailed above, the values of  $c_{UV}$  can be easily read from a suitably organized calculation of the free field 1-loop contributions of the various spins. For gauge bosons one must include the contribution of Faddeev-Popov ghosts.

**Ex. 32:** *Do this. Work directly in  $x$ -space at separated points. No integrals and no regularization is required. Organize the result in the form of the first term of (10.4).*

In *SUSY* gauge theories the stress tensor has a supersymmetric partner, the  $U(1)_R$  current  $R_i$ . There are anomalies when the theory is coupled to  $g_{ij}$  and/or an external vector  $V_i(x)$  with field strength  $V_{ij}$ . Including both sources one can write the combined anomalies as [226]

$$\begin{aligned}\langle T_i^i \rangle &= \frac{c}{16\pi^2} W_{ijkl}^2 - \frac{a}{16\pi^2} \tilde{R}_{ijkl}^2 + \frac{c}{6\pi^2} V_{ij}^2 \\ \langle \partial_i \sqrt{g} R^i \rangle &= \frac{c-a}{24\pi^2} R_{ijkl} \tilde{R}^{ijkl} + \frac{5a-3c}{9\pi^2} V_{ij} \tilde{V}^{ij}\end{aligned}\tag{10.12}$$

Anomalies in the coupling of a gauge theory to external sources may be called external anomalies. There are also internal or gauge anomalies for both  $\langle T_i^i \rangle$  and  $R_i$ . The gauge anomaly of  $R_i$  is described by an additional term in (10.12) proportional to  $\beta(g) F_{ij} \tilde{F}^{ij}$ , but this term vanishes in a CFT.

The formula (10.12) can be used to obtain  $c, a$  from 1-loop fermion triangle graphs for both the UV and IR critical theories. The triangle graph for  $\langle R^i T_{jk} T_{lm} \rangle$  is linear in the  $U(1)_R$  charges  $r_{\hat{\alpha}}$  of the fermions in the theory, while the graph for  $\langle R^i R_j R_k \rangle$  is cubic. We consider a general  $\mathcal{N} = 1$  theory with gauge group  $G$  and chiral multiplets in representations  $R_{\alpha}$  of  $G$ . Comparing standard results for the anomalous divergences of triangle graphs with (10.12), one finds (see [226, 230]),

$$c - a = -\frac{1}{16}(\dim G + \sum_{\alpha} \dim R_{\alpha}(r_{\alpha} - 1))\tag{10.13}$$

$$\tag{10.14}$$

$$5a - 3c = \frac{9}{16}(\dim G + \sum_{\alpha} \dim R_{\alpha}(r_{\alpha} - 1))^3\tag{10.15}$$

We incorporate the facts that the  $U(1)_R$  charge of the gaugino is  $r_{\lambda} = 1$  while the charge of a fermion  $\psi_{\alpha}$  in a chiral multiplet is related to the charge of the chiral superfield  $\Phi^{\alpha}$  by  $r_{\hat{\alpha}} = r_{\alpha} - 1$ .

If asymptotic freedom holds, then the  $\text{CFT}_{UV}$  is free, and one obtains its central charges  $c_{UV}, a_{UV}$  using the free field  $U(1)_R$  charges,  $r_{\lambda} = 1$  for the gaugino and  $r_{\alpha} = \frac{2}{3}$  for chiral multiplets. The situation is more complex for the  $\text{CFT}_{IR}$  since the central charges are corrected by interactions. Seiberg and others following his techniques have found a large set of *SUSY* gauge theories which do flow to critical points in the IR [18]. The  $\mathcal{N} = 1$  superconformal algebra  $SU(2, 2|1)$  contains a  $U(1)_R$  current  $S_i$  which is in the same composite multiplet as the stress tensor. In many models this current is uniquely determined as a combination of the free current  $R_i$  plus terms which cancel the internal (gauge) anomalies of the former. Of course, the current  $S_i$  must also be conserved classically. Thus the  $S$ -charges of each  $\Phi_{\alpha}$  arrange so that all terms in the superpotential  $U(\Phi_{\alpha})$  have charge 2. It is the  $S$ -current which is used to show that anomalies match between Seiberg duals. These anomalies can be calculated from 1-loop graphs because the external anomalies are 1-loop exact for currents with no gauge anomaly. This is just the standard procedure of 't Hooft anomaly matching. The charge assigned by the  $S_i$  current is  $r_{\lambda} = 1$  for gauginos and uniquely determined values  $r_{\alpha}$  for chiral multiplets. It can be shown [231, 226] that  $c_{IR}, a_{IR}$  are obtained by inserting these values in (10.13).

Given this theoretical background it is a matter of simple algebra to obtain the UV and IR central charges and subtract to deduce the following formulas for their change in an RG flow:

$$c_{UV} - c_{IR} = \frac{1}{384} \sum_{\alpha} \dim R_{\alpha} (2 - 3r_{\alpha}) [(7 - 6r_{\alpha})^2 - 17] \quad (10.16)$$

$$(10.17)$$

$$a_{UV} - a_{IR} = \frac{1}{96} \sum_{\alpha} \dim R_{\alpha} (3r_{\alpha} - 2)^2 (5 - 3r_{\alpha}) \quad (10.18)$$

These formulas were applied [230] to test the proposed  $c$ -theorem in the very many Seiberg models of  $SUSY$  gauge theories with IR fixed points. Results indicated that the sign of  $c_{UV} - c_{IR}$  is model-dependent, but  $a_{UV} - a_{IR} > 0$  in **all** models. Thus there is a wealth of evidence that the Euler central charge satisfies a  $c$ -theorem, even though a fundamental proof is lacking.

**Ex. 33:** *Serious readers are urged to verify as many statements about the anomalies as they can. For minimal credit on this exercise please obtain the flow formulas (10.16) from (10.13).*

Let us now apply some of these results to the field theories of most concern to us, namely the undeformed  $\mathcal{N} = 4$  theory and its  $\mathcal{N} = 1$  mass deformation. We can view the undeformed theory as the UV limit of the flow of its  $\mathcal{N} = 1$  deformation. The free  $R$ -current assigns the charges  $(1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$  to the gaugino and chiral matter fermions of the  $\mathcal{N} = 1$  description, while the  $S$ -current of the mass deformed theory with superpotential in (9.47) assigns  $(1, -\frac{1}{2}, -\frac{1}{2}, 0)$ . In both cases these are elements of the Cartan subalgebra of  $SU(4)_R$  and have vanishing trace. It is easy to see that the formula (10.13) for  $c - a$  is proportional to this trace and vanishes. The same observation establishes that both currents have no gauge anomaly. The formula (10.15) then becomes

$$a = c = \frac{9}{32} (N^2 - 1) (1 + \sum (r_{\alpha} - 1)^3). \quad (10.19)$$

Applied to the free current and then the  $S$ -current, this gives

$$\begin{aligned} a_{UV} = c_{UV} &= \frac{1}{4} (N^2 - 1) \\ a_{IR} = c_{IR} &= \frac{27}{32} \frac{1}{4} (N^2 - 1). \end{aligned} \quad (10.20)$$

The relation  $\Delta = \frac{3}{2}r$  between scale dimension and  $U(1)_R$  charge also leads to the assignment of charges we have used. In the UV limit we have the  $\mathcal{N} = 4$  theory with chiral superfields  $W_{\alpha}, \Phi^{\beta}$  with dimensions  $\frac{3}{2}, 1$ . In the IR limit we must consider the  $SU(2)$  invariant split  $W_{\alpha}, \Phi^{1,2}, \Phi^3$ , and the Leigh-Strassler argument for a conformal fixed point which requires  $\Delta = \frac{3}{2}, \frac{3}{4}, \frac{3}{4}, \frac{3}{2}$ . These values give the fermion charges used above. It is no accident that  $r = 0$  for the fermion  $\psi^3$ . The  $\Phi^3$  multiplet drops out at long distance and thus cannot contribute to IR anomalies.



## 10.2 Anomalies and the $c$ -theorem from AdS/CFT

One of the early triumphs of the AdS/CFT was the calculation of the central charge  $c$  for  $\mathcal{N} = 4$  SYM from the  $\langle TT \rangle$  correlator whose absorptive part was obtained from the calculation of the cross-section for absorption of a graviton wave by the D3-brane geometry, [2]. This was reviewed in [162] and we will take a different viewpoint here.

We will describe in some detail the general approach of Henningson and Skenderis [175] to the holographic Weyl anomaly. This leads to the correct values of the central charges and suggests a simple monotonic  $c$ -function.

We focus on the gravity part of the toy model action of Sec 9.1

$$S = \frac{-1}{16\pi G} \left[ \int d^5 z \sqrt{g} (R + \frac{12}{L^2}) + \int d^4 z \sqrt{\gamma} 2K \right] \quad (10.21)$$

in which we have added the Gibbons-Hawking surface term which we will explain further below. Lower spin bulk fields can be added and do not change the gravitational part of the conformal anomaly.

One solution of the Einstein equation is the  $\text{AdS}_{d+1}$  geometry which we previously wrote as

$$ds^2 = e^{\frac{2r}{L}} \delta_{ij} dx^i dx^j + dr^2 \quad (10.22)$$

We introduce the new radial coordinate  $\rho = e^{-\frac{2r}{L}}$  in order to follow the treatment of [175]. The boundary is now at  $\rho = 0$ .

**Ex. 34:** Show that the transformed metric is

$$ds^2 = L^2 \left[ \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} \delta_{ij} dx^i dx^j \right] \quad (10.23)$$

This is just  $\text{AdS}_5$  in new coordinates. We now consider more general solutions of the form

$$ds^2 = L^2 \left[ \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x, \rho) dx^i dx^j \right] \quad (10.24)$$

with non-trivial boundary data on the transverse metric, viz.

$$g_{ij}(x, \rho) \xrightarrow{\rho \rightarrow 0} \bar{g}_{ij}(x) \quad (10.25)$$

The reason for this generalization may be seen by thinking of the form  $\bar{g}_{ij}(x) = \delta_{ij} + h_{ij}(x)$ . The first term describes the flat boundary on which the  $\text{CFT}_4$  lives, while  $h_{ij}(x)$  is the source of the stress tensor  $T_{ij}$ . We can use the formalism to compute  $\langle T_{ij} \rangle$ ,  $\langle T_{ij} T_{kl} \rangle$ , etc.

**Ex. 35:** Consider the special case of (10.24) in which  $g_{ij}(x, \rho) = g_{ij}(x)$  depends only on the transverse  $x^i$ . Let  $R_{ijkl}$ ,  $R_{ij}$  and  $R$  denote Riemann, Ricci and scalar curvatures of the 4d metric  $g_{ij}(x)$ . Show that the 5D metric thus defined satisfies the EOM  $R_{\mu\nu} = -4g_{\mu\nu}$  if  $R_{ij} = 0$ . Show that the 5D curvature invariant is

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{\rho^2}{L^4} R_{ijkl} R^{ijkl} - \frac{4\rho}{L^2} R + \frac{40}{L^4} \quad (10.26)$$

Thus, as observed in [232], if  $R_{ij} = 0$ , we have a reasonably generic solution of the 5D EOM's with a curvature singularity on the horizon,  $\rho \rightarrow \infty$ .

As we will see shortly we will need to introduce a cutoff at  $\rho = \epsilon$  and restrict the integration in (10.21) to the region  $\rho \geq \epsilon$ . The induced metric at the cutoff is  $\gamma_{ij} = \frac{g_{ij}}{\rho}$ . The measure  $\sqrt{\gamma}$  appears in the surface term in (10.21) as does the trace of the second fundamental form

$$K = \gamma^{ij} K_{ij} = -g^{ij} \rho \frac{\partial}{\partial \rho} \left( \frac{g_{ij}(x, \rho)}{\epsilon} \right) \Big|_{\rho=\epsilon} \quad (10.27)$$

We now consider a particular type of infinitesimal 5D diffeomorphism first considered in this context in [233]:

$$\begin{aligned} \rho &= \rho'(1 - 2\sigma(x')) \\ x^i &= x'^i + a^i(x', \rho') \end{aligned} \quad (10.28)$$

with

$$a^i(x, \rho) = \frac{L^2}{2} \int_0^\rho d\hat{\rho} g^{ij}(x, \hat{\rho}) \partial_j \sigma(x) \quad (10.29)$$

**Ex. 36:** Show that  $g'_{55} = g_{55}$  and  $g'_{5i} = g_{5i} = 0$  under this diffeomorphism, but that

$$g_{ij} \rightarrow g'_{ij} = g_{ij} + 2\sigma \left(1 - \rho \frac{\partial}{\partial \rho}\right) g_{ij} + \nabla_i a_j + \nabla_j a_i \quad (10.30)$$

In the boundary limit,  $a_i \rightarrow 0$  and  $\rho \frac{\partial}{\partial \rho} g_{ij} \rightarrow 0$ , so that

$$\bar{g}_{ij}(x) \rightarrow \bar{g}'_{ij}(x) = (1 + 2\sigma(x)) \bar{g}_{ij}(x) \quad (10.31)$$

Hence the effect of the 5D diffeomorphism is a Weyl transformation of the boundary metric!

This raises a puzzle. Consider the on-shell action  $S[\bar{g}_{ij}]$  obtained by substituting the solution (10.24) into (10.21). Since the bulk action and the field equations are invariant under diffeomorphisms, we would expect  $S[\bar{g}_{ij}] = S[\bar{g}'_{ij}]$ . But AdS/CFT requires that  $S[\bar{g}_{ij}] = S_{eff}[\bar{g}_{ij}]$ , and we know that, due to the Weyl anomaly,  $S_{eff}[\bar{g}_{ij}] \neq S_{eff}[\bar{g}'_{ij}]$ .

The resolution of the puzzle is that  $S[\bar{g}_{ij}]$  as we defined it is meaningless since it diverges. This isn't the somewhat fuzzy-wuzzy divergence usually blamed on the functional integral for  $S_{eff}[\bar{g}_{ij}]$  in quantum field theory. It is very concrete; when you insert a solution of Einstein's equation with the boundary behavior above into (10.21), the radial integral diverges near the boundary.

Therefore we define a cutoff action  $S_\epsilon[\bar{g}_{ij}]$  as the on-shell value of (10.21) with radial integration restricted to  $\rho \geq \epsilon$ . One can study its dependence on the cutoff to obtain and subtract a counterterm action  $S_\epsilon[\bar{g}_{ij}]_{ct}$  to cancel singular terms as  $\epsilon \rightarrow 0$ .  $S_\epsilon[\bar{g}_{ij}]_{ct}$  is an integral over the hypersurface  $\rho = \epsilon$  of a local function of the induced metric  $\gamma_{ij}$  and its curvatures, and it is not Weyl invariant. The renormalized action is defined as

$$S_{ren}[\bar{g}] \equiv \lim_{\epsilon \rightarrow 0} (S_\epsilon[\bar{g}] - S_\epsilon[\bar{g}]_{ct}) \quad (10.32)$$

We now outline how the calculation of correlation functions and the conformal anomaly proceeds in this formalism and then discuss further necessary details. The variation of  $S_{ren}$

is

$$\delta S_{ren}[\bar{g}] \equiv \frac{1}{2} \int d^4x \sqrt{\bar{g}} \langle T_{ij} \rangle \delta \bar{g}^{ij}. \quad (10.33)$$

The variation defines the quantity  $\langle T_{ij}(x) \rangle$  which, in the light of (10.6), is interpreted as the expectation value of the field theory stress tensor in the presence of the source  $\bar{g}_{ij}$ , and it depends non-locally on the source. Correlation functions in the CFT are then obtained by further differentiation, *e.g.*

$$\langle T_{ij}(x) T_{kl}(y) \rangle = -\frac{2}{\sqrt{\bar{g}(y)}} \frac{\delta}{\delta \bar{g}^{kl}(y)} \langle T_{ij}(x) \rangle \Big|_{\bar{g}_{ij}=\delta_{ij}} \quad (10.34)$$

The contributions to  $\langle T_{ij}(x) \rangle$  come from the surface term in the radial integral in  $S_\epsilon[\bar{g}]$  and from  $S_\epsilon[\bar{g}]_{ct}$ . Possible contributions involving bulk integrals vanish by the equations of motion.

The variation  $\delta \bar{g}^{ij}$  is arbitrary; let's choose it to correspond to a Weyl transformation, *i.e.*  $\delta \bar{g}^{ij} = -2\bar{g}^{ij} \delta \sigma$ . Then (10.33) gives

$$\langle T_i^i \rangle = \bar{g}^{ij} \langle T_{ij} \rangle = -\frac{\delta S_{ren}[\bar{g}]}{\delta \sigma} \quad (10.35)$$

which is a standard result in quantum field theory in curved space. The quantity  $\langle T_i^i \rangle$  is to be identified with the conformal anomaly of the CFT and must therefore be local. It is local, and the holographic computation gives (as we derive below)

$$\langle T_i^i \rangle = \frac{L^3}{8\pi G} \left( \frac{1}{8} R^{ij} R_{ij} - \frac{1}{24} R^2 \right) \quad (10.36)$$

(The 2-point function (10.34) must be non-local, and it is. See [181, 182] for recent studies in the present formalism, and [214] for a closely related treatment.)

The holographic result may be compared with the field theory  $\langle T_i^i \rangle$  in (10.8). The absence of the invariant  $R_{ijkl}^2$  in (10.36) requires  $c = a$ . Thus we deduce that any CFT<sub>4</sub> which has a holographic dual in this framework must have central charges which satisfy  $c = a$  (at least as  $N \rightarrow \infty$  when the classical supergravity approximation is valid.) This is satisfied by  $\mathcal{N}=4$  SYM but not by the conformal invariant  $\mathcal{N}=2$  theory with an  $SU(N)$  gauge multiplet and  $2N$  fundamental hypermultiplets.

**Ex. 37:** Show that when  $c = a$  the QFT trace anomaly of (10.8) reduces to

$$\langle T_i^i \rangle = \frac{c}{8\pi^2} (R^{ij} R_{ij} - \frac{1}{3} R^2) \quad (10.37)$$

Thus agreement with the holographic result (10.36) requires  $c = \frac{\pi L^3}{8G}$ . To check this recall that  $G$  is the 5D Newton constant, so that  $G = \frac{G_{10}}{Vol S_5} = \frac{\pi L^3}{2N^2}$ , where the last equality incorporates the requirement that  $AdS_5 \times S^5$  with 5-form flux  $N$  is a solution of the field equations of  $D = 10$  Type IIB sugra. This gives the anomaly of undeformed  $\mathcal{N} = 4$  SYM theory on the nose!

The Henningson-Skenderis method is very elegant and has useful generalizations [176]. It is worth discussing in more detail. The treatment starts with the mathematical result [234] that the general solution of the Einstein equations can be brought to the form (10.24), and that the transverse metric can be expanded in  $\rho$  near the boundary as

$$g_{ij}(x, \rho) = \bar{g}_{ij} + \rho g_{(2)ij} + \rho^2 g_{(4)ij} + \rho^2 \ln \rho h_{(4)ij} + \dots \quad (10.38)$$

The tensor coefficients are functions of the transverse coordinates  $x^i$ . The tensors  $g_{(2)ij}$ ,  $h_{(4)ij}$  can be determined as local functions of the curvature  $\bar{R}_{ijkl}$  of the boundary metric  $\bar{g}_{ij}$ . One just needs to substitute the expansion (10.38) in the 5D field equations  $R_{\mu\nu} = -4g_{\mu\nu}$  and grind out a term-by-term solution.

**Ex. 38:** Do this and derive  $g_{(2)ij} = \frac{1}{2}(\bar{R}_{ij} - \frac{1}{6}\bar{R}\bar{g}_{ij})$ . Very serious readers are encouraged to obtain the more complicated result for  $h_{(4)ij}$  given in (A.6) of [176].

The tensor  $g_{(4)ij}$  is only partially determined by this process of near-boundary analysis. Specifically its divergence and trace are local in the curvature  $\bar{R}_{ijkl}$ , but transverse traceless components are left undetermined. This is sensible since the EOM's are second order, and the single Dirichlet boundary condition does not uniquely fix the solution. At the linearized level the extra condition of regularity at large  $\rho$  (the deep interior) is imposed. The transverse traceless part of  $g_{(4)ij}$  then depends non-locally on  $\bar{g}_{ij}$  and eventually contributes to  $n$ -point correlators of  $T_{ij}$  in the dual field theory.

The local tensors in (10.38) are sufficient to determine the divergent part of  $S_\epsilon[\bar{g}]$ . It is tedious, delicate (but straightforward!) to substitute the expansion in (10.21), integrate near the boundary and identify the counterterms which cancel divergences. The result is

$$S_\epsilon[\bar{g}]_{ct} = \frac{1}{4\pi G} \int d^4x \sqrt{\gamma} \left( \frac{3}{2L^2} - \frac{\hat{R}}{8} - \frac{L^2 \ln \epsilon}{32} (\hat{R}^{ij} \hat{R}_{ij} - \frac{1}{3} \hat{R}^2) \right) \quad (10.39)$$

where  $\hat{R}_{ij}$  and  $\hat{R}$  are the Ricci and scalar curvatures of the induced metric  $\gamma_{ij} = \frac{g_{ij}(x, \epsilon)}{\epsilon}$ . The first two terms in (10.39) thus have power singularities as  $\epsilon \rightarrow 0$ . Recall the discussion of cutoff dependence in Section 8.5. In the  $\rho = z_0^2$  coordinate, the bulk cutoff  $\epsilon$  should be identified with  $1/\Lambda^2$  where  $\Lambda$  is the UV cutoff in QFT. Thus we find the quartic, quadratic, and logarithmic divergences expected in QFT<sub>4</sub>! (See Appendix B of [176] for details of the computation of (10.39).)

The next step is to calculate

$$\langle T_i^i(x) \rangle = -\lim_{\epsilon \rightarrow 0} \frac{\delta}{\delta \sigma(x)} (S_\epsilon[\bar{g}] - S_\epsilon[\bar{g}]_{ct}). \quad (10.40)$$

However, one must vary the boundary data  $\delta \bar{g}_{ij} = 2\delta\sigma \bar{g}_{ij}$  while maintaining the fact that the interior solution corresponds to that variation. Thus one is really carrying out the diffeomorphism of (10.28) so that  $\delta\epsilon = 2\epsilon\delta\sigma(x)$ . All terms of  $S_\epsilon[\bar{g}]$  are invariant under the combined change of coordinates and change of shape of the cutoff hypersurface. The first two terms in  $S_\epsilon[\bar{g}]_{ct}$  are also invariant. There is the explicit variation  $\delta \ln \epsilon = -2\delta\sigma(x)$  in the logarithmic counterterm, and this is the only variation since the boundary integral is

the difference of the Weyl<sup>2</sup> and Euler densities and is invariant. Thus we find the result (10.36) stated earlier in a strikingly simple way!

The method just described may be applied to the calculation of holographic conformal anomalies in any **even** dimension [175, 176]. However for **odd** dimension the structure of the near-boundary expansion (10.38) changes. There is no  $\ln \rho$  term and no logarithmic counterterm either. Hence no conformal anomaly in agreement with QFT in odd dimension.

### 10.3 The Holographic $c$ -theorem

The method just discussed can be extended to apply to the Weyl anomalies of the critical theories at end-points of holographic RG flows. In general we can consider a domain wall interpolating between the region of an  $\text{AdS}_{UV}$  with scale  $L_{UV}$  and the deep interior of an  $\text{AdS}_{IR}$  with scale  $L_{IR}$ . The holographic anomalies are

$$c_{UV} = \frac{\pi}{8G} L_{UV}^3 \quad c_{IR} = \frac{\pi}{8G} L_{IR}^3 \quad (10.41)$$

The first result can be derived by including relevant scalar fields in the previous method, and latter by applying the method to an entire AdS geometry with scale  $L_{IR}$ .

For any bulk domain wall one can consider the following scale-dependent function (and its radial derivative):

$$\begin{aligned} C(r) &= \frac{\pi}{8G} \frac{1}{A^3} \\ C'(r) &= \frac{\pi}{8G} \frac{-3A''}{A^4} \end{aligned} \quad (10.42)$$

We have  $C'(r) \geq 0$  as a consequence of the condition  $A'' \leq 0$  derived from the domain wall *EOM*'s in Section 9.2. Thus  $C(r)$  is an essentially perfect holographic  $c$ -function:

1. It decreases monotonically along the flow from  $UV \rightarrow IR$ .
2. It interpolates between the central charges  $c_{UV}$  and  $c_{IR}$ .
3. If perfect, it would be stationary only if conformal symmetry holds. This is true if the domain wall is the solution of the first order flow equations discussed in Sec 9.3 and thus true for SUSY flows.

The moral of the story is that the  $c$ -theorem for RG flows, which has resisted proof by field theory methods, is trivial when the theory has a gravity dual since  $A'' \leq 0$ . See [179, 190].

Finally, we note that for the mass deformed  $\mathcal{N}=4$  theory the ratio  $(\frac{L_{IR}}{L_{UV}})^3 = (\frac{W_{UV}}{W_{IR}})^3 = \frac{27}{32}$ . Thus the holographic prediction of  $c_{IR} = a_{IR}$  agrees with the field theory result in (10.20)! See [235]

There is much more to be said about the active subject of holographic RG flows and many interesting papers that deserve study by interested theorists. We hope that the introduction to the basic ideas contained in these lecture notes will stimulate that study.

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