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Cohomological Yang-Mills Theory in Eight Dimensions*

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Abstract

We construct nearly topological Yang-Mills theories on eight dimensional manifolds with a special holonomy group. These manifolds are the Joyce manifold with $Spin(7)$ holonomy and the Calabi-Yau manifold with $SU(4)$ holonomy. An invariant closed four form $T_{\mu\nu\rho\sigma}$ on the manifold allows us to define an analogue of the instanton equation, which serves as a topological gauge fixing condition in BRST formalism. The model on the Joyce manifold is related to the eight dimensional supersymmetric Yang-Mills theory. Topological dimensional reduction to four dimensions gives non-abelian Seiberg-Witten equation.

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1 Introduction

Almost a decade ago topological quantum field theory (TQFT) was proposed and it was conjectured that TQFT describes "topological phase" where general covariance is unbroken [1]. The topological phase was supposed to be a key to the underlying principle of string theory. It has been shown that in two dimensional quantum gauge theory and quantum gravity we can see the topological phase realized by TQFT. Furthermore, TQFT, or more specifically, cohomological quantum field theory is related to supersymmetric (SUSY) quantum field theory by twisting. From mathematical viewpoint, TQFT provides a field theoretical description of intriguing topological invariants such as Donaldson invariants, Gromov-Witten invariants and Seiberg-Witten invariants. So far TQFT has been extensively studied in two, three and four dimensions. (See e.g. [2], [3] and references therein.)

Recent developments in dualities and non-perturbative dynamics of superstring theory have stimulated a renewed interest in SUSY field theories in diverse dimensions, which appear in various compactifications or limits of M-theory, F-theory, or any other hypothetical non-perturbative formulation of "string theory". From such a viewpoint it is an interesting problem to see how the idea of TQFT is extended to higher dimensions. In this article we will report a recent progress in this direction [4]. Our main results are summarized as follows; we have constructed cohomological Yang-Mills theories in eight dimensions. On the eight dimensional Joyce manifold and the Calabi-Yau fourfold, which have special holonomy groups, we can find a topological action and topological gauge fixing conditions in BRST formalism. Our covariant gauge conditions are the octonionic instanton equation and the complexified anti-self-dual (ASD) equation, respectively. The theory on the Joyce manifold is a twist of eight dimensional SUSY Yang-Mills theory and hence related to ten dimensional $N=1$ SUSY Yang-Mills theory. Moreover, an appropriate dimensional reduction to four dimensions gives topological field theory of non-abelian Seiberg-Witten (monopole) equation. For a full account of technical details and further references, we refer to the original paper [4].

The most crucial point for our construction in higher dimensions is to find an appropriate set of topological action and covariant gauge fixing conditions. To obtain a quantum field theory which explores a moduli space of gauge fixing conditions by a well-defined weak coupling expansion, all the dynamical fields should have non-degenerate kinetic terms after eliminating the auxiliary fields. This means mathematically we should have a complex of elliptic operators behind our choice of gauge fixing conditions. We have found that as covariant gauge fixing condition the following generalization of the instanton equation works well for pure Yang-Mills theory in higher dimensions;

$$\frac{1}{2} T^{\mu\nu\rho\sigma} F_{\rho\sigma} \ = \ \lambda F^{\mu\nu} \ , \tag{1}$$

where $T^{\mu\nu\rho\sigma}$ is a totally antisymmetric tensor and λ is a constant (an eigenvalue). In four dimensions we can choose the $SO(4)$ invariant tensor $\epsilon^{\mu\nu\rho\sigma}$, which leads to the self-dual ($\lambda = 1$) or the anti-self-dual ($\lambda = -1$) condition for the curvature two form $F_{\mu\nu}$. If the dimension N is higher than four, $T^{\mu\nu\rho\sigma}$ cannot be invariant under $SO(N)$ any more. Corrigan et al [5] classified possible choices of $T^{\mu\nu\rho\sigma}$ up to eight dimensions, requiring that $T^{\mu\nu\rho\sigma}$ is invariant under a maximal subgroup of $SO(N)$. They found that a choice of $Spin(7) \subset SO(8)$ gave seven first order conditions for the gauge field A_μ in eight dimensions. We adopt these conditions as topological gauge conditions for the Joyce manifold. In the following we will mainly be devoted to this case. The Calabi-Yau manifold gives another possibility, where the complexified ASD instanton equation is employed as covariant gauge fixing condition.

Twisting of SUSY quantum field theory is another approach to construct a model of TQFT [1] [6]. If the base manifold admits a covariantly constant spinor ζ, the supersymmetry transformation with the parameter ζ may be identified as a topological BRST transformation. But this cannot take place on generic curved manifolds. To circumvent the problem we use the trick of twisting that effectively produces a covariantly constant spinor parameter in lower dimensions. This is regarded as a result of gauging the R-symmetry, or global symmetry of a supersymmetric theory. More generally, we have a chance of having covariantly constant spinors due to a reduction of the holonomy group. This possibility of twisting the supersymmetry when the holonomy group is reduced was first pointed out in [7] and it was applied to the twisting of $N=1$ supersymmetry on four dimensional Kähler manifolds in [8]. Eight dimensional (oriented) Riemannian manifolds have the holonomy group $SO(8)$. On the Joyce manifold the holonomy is reduced to a maximal subgroup $Spin(7)$ [9]. This reduction allows exactly one covariantly constant spinor, in terms of which we can define topological BRST transformation. In this sense the cohomological Yang-Mills theory on the Joyce manifold is a natural generalization of the topological Yang-Mills theory on $K3$ surface which has $SU(2) \simeq Spin(3)$ holonomy [10].

2 Instanton Equation on the Joyce manifold

Four dimensional topological Yang-Mills theory can be obtained by the BRST formalism [11]. Taking $p_1 = \frac{1}{8\pi^2}\mathrm{Tr}\ (F \wedge F)$ as a topological Lagrangian, we gauge-fix its topological invariance with three covariant gauge conditions of (anti-) self-duality $F_{\mu\nu} \pm \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = 0$ and one Feynman-Landau type gauge condition $\partial_\mu A^\mu = 0$. It is crucial in this construction that the (anti-) self-duality of the curvature gives three conditions for four degrees of freedom of local deformations of the gauge field A_μ. More mathematically, we have an elliptic complex $0 \to \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2_\pm \to 0$, tensored with a Lie algebra $\mathcal{G}$. Topological Yang-Mills theory is also regarded as a twisted $N=2$ SUSY Yang-Mills theory in four dimensions. Hence, we expect that topological Yang-Mills theory probes the vacuum structure of supersymmetric gauge theory.

We want to extend this scheme of topological Yang-Mills theory to eight dimensions. We cannot do it for an arbitrary manifold. It is possible when the holonomy group $SO(8)$ is reduced to $Spin(7)$ or $SU(4)$, which allows an invariant closed four form Ω. We can consider a topological action

$$S_0 \ = \ \frac{1}{2}\int_{M_8} \ \Omega \wedge \mathrm{Tr}\ (F \wedge F)\ , \tag{2}$$

which is quadratic in the curvature F. Since Ω is closed, S_0 is invariant under an arbitrary local deformation of the gauge field A_μ; $\delta A_\mu = \epsilon_\mu$. A $Spin(7)$ structure on an eight dimensional (Riemannian) manifold M_8 is given by a closed self-dual $Spin(7)$ invariant four form Ω. Then, the holonomy group is a subgroup of $Spin(7)$ and the metric g is Ricci-flat. In this case (M_8, g) is called the Joyce manifold [9]. For the Calabi-Yau fourfolds with $SU(4)$ holonomy, we can take a covariantly constant $(4,0)$ form as Ω.

On eight dimensional manifolds, the vector, the chiral spinor and the anti-chiral spinor are all eight dimensional representations. They will be denoted by $\mathbf{8_v}, \mathbf{8_s}, \mathbf{8_c}$, respectively. The triality operation of $SO(8)$ interchanges these representations. The reduction of the holonomy group to $Spin(7)$ induces an invariant decomposition of the chiral spinor; $\mathbf{8_s} = \mathbf{1} \oplus \mathbf{7}$. The representations $\mathbf{8_v}$ and $\mathbf{8_c}$ remain irreducible under $Spin(7)$. The singlet in $\mathbf{8_s}$ means the existence of a covariantly constant spinor ζ. Then the following fourth rank antisymmetric tensor is $Spin(7)$ invariant;

$$T^{\mu\nu\rho\sigma} \ = \ \zeta^T \gamma^{\mu\nu\rho\sigma} \zeta\ , \tag{3}$$

where $\gamma^{\mu\nu\rho\sigma}$ is the totally antisymmetric product of γ matrices for the $SO(8)$ spinor representation. Eq.(3) gives a component representation of the four form Ω. On the

Joyce manifold with the invariant fourth rank tensor $T^{\mu\nu\rho\sigma}$, we can define the following generalization of the instanton equation [5];

$$F^{\mu\nu} = \frac{1}{2} T^{\mu\nu\rho\sigma} F_{\rho\sigma} \ . \tag{4}$$

The curvature 2-form $F_{\mu\nu}$ in 8 dimensions has 28 components, whose $Spin(7)$ decomposition is $\mathbf{28} = \mathbf{21} \oplus \mathbf{7}$. This is made explicit by the eigenspace decomposition of the action of $1/2\ T^{\mu\nu\rho\sigma}$ with the eigenvalues $\lambda = 1$ and $\lambda = -3$. Eq. (4) means the curvature has no components in the latter subspace which is 7-dimensional. Computed explicitly, eq. (4) leads the following seven linear relations among the curvature;

$$\begin{aligned}
\Phi_1 &\equiv F_{12} + F_{34} + F_{56} + F_{78} = 0 \ , \\
\Phi_2 &\equiv F_{13} + F_{42} + F_{57} + F_{86} = 0 \ , \\
\Phi_3 &\equiv F_{14} + F_{23} + F_{76} + F_{85} = 0 \ , \\
\Phi_4 &\equiv F_{15} + F_{62} + F_{73} + F_{48} = 0 \ , \\
\Phi_5 &\equiv F_{16} + F_{25} + F_{38} + F_{47} = 0 \ , \\
\Phi_6 &\equiv F_{17} + F_{82} + F_{35} + F_{64} = 0 \ , \\
\Phi_7 &\equiv F_{18} + F_{27} + F_{63} + F_{54} = 0 \ .
\end{aligned} \tag{5}$$

Behind these combinations there is the algebra of octonions [12]. In fact eq.(5) can be written as;

$$F_{8i} = c_{ijk}\ F_{jk} \ , \quad (1 \leq i, j, k \leq 7) \tag{6}$$

where c_{ijk} are the structure constants for octonions. For this reason, we refer to eq.(5) as the octonionic instanton equation. It is worth while to compare it with the anti-self-dual (ASD) equation;

$$F_{4i} = \epsilon_{ijk}\ F_{jk} \ , \quad (1 \leq i, j, k \leq 3) \tag{7}$$

where we have ϵ_{ijk}; the structure constants for quaternions. On the Calabi-Yau fourfold we can introduce four complex coordinates and the complexified ASD equation makes sense. We employ it as topological gauge fixing condition that gives six conditions on the gauge field.

It is straightforward to check the identity;

$$4 \sum_{i=1}^{7} \mathrm{Tr}\ (\Phi_i \Phi_i) \cdot\ (vol) = -\,\Omega \wedge \mathrm{Tr}\ (F \wedge F) + \mathrm{Tr}\ (F \wedge *F) \ , \tag{8}$$

where (vol) stands for the volume form. The first term in the righthand side is a density of a topological invariant, since Ω is closed. The second term is nothing but the standard action density for the Yang-Mills theory. Hence a solution to $\Phi_i = 0$, $(i = 1, \dots, 7)$ gives a stationary point of the eight dimensional Yang-Mills theory. In this sense, eq. (5) deserves to be called instanton equation on the Joyce manifold. It is known that the octonionic instanton exists, if the gauge group has a $Spin(7)$ subgroup. The construction is a simple generalization of the one for BPST instanton in four dimensions, which exists if the gauge group contains an $SU(2) \simeq Spin(3)$ factor.

Once we find the octonionic instanton equation which serves as seven covariant gauge conditions, it is straightforward to perform BRST gauge fixing of the topological action (2). Precisely speaking, however, the model is not a topological quantum field theory, since it involves a choice of the four form Ω. It is topological only in the gauge sector and the gravitation is treated as a background. The topological BRST transformations and the final action are in parallel to those of four dimensional case. Hence, we omit their expressions, refering to [4]. The mathematics of the moduli space of the octonionic instanton equation is still an interesting open problem.

3 Link to Supersymmetry

The field contents of the theory on the Joyce manifold are in parallel to those of four dimensional topological Yang-Mills theory. We have a gauge field A_μ with the topological ghost ψ_μ and the secondary ghost ϕ. According to our choice of the seven gauge fixing conditions, we introduce the anti-ghost χ_i $(1 \leq i \leq 7)$. Finally there also exists a pair of fields (η, λ) for the gauge fixing of the topological ghost ψ_μ. Four dimensional topological Yang-Mills theory is a twisted version of $N = 2$ super Yang-Mills theory. Moreover, it is related by dimensional reduction to the minimal six dimensional super Yang-Mills theory. The above field contents naturally lead a similar connection in eight dimensions. In fact this is understood as follows; the gauge supermultiplet in eight dimensions consists of one gauge field in $\mathbf{8_v}$, one chiral spinor in $\mathbf{8_s}$, one anti-chiral spinor in $\mathbf{8_c}$ and two scalars [13]. The reduction of the holonomy group to $Spin(7)$ defines a well-defined decomposition of the chiral spinor; $\mathbf{8_s} = \mathbf{1} \oplus \mathbf{7}$. Now it is natural to identify A_μ and ψ_μ in our topological theory as $\mathbf{8_v}$ and $\mathbf{8_c}$, respectively. Furthermore χ_i and η just correspond to the chiral spinor $\mathbf{8_s}$ according to the above decomposition. Finally ϕ and λ give the remaining two scalars. This exhausts all the dynamical fields in our action of eight dimensional cohomological Yang-Mills theory. Thus, the theory on the Joyce manifold is identified as a twisted super Yang-Mills theory in eight dimensions. One can show that the supersymmetry transformation with a covariantly constant spinor as its parameter recovers the BRST transformation.

The supersymmetric Yang-Mills theory in eight dimensions is obtained by dimensional reduction from the ten dimensional $N=1$ super Yang-Mills theory. It has been argued that the effective world volume theory of the D-brane is the dimensional reduction of the ten dimensional super Yang-Mills theory [14]. Hence the cohomological Yang-Mills theory constructed in this paper may arise as an effective action of 7-brane theory. We expect that the world volume theory of D-branes would provide a variety of higher dimensional TQFT's [15].

4 Topological Dimensional Reduction

It is amusing that by dimensional reduction the octonionic instanton equation (5) gives other intriguing equations in lower dimensions. For example, a dimensional reduction to six dimensions decomposes eq.(5) into two sets; a single condition $\Phi_1 = 0$ and others $\Phi_k = 0$, $(2 \leq k \leq 7)$. These are equivalent to the Donaldson-Uhlenbeck-Yau equation for the moduli problem of stable holomorphic vector bundles. More interestingly an appropriate topological dimensional reduction to four dimensions recovers a non-abelian version of Seiberg-Witten (monopole) equation. In this case the octonionic instanton equation can be separated into 3 plus 4 equations. We group A_5, A_6, A_7, A_8 into a four component field φ^α, $\alpha = 1, 2, 3, 4$, which can be interpreted as a commuting complex Weyl spinor on a Cayley submanifold C_4 of the Joyce manifold [15]. That is, the normal bundle to C_4 is isomorphic to (twisted) spin bundle on C_4. Of course, $A_\mu = A_1, A_2, A_3, A_4$ are gauge fields on C_4. Then the first three of eq.(5) imply

$$F_{\mu\nu} + \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} + \varphi^T \Sigma_{\mu\nu} \varphi = 0 \ , \tag{9}$$

where $\Sigma_{\mu\nu} = \frac{1}{4}[\Gamma_\mu, \Gamma_\nu]$ is the anti-symmetric product of the gamma matrices in four dimensions. The remaining four equations are written as a Dirac-type equation

$$D_\mu^{(A)} \Gamma^\mu \varphi = 0 \ . \tag{10}$$

We recognize eqs.(9) and (10) as non-abelian Seiberg-Witten equation with a matter φ in the adjoint representation. Actually this is not unexpected, if we recall the link to SUSY discussed in the last section, where the octonionic instanton equation in 8 dimensions is connected the dimensional reduction of the $N = 1$ super Yang-Mills theory in 10 dimensions. Therefore, we predict that the theory we obtain by dimensional reduction to 4 dimensions is related to a twisted version of the $N=4$ super Yang-Mills theory [10].

Table 1: 8 dimensional Cohomological Yang-Mills Theories

	Joyce	Calabi-Yau
holomomy	$Spin(7)$	$SU(4) \sim Spin(6)$
closed 4 form	$T^{\mu\nu\rho\sigma} = \zeta^T \gamma^{\mu\nu\rho\sigma} \zeta$	holomorphic (4,0) form Ω
division algebra	**O**	**C**$\times$**H**
topological gauge fixing	octonionic instanton 7 conditions	complexified ASD 6 conditions
ordinary gauge fixing	real Lorentz condition	complex Lorentz condition
topological BRST	$N_T = 1$	$N_T = 2$

We thus conclude that the fields of the eight dimensional cohomological Yang-Mills theory, the fields which are involved in the four dimensional Seiberg-Witten equations, and the fields of the $D = 4, N = 4$ super Yang-Mills theory are all connected by twist and dimensional reduction. The ten dimensional $N = 1$ super Yang-Mills theory underlies all these theories.

5 Conclusion

We have shown that we can construct cohomological Yang-Mills theories in eight dimensions. The reduction of the holomony group to $Spin(7)$ or $SU(4)$ allows an invariant closed four from Ω, which we have used for both topological action and covariant gauge fixing condition. A comparison of two cases is made in the Table 1. We expect that a model on the eight dimensional hyperKähler manifold with $Sp(2) \simeq Spin(5)$ holonomy is also interesting.

Going to the higher dimensions gives other types of topological models involving higher rank gauge fields. In fact we have constructed several models up to 12 dimensions. For more details, see [4].

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