Gravitational Degrees of Freedom and the Initial-Value Problem*

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It is shown that for every spacelike three-geometry there exists a symmetric tensor that is (1) defined locally using only the three-metric and its derivatives, (2) conformally invariant, (3) traceless, and (4) covariantly divergence free ("transverse"). As a result, the arbitrarily specifiable (unconstrained) initial-value data in the Einstein initial-value problem for gravity can be completely characterized by a pair of symmetric, transverse, traceless tensors.

One knows in electromagnetism what initial information to specify freely, thereby to determine the future behavior of the field. To acquire the same power of analysis and understanding in the dynamics of gravitation is an important and actively pursued issue. The present work shows that it is physically natural and simple to specify, not the initial three-geometry itself, but a certain conformal tensor that determines the three-geometry up to a position-dependent scale factor.

Conformally invariant properties of space-time structure have proved to be of great importance in studies of gravitational radiation and in other fundamental problems. Conformal mappings of spacelike three-geometries have been employed as part of techniques for construction of initial-value data for gravity.[1,2] However, these conformal transformations $\gamma_{ab} \to \bar{\gamma}_{ab} = \varphi^4 \gamma_{ab}$ of the three-dimensional metric γ_{ab} have not seemed to possess great physical significance, presumably because of the absence in spatial geometry of any structure like the null cones of space-time. Perhaps for this reason conformally invariant characterization of three-geometry has not been widely studied in a physical context. Yet these properties turn out to be fundamental in connection with the Einstein initial-value problem for gravity.

It is well known that the Weyl conformal curvature tensor vanishes identically for three-dimensional spaces. This vanishing is equivalent to the fact that, in three dimensions, the Riemann curvature tensor $R^a{}_{bcd}$ and the Ricci tensor $R_{bd} \equiv R^a{}_{bad}$ are related by

$$R^d{}_{abc} = \delta^b_d R_{ac} - \delta^d_c R_{ab} + \gamma_{ac} R^d{}_b - \gamma_{ab} R^d{}_c + \tfrac{1}{2} R(\delta^d_c \gamma_{ab} - \delta^d_b \gamma_{ac}). \qquad (1)$$

However, there is a conformally invariant tensor which in three dimensions plays a role analogous to that of the Weyl tensor in higher dimensions. This tensor is defined by[3]

$$R_{abc} \equiv \nabla_c R_{ab} - \nabla_b R_{ac} + \tfrac{1}{4}(\gamma_{ac} \nabla_b R - \gamma_{ab} \nabla_c R), \qquad (2)$$

where ∇_a indicates covariant differentiation. It can be shown that a three-geometry is conformally flat if and only if $R_{abc} = 0$. The following identities reduce to five the number of independent components of R_{abc}:

$$R^a{}_{ac} \equiv \gamma^{af} R_{fac} = 0,$$
$$R_{abc} + R_{acb} = 0,$$
$$R_{abc} + R_{cab} + R_{bca} = 0. \qquad (3)$$

The significance of this tensor is more readily perceived if we write it in the algebraically equivalent form

$$\beta^{ab} \equiv \epsilon^{aef} \nabla_e (R_f{}^b - \tfrac{1}{4}\delta^b_f R) = -\tfrac{1}{2} \epsilon^{aef} \gamma^{bm} R_{mef}, \qquad (4)$$

where ϵ^{abc} is the completely antisymmetric tensor density of weight +1, with $\epsilon^{123} = +1$. However, β^{ab} is not conformally invariant owing to the raising of an index of R_{abc} in the definition (4). If we set $\gamma \equiv \det(\gamma_{ab})$, then clearly $\tilde{\beta}^{ab} \equiv \gamma^{1/3} \beta^{ab}$ is in conformally invariant form. Thus $\tilde{\beta}^{ab}$ defines a conformal equivalence class of three-metrics and vanishes if and only if the three-space is conformally flat, just as does the Weyl conformal curvature tensor of higher dimensional spaces.

Further properties of β^{ab} can now be detailed. It is symmetric in its indices because of the contracted Bianchi identity $\nabla_b G^b{}_a \equiv 0$, where $G^b{}_a \equiv R^b{}_a - \tfrac{1}{2}\delta^b_a R$ is the three-dimensional Einstein tensor. It is traceless, $\gamma_{ab}\beta^{ab} \equiv 0$, because of the final identity of equations (3). Finally, it is not difficult to show that β^{ab} is transverse, $\nabla_b \beta^{ab} \equiv 0$. Inasmuch as β^{ab} involves third derivatives of the metric, one might not suspect it to have the transverse property. However, this property follows as a consequence of the Ricci identity[3] and the equivalence in three-space of the Riemann and Ricci tensors (1). For every three-geometry, therefore, there is a symmetric transverse traceless tensor (TT tensor) β^{ab} which can easily be written in conformally invariant form $\tilde{\beta}^{ab}$. We can think of $\tilde{\beta}^{ab} \equiv \tilde{\beta}_{\mathrm{TT}}^{ab}$ as giving a "pure spin-two" representation of intrinsic geometry. *Conformally equivalent three-geometries have equivalent spin-two representations.*

Now we turn to consideration of the Einstein initial-value equations, which in standard canonical variables have the form[4,5]

$$\nabla_b \pi^{ab} = 0, \tag{5}$$

$$\gamma^{-1/2}(\pi_{ab}\pi^{ab} - \tfrac{1}{2}\pi^2) - \gamma^{1/2} R = 0. \tag{6}$$

The scalar curvature of γ_{ab} is R, and π^{ab} is a symmetric tensor density of weight unity with trace π. In the Hamiltonian form of Einstein's theory π^{ab} is the momentum density of the field, conjugate to γ_{ab}. Geometrically, π^{ab} describes the bending of the spacelike slice as it is embedded in space-time ("extrinsic curvature"). That these equations contain implicitly all the dynamics of gravity has been spelled out by a number of workers in recent years.[6] This is the primary motivation for a continuing search for deeper understanding of the problem of initial conditions.

The momentum constraints (5) are first-order linear partial differential equations for the π^{ab}, if the γ_{ab} are assumed given as is usually done and as we do here. The chief difficulty comes from the Hamiltonian constraint (6) which is quadratic in the π^{ab} and is coupled to the momentum constraints. It was probably for the former reason that Lichnerowicz and others[1,2] used conformal transformations on γ_{ab} to convert (6) into a partial differential equation and put the problem into a more convenient mathematical form.

Here, also, a conformal transformation $\bar{\gamma}_{ab} = \varphi^4 \gamma_{ab}$ is performed on a given metric γ_{ab}. But, at the same time, it is essential for what follows to map in addition the momentum according to the rule $\bar{\pi}^{ab} = \varphi^{-4}\pi^{ab}$. For, supposing we have obtained the transverse and traceless momenta π_{TT}^{ab} relative to a metric γ_{ab}, then $\bar{\pi}^{ab}$ will have the TT property relative to the metric $\bar{\gamma}_{ab}$ for arbitrary $\varphi(x)$. The "traceless" requirement on π^{ab} is added to (5) and (6) here because it is needed in order that $\bar{\pi}^{ab}$ be transverse with respect to $\bar{\gamma}_{ab}$, as is easily seen by writing out $\bar{\nabla}_b \bar{\pi}^{ab}$. In other words, the TT property is preserved by the above mapping $\tilde{\mathrm{C}}$. Furthermore, it is clear that the momentum density of weight $\frac{5}{3}$ defined by[7] $\tilde{\pi}^{ab} \equiv \gamma^{1/3}\pi^{ab}$ transforms with zero conformal weight under the mapping, that is, $\tilde{\mathrm{C}}: \tilde{\pi}^{ab} \to \tilde{\pi}^{ab}$.

Therefore, $\tilde{\pi}_{\mathrm{TT}}^{ab}$ is the same for an entire conformal equivalence class of metrics. Our strategy now is to pick the conformal factor so as to satisfy the Hamiltonian constraint (6), assuming that $\tilde{\pi}_{\mathrm{TT}}^{ab}$ is given.

Let us begin with (6) written in terms of barred variables and the $\frac{5}{3}$-weight momentum. Mapping this equation under $\tilde{\mathrm{C}}$ and using the well-known transformation law for scalar curvature,

$$\bar{R} = \varphi^{-4}R - 8\varphi^{-5}\nabla^2\varphi,$$

we find that φ must satisfy the Lichnerowicz[1] equation

$$\nabla^2\varphi + \tfrac{1}{8}M\varphi^{-7} - \tfrac{1}{8}R\varphi = 0, \tag{7}$$

where $\nabla^2 \equiv \gamma^{ab}\nabla_a\nabla_b$ is the Laplacian operator and

$$M \equiv \gamma^{-7/6}\gamma_{ac}\gamma_{bd}\tilde{\pi}_{\mathrm{TT}}^{ab}\tilde{\pi}_{\mathrm{TT}}^{cd}.$$

All the coefficients are known in this elliptic equation for φ. Therefore, we may regard the Hamiltonian constraint as determining the conformal factor of the metric, with $\tilde{\pi}_{\mathrm{TT}}^{ab}$ unconstrained. The transformation $\tilde{\mathrm{C}}$ effectively decouples the construction of TT momenta from the Hamiltonian constraint.

The requirement that π^{ab} be traceless is often regarded as a coordinate or slicing condition,[6] that is, the three-geometry is to be embedded into space-time "maximally." As such, no additional physical constraints are thereby imposed. Moreover, Deser[8] has shown that π_{TT}^{ab} can be covariantly constructed without the imposition of coordinate conditions. He obtains the TT part of any symmetric tensor[9] in a manner analogous to the way one obtains covariantly the transverse part of an arbitrary vector field.

If one does not have transverse traceless momenta, the constraints (5) and (6) remain coupled in regard to construction of momenta.[1] However, it is still true that *the conformally invariant three-geometry can be arbitrarily specified and is independent of all constraints. This independence receives physical significance through the fact that all conformally equivalent three-geometries give the same spin-two representation of the gravitational field.* This interpretation is exact and valid regardless of the connectedness of the three-space and its topological properties in the large. Moreover, when the traceless momentum condition is achieved, the complete set of unconstrained initial data of pure gravitational fields may be specified by two TT tensors, one purely intrinsic to the three-geometry and the other extrinsic. Each of these contains in general two arbitrary functions of the three spatial coordinates, just as required for a field with two independent states of polarization.

The present work leads one to write the Einstein initial-value equations in "Maxwell form," analogous to $\nabla\cdot\vec{B}=0$, $\nabla\cdot\vec{E}=0$:

$$\nabla_b\tilde{\beta}^{ab}=0, \quad \nabla_b\tilde{\pi}^{ab}=0,$$
$$\tilde{\beta}_a{}^a=0, \quad \tilde{\pi}_a{}^a=0. \tag{8}$$

Giving any metric γ_{ab} and constructing from it $\tilde{\beta}^{ab}$ according to the prescription

$$\tilde{\beta}^{ab}=\epsilon^{aef}\nabla_e[\gamma^{1/3}(R_f{}^b-\tfrac{1}{4}\delta_f^bR)] \tag{9}$$

guarantees that the left-hand equations in (8) will be satisfied, just as setting $\vec{B}=\nabla\times\vec{A}$, for arbitrary $\vec{A}$, insures that $\nabla\cdot\vec{B}=0$. Just as $\vec{B}$ depends only on the transverse part of $\vec{A}$, $\tilde{\beta}^{ab}$ depends only on the conformally invariant part of the metric. Both $\tilde{\beta}^{ab}$ and $\tilde{\pi}^{ab}$ are invariant with respect to the mapping $\tilde{C}$, just as both $\vec{B}$ and $\vec{E}$ are gauge invariant. Gauge invariance in electrodynamics and $\tilde{C}$ invariance in geometrodynamics are therefore formally similar in several respects as regards the initial value equations. The Hamiltonian constraint was not explicitly written down in (8), because in this view it only serves to determine the conformal factor φ, to which the fields $\tilde{\beta}^{ab}$ and $\tilde{\pi}^{ab}$ are insensitive. However, if one wishes to know the final (conformally transformed) metric, one has to take the solution φ of the Hamiltonian constraint explicitly into account. There is no analog of this latter process in electrodynamics, of course. In Maxwell's theory, the gauge can only be determined by supplementary conditions. It is not determined by any of the field equations.

This method of characterization of gravitational degrees of freedom based on the initial-value problem suggests a number of further investigations which depend in part on detailed understanding of Deser's covariant decomposition of symmetric tensors. The connection of $\tilde{\beta}^{ab}$ and $\tilde{\pi}_{TT}^{ab}$ on spacelike hypersurfaces to the results obtained by analysis of gravitational fields on null hypersurfaces should be spelled out. The dynamical equations of gravity should be written in a form suitable to the present viewpoint. These and related issues are being actively investigated and will be reported in detail elsewhere.

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