Problem sets - General Relativity

Solutions by Sergei Winitzki

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Contents

A good textbook corresponding to the level of this course: *General Relativity: An Introduction for Physicists* by M. P. Hobson, G. P. Efstathiou, and A. N. Lasenby (Cambridge University Press, 2006).

See also the lecture notes of S. Carroll: http://preposterousuniverse.com/grnotes/

Part I
Problems

1 Coordinates and 1-forms

1.1 Invertible transformations

Under what conditions is a coordinate transformation $\xi^\alpha = \xi^\alpha(x^\alpha)$ invertible in a neighborhood of some point x^α?

1.2 Examples of coordinate transformations

The following coordinate transformations are given, mapping the standard Euclidean coordinates (x, y) or (x, y, z) into new coordinates.

1. In a two-dimensional plane, $(x, y) \to (u, v)$, where $-\infty < u, v < +\infty$:

$$x = u + uv^2 + \frac{1}{3}u^3,$$
$$y = v + vu^2 + \frac{1}{3}v^3.$$

2. In a three-dimensional space, $(x, y, z) \to (r, \theta, \phi)$, where $-\infty < r < +\infty$, $0 \le \theta < +\infty$, $0 \le \phi < 2\pi$:

$$x = r \sinh\theta \cos\phi,$$
$$y = r \sinh\theta \sin\phi,$$
$$z = r \cosh\theta.$$

3. In a three-dimensional space, $(x, y, z) \to (r, \theta, \phi)$, where $0 \le r < +\infty$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$:Solutions

$$x = -r \sin\theta \cos\phi,$$
$$y = -r \sin\theta \sin\phi,$$
$$z = r \cos\theta.$$

The following questions must be answered in all three cases:
(a) Find the subdomain covered by the new coordinates. Hint: Consider e.g. the range of x at constant value of y.
(b) Find the points where the new coordinates do not specify a one-to-one invertible transformation (*singular points*).
(c) If singular points exist, give a geometric interpretation.

1.3 Basis in tangent space

Prove that the vectors $\mathbf{e}_\alpha = \frac{\partial}{\partial x^\alpha}$ are linearly independent.

1.4 Differentials of functions as 1-forms

If $f(x^\alpha)$ is a function of coordinates x^α, then one defines the 1-form df (called the *differential* of the function f) as

$$df \equiv \sum_\alpha \frac{\partial f}{\partial x^\alpha} dx^\alpha. \tag{1}$$

Compute $d(x)$, $d(x^2)$, $d(xy)$, $d(x+y)$. Compute the 1-forms df, dg, dh, where the functions f, g, h are defined as follows,

$$\begin{aligned} f(x, y, z) =& 4x^2 y + x^3 z, \\ g(x, y) =& 3\sqrt{x^2 + y^2}, \\ h(x, y) =& \arctan(x + y) + \arctan(x - y) \\ &+ \arctan \frac{2x}{x^2 - y^2 - 1}. \end{aligned}$$

1.5 Basis in cotangent space

Show that the 1-forms $dx^1, ..., dx^n$ comprise a basis in the space of 1-forms at any point M. Show that

$$< dx^\alpha, \frac{\partial}{\partial x^\beta} >= \delta^\alpha_\beta.$$

1.6 Linearly independent 1-forms

Check whether the following sets of 1-forms are linearly independent at each point of the 2-dimensional or the 3-dimensional space respectively. If not, determine the points where these sets are linearly dependent.

1. Two 1-forms $d(e^x \cos y)$, $d(e^x \sin y)$.
2. Two 1-forms $(1+y)dx - 2xydy$, $8dx$.
3. Three 1-forms $dx + dy$, $dx + dz$, $dy + dz$.
4. Three 1-forms $dx - dy$, $dy - dz$, $dz - dx$.

1.7 Transformation law for 1-forms

Derive the transformation law for 1-forms,

$$d\tilde{x}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta}\, dx^\beta, \tag{2}$$

under a coordinate transformation $x^\alpha \to \tilde{x}^\alpha = \tilde{x}^\alpha(x^\beta)$.

1.8 Examples of transformations

Consider the coordinate transformation $(x, y) \to (u, v)$ defined in Problem 1.2(1). Transform the following 1-form,

$$\omega = d\frac{1}{x+y},$$

into the coordinates (u, v) in two ways:
(a) By a direct substitution of the new coordinates.
(b) By using the transformation law (2).

1.9 Supplementary Problem Sheet 1

2D surfaces embedded in 3D Euclidean space

1A Tangent plane

Consider the surface given by $z = -h \exp\left(-\frac{1}{2\sigma^2}(x^2+y^2)\right)$. If gravity acts in the negative z-direction, at what points will a ball rolling along this surface experience the greatest acceleration? Find the tangent plane at one of these points.

1B Induced metric

Find the metric for the surface given parametrically by

$$\begin{aligned} x &= a \sin^2\theta \cos\phi, \\ y &= a \sin^2\theta \sin\phi, \\ z &= a \cos\theta \sin\theta, \end{aligned}$$

where, as usual, $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$. Is the metric well defined at $\theta = 0$? Do you think the surface is well defined there?

1C Embedding waves

1. Sketch the surface given by

$$\begin{aligned} x &= \frac{\cos v}{\sqrt{2} - \sin u}, \\ y &= \frac{\sin v}{\sqrt{2} - \sin u}, \\ z &= \frac{\cos u}{\sqrt{2} - \sin u}, \end{aligned}$$

   where $u, v \in [0, 2\pi)$. (Hint: Consider the intersection of the surface with the plane $y = 0$. What happens for general v?)

2. Find the normal vector and the tangent plane to this surface at point (u, v).

3. Determine the induced metric on the surface. Then consider the 2D vector $V^a = (\cos v, \sin v)$, i.e.

$$V = \cos v \frac{\partial}{\partial v} + \sin v \frac{\partial}{\partial u},$$

defined within the surface. Is V^a a unit vector? What are the 3D Euclidean components of the vector V in the 3D space? Show that the 3D components of the vector V everywhere lie in the tangent plane to the surface.

2 Tensors

2.1 Definition of tensor product

If ω_1 and ω_2 are 1-forms, their tensor product $\omega_1 \otimes \omega_2$ is defined as a function on pairs of vectors:

$$(\omega_1 \otimes \omega_2)(\mathbf{v}_1, \mathbf{v}_2) = \langle \omega_1, \mathbf{v}_1 \rangle \langle \omega_2, \mathbf{v}_2 \rangle . \tag{3}$$

Let $\omega_1 = dx + 2ydy$, $\omega_2 = -2dy$ be 1-forms on a 2-dimensional space and $\mathbf{v}_1 = 3\partial/\partial x$, $\mathbf{v}_2 = -x\,(\partial/\partial x + \partial/\partial y)$ be vector fields (also defined in this 2-dimensional space). Just for this problem, let us denote $T \equiv \omega_1 \otimes \omega_2$.

(a) Compute $T(\mathbf{v}_1, \mathbf{v}_1)$.

(b) Compute $T(\mathbf{v}_1, \mathbf{v}_2)$.

(c) Show that

$$T(\mathbf{a} + \lambda \mathbf{b}, \mathbf{u}) = T(\mathbf{a}, \mathbf{u}) + \lambda T(\mathbf{b}, \mathbf{u}), \tag{4}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{u}$ are vectors and λ is a number). The same property holds for the second argument of T. Such functions T are called **bilinear**.

(d) Show that all bilinear functions of pairs of 2-dimensional vectors belong to a vector space of such functions. Show that the tensor products $dx \otimes dx$, $dx \otimes dy$, $dy \otimes dx$, $dy \otimes dy$ form a basis in that space. (That space is called the space of tensors of rank $0 + 2$.)

2.2 General tensors

(a) A general tensor of rank $r+s$ is defined as a multilinear function on sets of r 1-forms f_j and s vectors $\mathbf{v}_j$ (**multilinear** means linear in every argument). An example of a tensor of rank $r + s$ is a tensor product of r vectors $\mathbf{e}_1$, ..., $\mathbf{e}_r$, and s 1-forms ω_1, ..., ω_s, denoted by $\mathbf{e}_1 \otimes ... \otimes \mathbf{e}_r \otimes \omega_1 \otimes ... \otimes \omega_s$. This tensor is a function that acts on a set of r 1-forms f_j and s vectors $\mathbf{v}_j$ via the formula

$$\begin{aligned} &\mathbf{e}_1 \otimes ... \otimes \mathbf{e}_r \otimes \omega_1 \otimes ... \otimes \omega_s(f_1, ..., f_r, \mathbf{v}_1, ..., \mathbf{v}_s) \\ &\quad = \langle f_1, \mathbf{e}_1 \rangle ... \langle f_r, \mathbf{e}_r \rangle \langle \omega_1, \mathbf{v}_1 \rangle ... \langle \omega_s, \mathbf{v}_s \rangle . \end{aligned}$$

(This is a generalization of Eq. (3) to tensors of rank $r + s$.) Show that this function is linear in every argument. Such functions are called $r + s$-linear functions. Show that all $r + s$-linear functions form a vector space. This vector space is called the space of tensors of rank $r + s$.

(b) Let $\mathbf{e}_j$, $j = 1, ..., N$, and ω^j, $j = 1, ..., N$ are bases in the space of vectors and in the space of 1-forms respectively (both spaces have dimension N). Show that the set of tensors

$$\mathbf{e}_{\alpha_1} \otimes ... \otimes \mathbf{e}_{\alpha_r} \otimes \omega^{\beta_1} \otimes ... \otimes \omega^{\beta_s} \tag{5}$$

form a basis in the space of $r + s$-tensors (where α_j and β_j exhaust all possible combinations of indices). Note that this set contains n^{r+s} basis tensors.

2.3 Example

(a) Let $\mathbf{T}$ be a bilinear function of two vectors with vector values, i.e. $\mathbf{T}(\mathbf{v}_1, \mathbf{v}_2)$ is a vector if $\mathbf{v}_1, \mathbf{v}_2$ are vectors. Give a simple example of such $\mathbf{T}$ as a tensor and determine its rank.

(b) A particular example of such a tensor $\mathbf{T}$ in 3-dimensional Euclidean space is the following function,

$$\mathbf{T}(\mathbf{v}_1, \mathbf{v}_2) = 2\mathbf{v}_1 \times \mathbf{v}_2 - \mathbf{v}_1(\mathbf{n} \cdot \mathbf{v}_2), \tag{6}$$

where $\mathbf{n}$ is a fixed vector. Show that the function $\mathbf{T}$ is bilinear in $\mathbf{v}_1, \mathbf{v}_2$. Determine the components $T^{\alpha}{}_{\beta\gamma}$ of the tensor $\mathbf{T}$ in an orthogonal basis where the vector $\mathbf{n}$ has the components $n^\alpha \equiv (n^1, n^2, n^3)$.

2.4 Transformation law

Derive the transformation law for the components $T_{\alpha_1 ... \alpha_s}{}^{\beta_1 ... \beta_r}$ of a tensor of rank $r + s$.

2.5 Contractions of tensor indices

(a) Show that the results of addition, multiplication by scalar, tensor multiplication, and index contraction of tensors are again tensors. Use the definition of tensor from Problem 2.2.

(b) Show that a contraction of indices in the same position (e.g. lower indices with lower indices, $T_{\alpha\alpha\beta}$) does not generally yield a tensor.

(c) Consider the tensor $T^{\alpha}{}_{\beta\gamma}$ defined in Problem 2.3(b) and compute the contraction $T^{\alpha}{}_{\alpha\beta}$. Is the result a tensor? If so, determine its rank.

2.6 Invariance of the interval

Show that the spacetime interval $ds^2 \equiv g_{\alpha\beta}dx^\alpha dx^\beta$ is invariant under coordinate transformations $x^\alpha \to \tilde{x}^\alpha$ if $g_{\alpha\beta}$ are components of a tensor transforming according to the tensor transformation law

$$g_{\alpha\beta} \to \tilde{g}_{\alpha\beta} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial x^\nu}{\partial \tilde{x}^\beta} g_{\mu\nu}. \tag{7}$$

2.7 Correspondence between vectors and 1-forms

For a given metric $g_{\alpha\beta}$, each vector $\mathbf{v}$ has a corresponding 1-form which we shall denote $\omega_{\mathbf{v}}$. This 1-form is defined by its action on an arbitrary vector $\mathbf{x}$ as follows,

$$\langle \omega_{\mathbf{v}}, \mathbf{x} \rangle = \mathbf{v} \cdot \mathbf{x}, \tag{8}$$

where the scalar product $\mathbf{v} \cdot \mathbf{x}$ is defined through the metric $g_{\alpha\beta}$. Show that the components of the 1-form $\omega_{\mathbf{v}}$ in the basis dx^α are related to the components of the vector $\mathbf{v}$ in the basis $\partial/\partial x^\alpha$ by

$$\omega_\alpha = g_{\alpha\beta} v^\beta. \tag{9}$$

2.8 Examples of spaces with a metric

(a) Consider the usual, Euclidean 3-dimensional space with the metric

$$g(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{v}_1 \cdot \mathbf{v}_2 - (\mathbf{n} \times \mathbf{v}_1) \cdot (\mathbf{n} \times \mathbf{v}_2), \tag{10}$$

where $\mathbf{v}_1 \cdot \mathbf{v}_2$ is the usual scalar product, $\mathbf{a} \times \mathbf{b}$ is the cross product, and $\mathbf{n}$ is a fixed vector with components n^α. Compute the components of the tensor $g_{\alpha\beta}$. For which vectors $\mathbf{n}$ is the metric g nondegenerate (i.e. $\det g_{\alpha\beta} \neq 0$)?

(b) Answer the same questions for the 2-dimensional Euclidean space with the metric

$$g(\mathbf{v}_1, \mathbf{v}_2) = 3\mathbf{v}_1 \cdot \mathbf{v}_2 + (\mathbf{n} \cdot \mathbf{v}_1)(\mathbf{n} \cdot \mathbf{v}_2). \tag{11}$$

Note that the cross product is undefined in the 2-dimensional space.

(c)* Answer the same questions for the metric (11) now defined in an r-dimensional Euclidean space, $r \geq 3$.

(d)* Consider a 2-dimensional surface embedded in the 3-dimensional Euclidean space,

$$x = R \cosh u \cos v, \tag{12}$$
$$y = R \cosh u \sin v, \tag{13}$$
$$z = R \sinh u. \tag{14}$$

Determine the 2-dimensional metric $g_{\alpha\beta}$ in the basis du, dv.

2.9 Supplementary Problem Sheet 2

Calculations with tensor indices

2A Vector equations

In the following equations, the vector x^α is unknown and all other quantities are known. The symbol $\varepsilon_{\alpha\beta\gamma}$ denotes the completely antisymmetric tensor. Determine the unknown vector x^α from the given data. In every case, assume the "generic" choice of data. This means that every given scalar, vector and tensor is nonzero (k, A_α, B^β, ...), there are no accidental cancellations or linear dependence between given vectors, matrices are nondegenerate, etc.

(a) $kx^\alpha + \varepsilon^{\alpha\beta\gamma} x_\beta A_\gamma = B^\alpha$ (3-dimensional vectors). The assumption of the "generic" case is $k \neq 0$ and A_α and B_α linearly independent.

(b) $\varepsilon_{\alpha\beta\gamma} x^\beta A^\gamma = B_\alpha$, $x^\alpha C_\alpha = k$ (3-dimensional vectors).

(c) $x^\alpha A_\alpha = k$, $x^\beta B_\beta = l$ (2-dimensional vectors).

(d) $x^\alpha A_{\alpha\beta} = B_\beta$ (3-dimensional vectors and a given tensor $A_{\alpha\beta}$).

2B Tensor equations

In the following equations, the tensor $X^{\alpha\beta}$ is unknown and all other quantities are known. The dimensionality of the (Euclidean) space is indicated. Determine $X^{\alpha\beta}$ under the assumption that all given quantities are generic.

(a) $X^{\alpha\beta} = X^{\beta\alpha}$, $X^{\alpha\beta}A_\alpha = B^\beta$, $X^\alpha_\alpha = 0$, where $A^\alpha B_\alpha = 0$ (2-dimensional).

(b) $X^{\alpha\beta} = -X^{\beta\alpha}$, $X^{\alpha\beta}A_\alpha = B^\beta$, $X^{\alpha\beta}B_\alpha = 0$, where $A^\alpha B_\alpha = 0$ (3-dimensional vectors).

2C Degeneracy of the metric

(a) A two-dimensional space with coordinates (x, y) has the metric given as a bilinear form

$$g = y^2 dx \otimes dx + (x^2 + 1)(dx \otimes dy + dy \otimes dx). \tag{15}$$

Is the metric nondegenerate at all points (x, y)?

(b) The same question for the n-dimensional metric of the form

$$g_{\alpha\beta} = \delta_{\alpha\beta} - \left(1 + r^2\right) A_\alpha A_\beta, \tag{16}$$

where A_α is a given vector and $r^2 \equiv \delta_{\alpha\beta}x^\alpha x^\beta$ is the squared Euclidean distance.

3 The Christoffel symbol $\Gamma^\mu_{\alpha\beta}$

3.1 Transformations 1

In flat space with standard Euclidean coordinates ξ^μ and arbitrary coordinates $x^\mu = x^\mu(\xi^\alpha)$, the Christoffel symbol can be found as

$$\Gamma^\mu_{\alpha\beta} = \frac{\partial^2 \xi^\nu}{\partial x^\alpha \partial x^\beta} \frac{\partial x^\mu}{\partial \xi^\nu}.$$

Derive the transformation law for $\Gamma^\mu_{\alpha\beta}$ between arbitrary coordinate systems x^μ and $\tilde{x}^\mu$:

$$\tilde{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\mu_{\rho\sigma} \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial x^\rho}{\partial \tilde{x}^\beta} \frac{\partial x^\sigma}{\partial \tilde{x}^\gamma} + \frac{\partial^2 x^\sigma}{\partial \tilde{x}^\beta \partial \tilde{x}^\gamma} \frac{\partial \tilde{x}^\alpha}{\partial x^\sigma}. \tag{17}$$

3.2 Transformations 2

Show that the Christoffel symbol must transform according to Eq. (17) not only in flat space but also in arbitrary space. *Hint:* consider the covariant derivative of a vector field,

$$A_{\alpha;\beta} = \frac{\partial A_\alpha}{\partial x^\beta} - \Gamma^\mu_{\alpha\beta} A_\mu,$$

and demand that the components $A_{\alpha;\beta}$ transform as a tensor.

3.3 Covariant derivatives

Derive the explicit form of the covariant derivative

$$T^{\alpha\beta}{}_{\gamma\delta\mu;\nu}$$

for a tensor field $T^{\alpha\beta}{}_{\gamma\delta\mu}$.

3.4 The Leibnitz rule

Prove the Leibnitz rule in the following specific case,

$$\left(A_\alpha B^\beta\right)_{;\gamma} = A_{\alpha;\gamma} B^\beta + A_\alpha B^\beta_{;\gamma}.$$

3.5 Locally inertial reference frame

Suppose that the Christoffel symbol at a point $x^\alpha_{(0)}$ in some coordinate system x^α has the value $\Gamma^\alpha_{(0)\mu\nu}$ and is symmetric, $\Gamma^\alpha_{(0)\mu\nu} = \Gamma^\alpha_{(0)\nu\mu}$. Then a locally inertial system at point x_0 can be constructed by defining the new coordinates

$$\xi^\alpha(x) = x^\alpha - x^\alpha_{(0)} + \frac{1}{2}\left(x^\mu - x^\mu_{(0)}\right)\left(x^\nu - x^\nu_{(0)}\right)\Gamma^\alpha_{(0)\mu\nu}.$$

The point x_0 in the new coordinates is the origin $\xi^\alpha = 0$. Prove explicitly that the Christoffel symbol, when transformed to the new coordinates, is equal to zero at the point $\xi^\alpha = 0$.

4 Geodesics and curvature

4.1 Geodesics

(a) Show that the geodesic equation can be written in the following form,

$$\frac{du_\alpha}{ds} - \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} u^\beta u^\gamma = 0. \tag{18}$$

(b) Show that $g_{\alpha\beta}u^\alpha u^\beta$ is constant along a geodesic.

4.2 Commutator of covariant derivatives

Show that

$$u^\alpha{}_{;\beta;\gamma} - u^\alpha{}_{;\gamma;\beta} = R^\alpha{}_{\delta\gamma\beta}u^\delta, \tag{19}$$

where the Riemann tensor is defined by

$$R^\alpha{}_{\delta\gamma\beta} = \frac{\partial\Gamma^\alpha{}_{\delta\beta}}{\partial x^\gamma} - \frac{\partial\Gamma^\alpha{}_{\delta\gamma}}{\partial x^\beta} + \Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\delta\beta} - \Gamma^\alpha{}_{\sigma\beta}\Gamma^\sigma{}_{\delta\gamma}. \tag{20}$$

4.3 Parallel transport

Consider a vector A_α parallel-transported along a small closed curve $x^\mu(s)$. Show that the change in A_α after the parallel transport can be approximately expressed as

$$\delta A_\alpha \equiv \oint \Gamma^\beta_{\alpha\gamma}(x)A_\beta dx^\gamma \approx \frac{1}{2}R^\delta{}_{\alpha\beta\gamma}A_\delta \oint x^\beta dx^\gamma, \tag{21}$$

where it is assumed that the area within the closed curve is very small.

Hint: Use a locally inertial coordinate system where $\Gamma^\alpha_{\beta\gamma} = 0$ at one point. Also, show that

$$\oint x^\alpha dx^\beta = -\oint x^\beta dx^\alpha. \tag{22}$$

4.4 Riemann tensor

(a) Using the symmetry properties of the Riemann tensor $R_{\alpha\beta\gamma\delta}$, compute the number of independent components of $R_{\alpha\beta\gamma\delta}$ in an n-dimensional space ($n \geq 2$).

(b) Prove the Bianchi identity: $R^\alpha{}_{\beta\gamma\delta;\sigma} + R^\alpha{}_{\beta\sigma\gamma;\delta} + R^\alpha{}_{\beta\delta\sigma;\gamma} = 0$.

(c) Compute the Einstein tensor $G^\alpha{}_\beta$ in an arbitrary two-dimensional space. *Hint:* First determine the independent components of $R_{\alpha\beta\gamma\delta}$.

4.5 Lorentz transformations

Determine the number of independent parameters in Lorentz transformations $\tilde{x}^\mu = \Lambda^\mu_\alpha x^\alpha$, given by matrices Λ^β_α, and interpret these parameters. Hint: It is easier to consider infinitesimal Lorentz transformations $\Lambda^\alpha_\beta = \delta^\alpha_\beta + \varepsilon H^\alpha_\beta$, where $\varepsilon \ll 1$ and so ε^2 can be disregarded.

5 Gravitation theory applied

5.1 Redshift

Calculate the gravitational redshift at the surface of the Earth for the vertical distance of 1m between the sender and the receiver. Same question for 1 km.

5.2 Energy-momentum tensor 1

Rewrite the conservation law $T^{\alpha\beta}{}_{;\beta} = 0$ explicitly in the nonrelativistic limit for an ideal fluid, and show that these equations coincide with the continuity equation and the Euler equation.

5.3 Energy-momentum tensor 2

The EMT for a massless scalar field is

$$T^\alpha{}_\beta = \Phi^{;\alpha}\Phi_{;\beta} - \frac{1}{2}\delta^\alpha{}_\beta\Phi^{;\gamma}\Phi_{;\gamma}\,.$$

Show (using the conservation law) that the equation of motion the field is $\Phi^{;\alpha}{}_{;\alpha} = 0$.

5.4 Weak gravity

Show that in the limit of weak static gravitational field ($g_{00} = 1 + 2\Phi(x, y, z)$, and $g_{\mu\nu}$ is independent of t) the following relation holds,

$$R_{00} \approx \Delta\Phi + O(\Phi^2)\,,$$

where Δ is the ordinary Laplace operator, $\Delta \equiv \partial_{xx} + \partial_{yy} + \partial_{zz}$.

5.5 Equations of motion from conservation law

The EMT for a point particle of mass m_0 moving along a worldline $x^\sigma(s)$ can be expressed as

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} m_0 \int ds \frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\delta^{(4)}(x^\sigma - x^\sigma(s))\,.$$

Show that the conservation law $T^{\mu\nu}{}_{;\nu} = 0$ implies the geodesic equation for $x^\sigma(s)$.

Hint: First derive the relations

$$\Gamma^\nu{}_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}(\sqrt{-g})\,,$$
$$T^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}(\sqrt{-g}T^{\mu\nu}) + \Gamma^\mu{}_{\rho\sigma}T^{\rho\sigma}\,.$$

6 The gravitational field

6.1 Degrees of freedom

Using the scheme developed in the lecture, compute the number of degrees of freedom in the electromagnetic field, taking into account the presence of charges and currents.

6.2 Spherically symmetric spacetime

Compute the Ricci tensor R^α_β and the curvature scalar R for a spherically symmetric gravitational field. Assume that the metric has the form

$$g_{\mu\nu} = \begin{pmatrix} e^{\nu(t,r)} & 0 & 0 & 0 \\ 0 & -e^{\lambda(t,r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}.$$

Write the corresponding Einstein equations in vacuum ($T_{\mu\nu} = 0$).

Comment: This computation is *extremely long* when performed by the methods explained in this course (Christoffel symbols, energy-momentum tensor). There exist faster methods for computing curvature, for example methods based on the tetrad formalism, but this is beyond the scope of this introductory course on GR. In this course, it would be more appropriate to ask for an easier computation. For example, to compute the curvature in two dimensions of the metric $g_{\mu\nu}$ =diag(1,$\cos^2\theta$), or another diagonal metric in a two-dimensional spacetime.

6.3 Motion in Schwarzschild spacetime

Derive the equation for the covariant component u_1 of the 4-velocity of a particle in Schwarzschild spacetime ($u_1(\lambda) \equiv -f^{-1}(r)\dot{r}$, $f(r) = 1 - r_g/r$). Verify that this equation follows from Eqs. (23)–(26) given in the lecture:

$$f\dot{t}^2 - f^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2 = \mathcal{K} \quad (= u_\alpha u^\alpha), \tag{23}$$

$$\frac{d}{d\lambda}\left(f\dot{t}\right) = 0, \quad (u_0) \tag{24}$$

$$\frac{d}{d\lambda}\left(-r^2\dot{\theta}\right) + r^2\sin\theta\cos\theta\dot{\phi}^2 = 0, \quad (u_2) \tag{25}$$

$$\frac{d}{d\lambda}\left(r^2\sin^2\theta\dot{\phi}\right) = 0, \quad (u_3) \tag{26}$$

where the overdot ($\dot{}$) denotes $d/d\lambda$ and the spherical coordinates are $\{x^0, x^1, x^2, x^3\} \equiv \{t, r, \theta, \phi\}$.

6.4 Equations of motion

Verify that Eq. (30) follows from Eqs. (27)–(29) given in the lecture:

$$-e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)+\frac{1}{r^2}=0, \tag{27}$$

$$-e^{-\lambda}\frac{\dot{\lambda}}{r}=0, \tag{28}$$

$$-e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2}=0, \tag{29}$$

$$\begin{aligned}-\frac{1}{2}e^{-\lambda}&\left(\nu''+\frac{\nu'^2}{2}+\frac{\nu'-\lambda'}{r}-\frac{\nu'\lambda'}{2}\right)\\&+\frac{1}{2}e^{-\nu}\left(\ddot{\lambda}+\frac{\dot{\lambda}^2}{2}-\frac{\dot{\lambda}\dot{\nu}}{2}\right)=0.\end{aligned} \tag{30}$$

Here the prime ($'$) denotes $\partial/\partial r$ and the overdot ($\dot{}$) denotes $\partial/\partial t$.

7 Weak gravitational fields

7.1 Gravitational bending of light

Verify that the gravitational bending of light passing near the Sun is

$$\delta = 1.75'' \frac{R_\odot}{R}$$

where R is the distance at which the light passes from the center of the Sun and $R_\odot$ is the radius of the Sun.

7.2 Einstein tensor for weak field

Derive the following expression for the Einstein tensor due to a weak gravitational field,

$$G^\mu{}_\nu = \frac{1}{2}\left(-\bar{h}^\mu{}_\nu{}^{,\alpha}{}_{,\alpha} - \delta^\mu{}_\nu \bar{h}^\alpha{}_\beta{}^{,\beta}{}_{,\alpha} + \bar{h}^\mu{}_\alpha{}^{,\alpha}{}_{,\nu} + \bar{h}^\alpha{}_\nu{}^{,\mu}{}_{,\alpha}\right) + O(h^2), \tag{31}$$

where $\bar{h}^\mu{}_\nu = h^\mu{}_\nu - \frac{1}{2}\,\delta^\mu{}_\nu\, h$.

7.3 Gravitational perturbations I

Derive the expressions (shown in the lecture) for the Einstein tensor $G^\mu{}_\nu$ in terms of the scalar, vector, and tensor perturbations of the gravitational field. The background is the flat Minkowski spacetime, ${}^{(0)}g_{\alpha\beta} = \eta_{\alpha\beta}$, and the metric is

$$g_{00} = 1 + 2\Phi,\ g_{0i} = B_{,i} + S_i, \quad g_{ij} = -\delta_{ij} + 2\Psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij}. \tag{32}$$

7.4 Gravitational perturbations II

Derive the transformation laws for the scalar, vector, and tensor perturbations of the gravitational field, under an infinitesimal change of the coordinates,

$$\tilde{x}^\mu = x^\mu + \xi^\mu(x). \tag{33}$$

Note: It is convenient to decompose ξ^μ as $\xi^\mu = \left(\xi^0, \xi^i_\perp + \zeta^{,i}\right)$, where ξ^0 and ζ are scalar functions and $\xi^i_{\perp,i} = 0$.

8 Gravitational radiation I

8.1 Gauge invariant variables

Verify that the following combinations of metric perturbations, $D = \Phi - \Psi - \dot{B} + \ddot{E}$ and $S_i - \dot{F}_i$, are gauge-invariant.

8.2 Detecting gravitational waves

Light noninteracting particles are situated in the $x-y$ plane in free space. A plane gravitational wave propagating in the z direction passes through the ring. The metric is of the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu n}$ contains only the pure tensor component,

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \exp\left[-i\omega\,(t-z)\right]. \tag{34}$$

Describe the deformation of the shape of the ring due to the gravitational wave. Consider cases $A_+ \neq 0, A_\times = 0$ and $A_+ = 0, A_\times \neq 0$.

8.3 Poisson equation

Derive the solution of the following differential equation,

$$\Delta\phi(\mathbf{x}) = 4\pi\rho(\mathbf{x}), \tag{35}$$

with boundary conditions $\phi \to 0$ at $|\mathbf{x}| \to \infty$.

8.4 Metric perturbations 1

Determine an explicit expression for α through $T^0{}_{i,i}$, where $\partial_i\alpha$ represents the scalar part of $T^0{}_i$.

8.5 Metric perturbations 2

Verify that the equation

$$-\frac{1}{2}(\dot{S}_i - \ddot{F}_i) = 8\pi G\sigma_i \tag{36}$$

which follows from vector part of the spatial Einstein equation, also follows from other components of the Einstein equation and from the conservation law (as derived in the lecture).

9 Gravitational radiation II

9.1 Projection of the matter tensor

The projection operator P_{ij} is defined by

$$P_{ij} = \delta_{ij} - n_i n_j, \quad n_i n^i = 1, \quad n_i \equiv \frac{R_i}{R}. \tag{37}$$

Show that the projected tensor ${}^{(T)}X_{ik}(t, |\vec{R}|)$ defined by

$${}^{(T)}X_{ik} = P_{ia}X_{ab}P_{bk} - \frac{1}{2}P_{ik}P_{ab}X_{ab}, \quad X_{ik}(t, \mathbf{R}) \equiv \int d^3\mathbf{r}\, r_i r_k T_{00}(t - |\mathbf{R}|, \mathbf{r}), \tag{38}$$

has the following properties,

$$\text{a) } {}^{(T)}X_{ii} = 0; \tag{39}$$

$$\text{b) } {}^{(T)}X_{ik,i} = O(X\,|\mathbf{R}|^{-1}), \tag{40}$$

that is, ${}^{(T)}X_{ik}$ is transverse-traceless up to terms of order $|\mathbf{R}|^{-1}$.

9.2 Matter sources

Verify thafullyt ${}^{(T)}X_{ik} = {}^{(T)}Q_{ik}$, where ${}^{(T)}X_{ik}(t - |\vec{R}|)$ is the projected tensor defined in Problem 9.1 and

$${}^{(T)}Q_{ik} = P_{ia}Q_{ab}P_{bk} - \frac{1}{2}P_{ik}P_{ab}Q_{ab}, \tag{41}$$

$$Q_{ik} \equiv \int (r_i r_k - \frac{1}{3}\delta_{ik}\, r^2)\, T^0{}_0\, d^3r. \tag{42}$$

9.3 Energy-momentum tensor of gravitational waves

Compute the second-order terms $G^{(2)\alpha}{}_\beta$, i.e. terms quadratic in $h_{\mu\nu}$, of the Einstein tensor G^α_β for small perturbations in flat space, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where only the transverse and traceless part ${}^{(T)}h_{ik}$ is nonzero. Verify that the energy-momentum tensor of gravitational waves in vacuum ($T_{\mu\nu} = 0$ for matter) is

$${}^{(\mathrm{GW})}T^\alpha{}_\beta \equiv -\frac{1}{8\pi G}\left\langle G^{(2)\alpha}{}_\beta \right\rangle = \frac{1}{32\pi G}\left\langle {}^{(T)}h^i{}_k{}^{,\alpha}\, {}^{(T)}h_i{}^k{}_{,\beta} \right\rangle. \tag{43}$$

9.4 Power of emitted radiation

Show that the rate of energy loss (energy lost per unit time) is

$$\frac{dE}{dt} = -\frac{G}{8\pi}\int d^2\Omega^{(T)}\dddot{Q}_{ik}{}^{(T)}\dddot{Q}_{ik} = -\frac{G}{5}\dddot{Q}_{ik}\dddot{Q}_{ik}. \tag{44}$$

Here the integration goes over all directions n^i in 2-sphere. In the calculation, derive and use the following relations,

$$\int n^l n^m \frac{d^2\Omega}{4\pi} = \frac{1}{3}\delta^{lm}, \tag{45}$$

$$\int n^l n^m n^k n^r \frac{d^2\Omega}{4\pi} = \frac{1}{15}(\delta^{lm}\delta^{kr} + \delta^{lk}\delta^{mr} + \delta^{lk}\delta^{mr}) \ . \tag{46}$$

10 Sample exam problems

These problems were at some time given at the exams. If some of these problems are again given at an exam, it means that the professor is not doing his job properly. Professors are paid for teaching, so they must be able to invent new exam problems each time.

10.1 Metric and curvature

1. A two-dimensional torus with coordinates (θ, ϕ) is described as the surface

$$\begin{aligned} x &= (b + a\cos\phi)\cos\theta, \\ y &= (b + a\cos\phi)\sin\theta, \\ z &= a\sin\phi, \end{aligned}$$

   embedded in the three-dimensional Euclidean space with the metric $ds^2 = dx^2+dy^2+dz^2$. Compute the induced metric on the torus,

$$ds^2 = (...)\, d\theta^2 + (...)\, d\theta d\phi + (...)\, d\phi^2.$$

2. In a two-dimensional space, the basis vectors (in polar coordinates) are $e_r = \frac{\partial}{\partial r}$, $e_\phi = \frac{1}{r}\frac{\partial}{\partial\phi}$. Consider the dual basis of 1-forms ω^r, ω^ϕ:

$$\omega^i(e_k) = \delta^i_k, \quad \text{where } \ i, k = \phi, r.$$

   Find a function $f_r(\phi, r)$ such that the 1-form ω^r is the differential of f_r, that is, $\omega^r = df_r$. Show that the 1-form ω^ϕ is not a differential.

3. A metric in a two-dimensional spacetime with coordinates (u, v) is

$$ds^2 = du^2 - u^2 dv^2.$$

   - Transform the line element ds^2 from (u, v) to new coordinates (x, t) defined by

$$x = u\cosh v, \quad t = u\sinh v.$$

   - Determine the curvature tensor $R_{\alpha\beta\mu\nu}$ for this spacetime.

10.2 Geodesics

1. Consider a two-dimensional spacetime with coordinates (t, x) and the metric

$$ds^2 = dt^2 - e^{2Ht}dx^2, \tag{47}$$

   where H is a known constant. Determine the Christoffel symbols and the equation for a geodesic $t(s), x(s)$. Solve this equation for the case of a light-like geodesic with initial conditions $t(0) = t_0$, $x(0) = x_0$ and obtain $x(s), y(s)$ explicitly. *Hint:* Use the property of light-like geodesics,

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0. \tag{48}$$

2. Suppose that the metric in a certain coordinate system $\{x^\mu\}$ has the form

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & g_{13} \\ 0 & g_{21} & g_{22} & g_{23} \\ 0 & g_{31} & g_{32} & g_{33} \end{pmatrix}, \tag{49}$$

where the spatial components $g_{ij}(x^\mu)$, $i, j = 1, 2, 3$ are arbitrary functions of space and time. Consider the worldline $x^\mu(s)$ describing a particle with constant values of the spatial coordinates,

$$x^0(s) = s, \quad x^{1,2,3}(s) = \text{const.} \tag{50}$$

Is the worldline $x^\mu(s)$ a geodesic?

10.3 Motion in central field

The motion of a particle in spacetime is given by the geodesic equation. For the Schwarzschild metric, the radial equation of motion is

$$\dot{r}^2 + V(r) = C^2,$$

where $\dot{} = d/d\tau$ and $V(r)$ is the effective potential given by

$$V(r) = \left(1 - \frac{2m}{r}\right)\left(1 + \frac{h^2}{r^2}\right).$$

The details of the motion are governed by the constants C and h; they are a measure of the particle's total energy and angular momentum respectively.
(a) For what values of h^2 are there circular orbits? Given that m and h^2 are positive, show that the radii of these orbits are always larger than $3m$.
(b) A circular orbit will be stable if $V''(r) > 0$. Show that when there are two circular orbits, the one with the larger radius is stable. It follows that the other orbit is unstable.
(c) The radius of the unstable orbit gives the position of the potential barrier. In the limit of $h \gg m$, show that the height of the barrier is approximately $\frac{h^2}{27m^2}$. Sketch the potential.
(d) A particle coming in from infinity must have $C^2 \geq 1$. What happens to this particle if C^2 is also larger than the barrier height? How is this result different from the case in Newtonian gravity?

10.4 Gravitational radiation

A light planet of mass m is revolving around a heavy star of mass M on a circular orbit with radius R. Assume that the motion of the planet is non-relativistic, the star is approximately motionless, and both the star and the planet can be treated as point masses.

1. Calculate the period T of the motion of the planet (in the Newtonian approximation). Determine the power L_{GW} of gravitational radiation emitted by the planet using the known formulae

$$L_{GW} = \frac{G}{5c^5}\left\langle \sum_{ij} \dddot{Q}_{ij}(t)\dddot{Q}_{ij}(t) \right\rangle, \quad \text{where } \langle\rangle \text{ means time average,}$$

$$Q_{ij}(t) = \int d^3x \left(x_i x_j - \frac{1}{3}x^2\delta_{ij}\right)\rho(\mathbf{x}, t).$$

Hint: Assume that the star is at the origin, write the trajectory of the planet as a function $\mathbf{x}_{\rm pl}(t)$ and express the corresponding $\rho(\mathbf{x}, t)$ using δ-functions,

$$\rho(\mathbf{x}, t) = M\delta(\mathbf{x}) + m\delta(\mathbf{x} - \mathbf{x}_{\rm pl}(t)).$$

2. Estimate the timescale ΔT for a significant change in the kinetic energy of the planet due to gravitational radiation. Express the dimensionless result, $\Delta T/T$, in terms of the ratios M/m and R/R_s, where $R_s = \frac{GM}{c^2}$ is the Schwarzschild radius of the star. Estimate the value $\Delta T/T$ for the orbit of the Earth around the Sun ($M/m \sim 343000$, $R_s \sim 3$km, $R \sim 1.5 \cdot 10^{11}$m, $T = 1$ year).

Part II
Solutions

1 Coordinates and 1-forms

1.1 Invertible transformations

The inverse function theorem guarantees that the equations $\xi^\alpha = \xi^\alpha(x)$ are solvable near a point x_0 if $\det\left(\partial\xi^\alpha(x)/\partial x^\beta\right) \neq 0$ at x_0. Under this condition, the coordinate transformation is invertible at x_0. Note: we are inverting not just one function $\xi = \xi(x)$, but we are determining x from a system of n equations, say $\xi^\alpha(x) = C^\alpha$, where C^α are n given values.

1.2 Examples of coordinate transformations

1. a) Since $x = u\left(1+v^2\right) + u^3/3$, it is clear that x has range $(-\infty, +\infty)$ for any fixed value of v as u varies in the range $(-\infty, +\infty)$. Similarly, y has the range $(-\infty, +\infty)$. To verify that the coordinate system (x, y) covers the entire plane, it is sufficient to show that x has the full range at every fixed value of y. It is sufficient to consider $y_0 > 0$ (else change $v \to -v$). At fixed $y = y_0 > 0$, we have $y_0 = v + vu^2 + v^3/3$ and thus the admissible values of u are from $-\infty$ to $+\infty$, while the admissible values of v are from 0 to $v = v_{\max}$ such that $y_0 = v_{\max} + \frac{1}{3}v_{\max}^3$. Then

$$
\begin{aligned}
u &= \pm\sqrt{\frac{y_0 - v - \frac{1}{3}v^3}{v}} \quad \left(\text{we have } \frac{y_0 - v - \frac{1}{3}v^3}{v} \geq 0 \text{ for } 0 < v < v_{\max}\right), \\
x &= \pm\left(1 + v^2 + \frac{y_0 - v - \frac{1}{3}v^3}{3v}\right)\sqrt{\frac{y_0 - v - \frac{1}{3}v^3}{v}} \\
&= \pm\left(\frac{2}{3} + \frac{8}{9}v^2 + \frac{y_0}{3v}\right)\sqrt{\frac{y_0 - v - \frac{1}{3}v^3}{v}}.
\end{aligned}
$$

We have now expressed x as a function of v, i.e. $x = x(v)$. When v varies from 0 to $v_{\max}$, $x(v)$ varies from $\pm\infty$ to 0. Since $x(v)$ is nonsingular for $v > 0$, it follows that x has the full range. Therefore, the coordinates (x, y) cover the entire two-dimensional plane.

b) The coordinate transformation is nonsingular if

$$\det\frac{\partial(x, y)}{\partial(u, v)} \neq 0.$$

Compute:

$$
\begin{aligned}
\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} &= \det\begin{pmatrix} 1 + u^2 + v^2 & 2uv \\ 2uv & 1 + u^2 + v^2 \end{pmatrix} \\
&= 1 + 2\left(u^2 + v^2\right) + \left(u^2 - v^2\right)^2 > 0.
\end{aligned} \tag{51}
$$

Therefore there are no singular points.

2. a) To determine the range, first consider $\phi = 0$. Then $x = r\sinh\theta$, $y = 0$, $z = r\cosh\theta$. It is clear that $z^2 - x^2 = r^2$. Since $r \geq 0$, the coordinates (x, y, z) cover only the domain $|z| > |x|$. With arbitrary ϕ, it is clear that the coordinates (x, y, z) cover the domain $|z| > \sqrt{x^2 + y^2}$.

b) Compute the determinant:

$$\det\begin{pmatrix} \sinh\theta\cos\phi & r\cosh\theta\cos\phi & -r\sinh\theta\sin\phi \\ \sinh\theta\sin\phi & r\cosh\theta\sin\phi & r\sinh\theta\cos\phi \\ \cosh\theta & r\sinh\theta & 0 \end{pmatrix} = r^2\sinh\theta.$$

The coordinates are singular if $r = 0$ or $\theta = 0$.

c) The singularity at $r = 0$ is due to the fact that the set $\{r = 0, \theta, \phi\}$ corresponds to a single point $x = y = z = 0$. This is similar to the singularity of the spherical coordinates at $r = 0$. Points along the cone $|z| = \sqrt{x^2 + y^2}$ are not covered because they correspond to $\theta \to \infty$, $r \to 0$. The singularity at $\theta = 0$, $r \neq 0$ is due to the fact that the set $\{r, \theta = 0, \phi\}$ corresponds to the point $x = y = 0$, $z = r$ at fixed $r \neq 0$. This is similar to the polar coordinate singularity.

3. a) To determine the range, note that $r\sin\theta \geq 0$ for the given range of θ and r. However, this is immaterial since the factors $\cos\phi$ and $\sin\phi$ will make x, y cover the full range $(-\infty, +\infty)$. The coordinates (x, y, z) are a slight modification of the standard spherical coordinates. These coordinates cover the whole space (x, y, z).

b) Compute the determinant:

$$\det\begin{pmatrix} -\sin\theta\cos\phi & -r\cos\theta\cos\phi & r\sin\theta\sin\phi \\ -\sin\theta\sin\phi & -r\cos\phi\sin\phi & -r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} = r^2\sin\theta.$$

This is nonzero unless $r = 0$ or $\theta = 0$.

c) The singularities are completely analogous to those in the spherical coordinates.

1.3 Basis in tangent space

Suppose that the vectors $\mathbf{e}_\alpha = \frac{\partial}{\partial x^\alpha}$ are linearly dependent, then there exist constants c^α, not all zero, such that the vector $c^\alpha \mathbf{e}_\alpha$ equals zero. Act with this vector on the coordinate function x^1:

$$c^\alpha \mathbf{e}_\alpha x^1 = c^\alpha \frac{\partial}{\partial x^\alpha} x^1 = c^1.$$

By assumption, $c^\alpha \mathbf{e}_\alpha = 0$, therefore $c^1 = 0$. It follows that every c^α equals zero, contradicting the assumption.

1.4 Differentials of functions as 1-forms

$$\begin{aligned} d(x) &= dx, & d\left(x^2\right) &= 2xdx, \\ d(xy) &= xdy + ydx, & d(x+y) &= dx + dy, \\ d\left(4x^2y + x^3z\right) &= \left(8xy + 3x^2z\right)dx + 4x^2dy + x^3dz, & d\left(3\sqrt{x^2+y^2}\right) &= 3\frac{xdx + ydy}{\sqrt{x^2+y^2}}. \end{aligned}$$

Now let us compute dh by first finding

$$\begin{aligned} d\left(\arctan\left(x \pm y\right)\right) &= \frac{dx \pm dy}{1 + \left(x \pm y\right)^2}, \\ d\left(\arctan \frac{2x}{x^2 - y^2 - 1}\right) &= \frac{1}{1 + \frac{4x^2}{(x^2-y^2-1)^2}} \left[\frac{2dx}{x^2 - y^2 - 1} - \frac{4x\left(xdx - ydy\right)}{\left(x^2 - y^2 - 1\right)^2}\right] \\ &= \frac{-2\left(x^2 + y^2 + 1\right)dx + 4xydy}{\left(x^2 - y^2 - 1\right)^2 + 4x^2}. \end{aligned}$$

Adding these together and noting that

$$\left(1 + (x+y)^2\right)\left(1 + (x-y)^2\right) = \left(x^2 - y^2 - 1\right)^2 + 4x^2,$$

we get

$$dh(x,y) = d\left(\arctan\left(x+y\right) + \arctan(x-y) + \arctan\frac{2x}{x^2 - y^2 - 1}\right) = 0.$$

This means that $h(x,y)$ is a constant. By using the tangent sum rule, we can easily show that $h(x,y) = 0$.

1.5 Basis in cotangent space

Note that the relation

$$\left\langle dx^\alpha, \frac{\partial}{\partial x^\beta}\right\rangle = \delta^\alpha_\beta$$

is the definition of how the 1-form dx^α acts on vectors $\partial/\partial x^\beta$. Now, it is clear that any 1-form is decomposed as a linear combination of the 1-forms $dx^1, ..., dx^n$. It remains to show that all these forms are linearly independent. If this were not so, there would exist a linear combination $c_\alpha dx^\alpha = 0$ such that not all $c_\alpha = 0$. Act with this on a vector $\partial/\partial x^1$ and obtain

$$0 = \left\langle 0, \frac{\partial}{\partial x^1}\right\rangle = \left\langle c_\alpha dx^\alpha, \frac{\partial}{\partial x^1}\right\rangle = c_1.$$

Therefore $c_1 = 0$. Similarly, we find that every other $c_\alpha = 0$, which contradicts the assumption.

1.6 Linearly independent 1-forms

1. Two 1-forms $d(e^x \cos y)$, $d(e^x \sin y)$ are linearly independent for every x, y because

$$\begin{aligned} d\left(e^x \cos y\right) &= e^x \cos y dx - e^x \sin y dy, \\ d\left(e^x \sin y\right) &= e^x \sin y dx + e^x \cos y dy, \end{aligned}$$

   and the following determinant is always nonzero,

$$\det\begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} = e^{2x} \neq 0.$$

2. Two 1-forms $(1+y)dx - 2xydy$, $8dx$ are linearly independent if the following determinant is nonzero,

$$\det\begin{pmatrix} 1+y & -2xy \\ 8 & 0 \end{pmatrix} = 16xy.$$

   This happens for $xy \neq 0$.

3. Three 1-forms $dx + dy$, $dx + dz$, $dy + dz$ are always linearly independent.

4. Three 1-forms $dx - dy$, $dy - dz$, $dz - dx$ are always linearly dependent (their sum is zero).

1.7 Transformation law for 1-forms

The transformation law for 1-forms,

$$d\tilde{x}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta}\, dx^\beta,$$

under a coordinate transformation $x^\alpha \to \tilde{x}^\alpha = \tilde{x}^\alpha(x^\beta)$, is merely a different interpretation of the definition of the 1-form $d\tilde{x}^\alpha$ (see Problem 1.4), where $\tilde{x}^\alpha(x)$ is considered a scalar function in the coordinates $\{x^\beta\}$.

1.8 Examples of transformations

a) First compute dx and dy:

$$\begin{aligned} dx &= \left(1+u^2+v^2\right) du + (2uv)\, dv, \\ dy &= (2uv)\, du + \left(1+u^2+v^2\right) dv. \end{aligned}$$

Then it is easy to compute $xdx + ydy$,etc. For instance,

$$d\frac{1}{x+y} = -\frac{dx+dy}{(x+y)^2} = -\frac{(du+dv)\left(1+(u+v)^2\right)}{(u+v)^2\left(1+\frac{1}{3}(u+v)^2\right)^2}.$$

b) The component transformation matrix is given in Eq. (51).

1.9 Supplementary problem sheet

1A Tangent plane

If the tangent plane is at angle α with the horizontal, then the acceleration is $g\sin\alpha$ (from elementary mechanics). Since $0 \le \alpha < \frac{\pi}{2}$, we need to maximize α or, equivalently, $\tan\alpha$, which equals

$$\frac{\partial z}{\partial r} = \frac{rh}{\sigma^2} e^{-\frac{1}{2}r^2\sigma^{-2}}, \quad r \equiv \sqrt{x^2+y^2}.$$

The maximum of $\partial z/\partial r$ is at $r_0 = \sigma$. For example, a point with maximum acceleration is $x_0 = \sigma$, $y_0 = 0$, $z_0 = -he^{-\frac{1}{2}}$. The tangent plane at a point (x_0, y_0, z_0) is given by the equation

$$n_x(x-x_0) + n_y(y-y_0) + n_z(z-z_0) = 0,$$

where (n_x, n_y, n_z) are the components of the normal vector,

$$(n_x, n_y, n_z) = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, 1\right)\Big|_{x_0,y_0,z_0} = \left(e^{-\frac{1}{2}}\frac{h}{\sigma}, 0, 1\right).$$

Therefore the equation of the tangent plane is

$$\frac{x}{\sigma}e^{-\frac{1}{2}} + \frac{z}{h} = 0.$$

1B Induced metric

The induced metric is found by taking the expression for the bulk metric, $g = dx^2 + dy^2 + dz^2$, and computing dx, dy, dz through the forms $d\theta$ and $d\phi$:

$$\begin{aligned} dx &= 2a\sin\theta\cos\theta\cos\phi d\theta - a\sin^2\theta\sin\phi d\phi, \\ dy &= 2a\sin\theta\cos\theta\sin\phi d\theta + a\sin^2\theta\cos\phi d\phi, \\ dz &= 2a\cos 2\theta d\theta. \end{aligned}$$

Therefore

$$g = dx^2 + dy^2 + dz^2 = a^2 d\theta^2 + a^2(\sin\theta)^4 d\phi^2.$$

The metric is degenerate at $\theta = 0$ and $\theta = \frac{\pi}{2}$. The singularities at these points are *not* merely coordinate singularities that disappear when choosing a different coordinate system; but the reason is subtle.

To figure out the nature of these singularities, let us visualize the surface in a neighborhood of $\theta = 0$. The $y = 0$ section of the surface corresponds to $\sin\phi = 0$, so $\phi = 0$ or $\phi = \pi$. Then

$$x = \pm\sin^2\theta, \quad y = \sin\theta\cos\theta, \quad x = \pm\frac{1-\cos 2\theta}{2} = \pm\frac{1-\sqrt{1-4z^2}}{2}.$$

This is a union of two circles touching at $x = z = 0$. Hence, the surface is a torus with zero inner radius, i.e. intersecting itself at $x = y = z = 0$. The rotational symmetry around the z axis leads to a "cusp" at $\theta = 0$: the surface has a sharp corner and the metric cannot be made smooth and nondegenerate by any choice of local coordinates. The situation near $\theta = \frac{\pi}{2}$ is similar.

1C Embedded waves

The surface is defined by

$$x = \frac{\cos v}{\sqrt{2} - \sin u}, \quad y = \frac{\sin v}{\sqrt{2} - \sin u}, \quad z = \frac{\cos u}{\sqrt{2} - \sin u}.$$

1. For $y = 0$, we have $v = 0$ and thus
$$x = \frac{1}{\sqrt{2} - \sin u}, z = \frac{\cos u}{\sqrt{2} - \sin u}.$$
To visualize this line in the (x, z) plane, we eliminate u from these equations and find
$$\sin u = \sqrt{2} - \frac{1}{x}, \quad \left(x - \sqrt{2}\right)^2 + z^2 = 1.$$
Therefore the line is a circle of radius 1 centered at $\left(x = \sqrt{2}, z = 0\right)$. This circle does not intersect the z axis since $\sqrt{2} > 1$. Now we see that (x, y) is obtained from $\left(\sqrt{2} - \sin u\right)^{-1}$ by multiplying with $\cos v$ and $\sin v$. Therefore, the full surface is a rotation surface, where we need to use the x coordinate as the radius. Therefore, the figure in the (x, z) plane needs to be rotated around the z axis. The resulting surface is a torus. It may be described by the equation
$$\left(\sqrt{x^2 + y^2} - \sqrt{2}\right)^2 + z^2 - 1 = 0.$$
Note that
$$\sqrt{x^2 + y^2} = \frac{1}{\sqrt{2} - \sin u} > 0.$$
Also
$$\sqrt{x^2 + y^2} - \sqrt{2} = \frac{1 - \sqrt{2}\left(\sqrt{2} - \sin u\right)}{\sqrt{2} - \sin u} = \frac{\sqrt{2}\sin u - 1}{\sqrt{2} - \sin u}.$$

2. Since the surface is now given by an equation of the form $F(x, y, z) = 0$, the normal vector (up to a constant factor C) can be found as
$$(n_x, n_y, n_z) = C\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = C\left(2\frac{\sqrt{x^2 + y^2} - \sqrt{2}}{\sqrt{x^2 + y^2}}x, 2\frac{\sqrt{x^2 + y^2} - \sqrt{2}}{\sqrt{x^2 + y^2}}y, 2z\right).$$
Multiplying by $C \equiv \frac{1}{2}\sqrt{x^2 + y^2}$ (this factor is chosen for simplicity), we have
$$(n_x, n_y, n_z) = \left(x\left(\sqrt{x^2 + y^2} - \sqrt{2}\right), y\left(\sqrt{x^2 + y^2} - \sqrt{2}\right), z\sqrt{x^2 + y^2}\right).$$
Expressed through the coordinates (u, v), this becomes
$$n_\alpha = (n_x, n_y, n_z) = \left(\cos v\frac{\sqrt{2}\sin u - 1}{\left(\sqrt{2} - \sin u\right)^2}, \sin v\frac{\sqrt{2}\sin u - 1}{\left(\sqrt{2} - \sin u\right)^2}, \frac{\cos u}{\left(\sqrt{2} - \sin u\right)^2}\right).$$
The equation of the tangent plane at point x_0 is
$$n_\alpha\left(x^\alpha - x^\alpha_{(0)}\right) = 0,$$
where n_α must be computed at $x = x_0$.

3. We compute
$$\begin{aligned}
\frac{\partial x}{\partial u} &= \frac{\cos v\cos u}{\left(\sqrt{2} - \sin u\right)^2}, & \frac{\partial x}{\partial v} &= -\frac{\sin v}{\sqrt{2} - \sin u}, \\
\frac{\partial y}{\partial u} &= \frac{\sin v\cos u}{\left(\sqrt{2} - \sin u\right)^2}, & \frac{\partial y}{\partial v} &= \frac{\cos v}{\sqrt{2} - \sin u}, \\
\frac{\partial z}{\partial u} &= \frac{1 - \sqrt{2}\sin u}{\left(\sqrt{2} - \sin u\right)^2}, & \frac{\partial z}{\partial v} &= 0.
\end{aligned}$$
Now we can expand
$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv, \quad \text{etc.}$$

Therefore the induced metric is

$$g = dx^2 + dy^2 + dz^2 = \left[\left(\frac{\cos u}{\left(\sqrt{2}-\sin u\right)^2}\right)^2 + \left(\frac{1-\sqrt{2}\sin u}{\left(\sqrt{2}-\sin u\right)^2}\right)^2\right] du^2 + \frac{dv^2}{\left(\sqrt{2}-\sin u\right)^2}$$
$$= \frac{du^2+dv^2}{\left(\sqrt{2}-\sin u\right)^2}.$$

The vector $V^a = (\cos v, \sin v)$ is not a unit vector because

$$g(V,V) = \frac{\cos^2 v + \sin^2 v}{\left(\sqrt{2}-\sin u\right)^2} = \frac{1}{\left(\sqrt{2}-\sin u\right)^2} \neq 1.$$

The Cartesian components of the vectors $\partial/\partial u$, $\partial/\partial v$ are found from

$$\frac{\partial}{\partial u} = \frac{\partial x(u,v)}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y(u,v)}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z(u,v)}{\partial u}\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial v} = \frac{\partial x(u,v)}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y(u,v)}{\partial v}\frac{\partial}{\partial y} + \frac{\partial z(u,v)}{\partial v}\frac{\partial}{\partial z}.$$

Therefore, the vector V^a has the following Cartesian components,

$$\begin{aligned}
V &= \cos v\left(-\frac{\sin v}{\sqrt{2}-\sin u}\frac{\partial}{\partial x} + \frac{\cos v}{\sqrt{2}-\sin u}\frac{\partial}{\partial y}\right) \\
&\quad + \sin v\left(\frac{\cos v\cos u}{\left(\sqrt{2}-\sin u\right)^2}\frac{\partial}{\partial x} + \frac{\sin v\cos u}{\left(\sqrt{2}-\sin u\right)^2}\frac{\partial}{\partial y} + \frac{1-\sqrt{2}\sin u}{\left(\sqrt{2}-\sin u\right)^2}\frac{\partial}{\partial z}\right) \\
&= \sin v\cos v\left(\frac{\sin u+\cos u-\sqrt{2}}{\left(\sqrt{2}-\sin u\right)^2}\right)\frac{\partial}{\partial x} + \left(\frac{\sin^2 v\cos u + \sqrt{2}\cos^2 v - \sin u\cos^2 v}{\left(\sqrt{2}-\sin u\right)^2}\right)\frac{\partial}{\partial y} \\
&\quad + \sin v\frac{1-\sqrt{2}\sin u}{\left(\sqrt{2}-\sin u\right)^2}\frac{\partial}{\partial z}.
\end{aligned}$$

This vector is within the tangent plane because $n_\alpha V^\alpha = 0$,

$$\begin{aligned}
n_\alpha V^\alpha &= \cos v\frac{\sqrt{2}\sin u - 1}{\left(\sqrt{2}-\sin u\right)^2}\sin v\cos v\left(\frac{\sin u+\cos u-\sqrt{2}}{\left(\sqrt{2}-\sin u\right)^2}\right) \\
&\quad + \sin v\frac{\sqrt{2}\sin u - 1}{\left(\sqrt{2}-\sin u\right)^2}\left(\frac{\sin^2 v\cos u + \sqrt{2}\cos^2 v - \sqrt{2}\sin u\cos^2 v}{\left(\sqrt{2}-\sin u\right)^2}\right) \\
&\quad + \frac{\cos u}{\left(\sqrt{2}-\sin u\right)^2}\sin v\frac{1-\sqrt{2}\sin u}{\left(\sqrt{2}-\sin u\right)^2} \\
\text{(after simplification)} &= 0.
\end{aligned}$$

2 Tensors

2.1 Definition of tensor product

a),b) A direct calculation using the property $\langle dx^i, \partial/\partial x^k\rangle = \delta^i_k$ gives:

$$\langle \omega_1, \mathbf{v}_1\rangle = \left\langle dx + 2ydy, 3\frac{\partial}{\partial x}\right\rangle = 3, \quad \text{etc.}$$

The results:

$$T(\mathbf{v}_1,\mathbf{v}_1) = 0, \quad T(\mathbf{v}_1,\mathbf{v}_2) = 6x.$$

c) d) First, show that the sum of two linear functions is again a linear function: If $A(\mathbf{x})$ and $B(\mathbf{x})$ are linear functions, i.e. if

$$A(\mathbf{x}+\lambda\mathbf{y}) = A(\mathbf{x}) + \lambda A(\mathbf{y})$$

and likewise for B, then $A+B$ obviously has the same property. Now, since a tensor is defined as a multi-linear function, it is clear that tensors form a vector space.

2.3 Example of tensor

a) An obvious example of such T is the vector product, $T(\mathbf{u},\mathbf{v}) = \mathbf{u}\times\mathbf{v}$, defined in three-dimensional space. To determine the rank of T, we need to represent T as a multilinear number-valued function of some number of 1-forms and vectors, e.g. $A(\mathbf{f}_1,...,\mathbf{f}_r,\mathbf{v}_1,...,\mathbf{v}_s)$. It is clear that $T(\mathbf{v}_1,\mathbf{v}_2)$ itself is not such a function because it has *vector* values instead of scalar (number) values. So we need to add a 1-form to the list of arguments. We can define

$$A(\mathbf{f}_1,\mathbf{v}_1,\mathbf{v}_2) = \langle \mathbf{f}_1, T(\mathbf{v}_1,\mathbf{v}_2)\rangle$$

and then it's clear that A is multilinear. Therefore T is a tensor of rank 1+2.

b) The calculation may go as follows. We need to determine the components $T^{\alpha}_{\beta\gamma}$ such that

$$[T(\mathbf{u},\mathbf{v})]^{\alpha} = T^{\alpha}_{\beta\gamma}u^{\beta}v^{\gamma}.$$

So we rewrite the given definition of $T(\mathbf{u},\mathbf{v})$ in the index notation, e.g. like this:

$$[T(\mathbf{u},\mathbf{v})]^{\alpha} = 2\varepsilon^{\alpha}{}_{\beta\gamma}u^{\beta}v^{\gamma} - u^{\alpha}n_{\beta}v^{\beta}.$$

Now we would like to move $u^{\beta}v^{\gamma}$ out of the brackets and so determine $T^{\alpha}_{\beta\gamma}$. However, the expression above contains $u^{\alpha}v^{\beta}$ instead of $u^{\beta}v^{\gamma}$. Therefore we rename the index β to γ and also introduce a Kronecker symbol δ^{α}_{β}, so as to rewrite identically

$$u^{\alpha}n_{\beta}v^{\beta} = u^{\beta}v^{\gamma}n_{\gamma}\delta^{\alpha}_{\beta}.$$

Therefore

$$\begin{aligned}[T(\mathbf{u},\mathbf{v})]^{\alpha} &= 2\varepsilon^{\alpha}{}_{\beta\gamma}u^{\beta}v^{\gamma} - u^{\beta}v^{\gamma}n_{\gamma}\delta^{\alpha}_{\beta} = \left(2\varepsilon^{\alpha}{}_{\beta\gamma} - n_{\gamma}\delta^{\alpha}_{\beta}\right)u^{\beta}v^{\gamma},\\ T^{\alpha}_{\beta\gamma} &= 2\varepsilon^{\alpha}{}_{\beta\gamma} - n_{\gamma}\delta^{\alpha}_{\beta}.\end{aligned}$$

2.5 Contraction of tensor indices

a) Using the definition of a tensor as a multilinear function, it is easy to show that linear combinations of tensors are also multilinear functions. Tensor products and contractions are also multilinear. The arguments are much simpler than the proof of tensor transformation law for components.

b) Contracting two lower indices, e.g. $T_{\alpha\alpha\beta}$, gives components of a quantity which is not a tensor because these components do not transform correctly under changes of basis. If $T_{\alpha\alpha\beta}$ were a tensor it would transform as

$$\tilde{T}_{\alpha\alpha\gamma} = \frac{\partial x^{\lambda}}{\partial \tilde{x}^{\gamma}}T_{\mu\mu\lambda}.$$

However, this does not agree with the contraction of the tensor $T_{\alpha\beta\gamma}$, which transforms as

$$\tilde{T}_{\alpha\beta\gamma} = \frac{\partial x^{\lambda}}{\partial \tilde{x}^{\alpha}}\frac{\partial x^{\mu}}{\partial \tilde{x}^{\beta}}\frac{\partial x^{\nu}}{\partial \tilde{x}^{\gamma}}T_{\lambda\mu\nu}.$$

The contraction over $\alpha = \beta$ yields

$$\sum_{\alpha}\tilde{T}_{\alpha\alpha\gamma} = \sum_{\alpha}\frac{\partial x^{\lambda}}{\partial \tilde{x}^{\alpha}}\frac{\partial x^{\mu}}{\partial \tilde{x}^{\alpha}}\frac{\partial x^{\nu}}{\partial \tilde{x}^{\gamma}}T_{\lambda\mu\nu} \neq \frac{\partial x^{\lambda}}{\partial \tilde{x}^{\gamma}}T_{\mu\mu\lambda}.$$

c) Calculation gives

$$T^{\alpha}_{\alpha\gamma} = 2\varepsilon^{\alpha}{}_{\alpha\gamma} - n_{\gamma}\delta^{\alpha}_{\alpha} = -3n_{\gamma}$$

because $\varepsilon^{\alpha}{}_{\alpha\gamma} = 0$ and $\delta^{\alpha}_{\alpha} = 3$.

2.8 Examples of spaces with a metric

a) We perform the calculation in components,

$$g(\mathbf{u},\mathbf{v}) = u_{\alpha}v^{\alpha} - \varepsilon_{\alpha\beta\gamma}\varepsilon^{\alpha}{}_{\lambda\mu}u^{\beta}n^{\gamma}v^{\lambda}n^{\mu}.$$

We would like to write $g(\mathbf{u},\mathbf{v}) = g_{\alpha\beta}u^{\alpha}v^{\beta}$, where $g_{\alpha\beta}$ are the components of the metric tensor. Using the known identity for the ε-symbol,

$$\varepsilon_{\alpha\beta\gamma}\varepsilon^{\alpha}{}_{\lambda\mu} = \delta_{\beta\lambda}\delta_{\gamma\mu} - \delta_{\beta\mu}\delta_{\gamma\lambda},$$

we find

$$\begin{aligned}g(\mathbf{u},\mathbf{v}) &= u_{\alpha}v^{\alpha} - \left(\delta_{\beta\lambda}\delta_{\gamma\mu} - \delta_{\beta\mu}\delta_{\gamma\lambda}\right)u^{\beta}n^{\gamma}v^{\lambda}n^{\mu}\\ &= u_{\alpha}v^{\alpha} - u_{\lambda}v^{\lambda}n_{\mu}n^{\mu} + u_{\lambda}n^{\lambda}v_{\mu}n^{\mu}.\end{aligned}$$

Denote $n^2 \equiv n_\mu n^\mu \equiv g(\mathbf{n}, \mathbf{n})$, and then we need to relabel indices such that $v^\alpha u^\beta$ can be moved outside the parentheses. The result is

$$g(\mathbf{u}, \mathbf{v}) = \left(\delta_{\alpha\beta} - n^2 \delta_{\alpha\beta} + n_\alpha n_\beta\right) v^\alpha u^\beta.$$

Therefore

$$g_{\alpha\beta} = \delta_{\alpha\beta} - n^2 \delta_{\alpha\beta} + n_\alpha n_\beta.$$

To analyze the conditions under which $\det g_{\alpha\beta} \neq 0$, we can choose an orthonormal basis such that n_α is parallel to the first basis vector. Then the components of the vector $\mathbf{n}$ in this basis are $(|\mathbf{n}|, 0, 0)$ and the matrix $g_{\alpha\beta}$ has the following simple form:

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - n^2 & 0 \\ 0 & 0 & 1 - n^2 \end{pmatrix}.$$

Then it is clear that $\det g_{\alpha\beta} = \left(1 - n^2\right)^2$. Therefore, the matrix $g_{\alpha\beta}$ is nondegenerate if $n^2 \neq 1$.

b) Similar calculations give

$$g_{\alpha\beta} = \begin{pmatrix} 3 + n^2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Therefore the determinant of $g_{\alpha\beta}$ is always nonzero.

c) Considerations are analogous to b), except that the size of the matrix $g_{\alpha\beta}$ is larger.

d) The metric is $(dx)^2 + (dy)^2 + (dz)^2$, and we need to express dx, dy, dz through du and dv. A calculation gives

$$g = dx^2 + dy^2 + dz^2 = R^2 \left(\sinh^2 u + \cosh^2 u\right) du^2 + R^2 \cosh^2 u \, dv^2.$$

2.9 Supplementary problem sheet

2A Vector equations

a) The equation contains two given vectors A_α and B_α. The solution x_α can be found as a linear combination of A_α, B_α, and the cross product $\varepsilon_{\alpha\beta\gamma} A^\beta B^\gamma$ with unknown coefficients. Using vector notation, we have

$$\mathbf{x} = \alpha \mathbf{A} + \beta \mathbf{B} + \gamma \left(\mathbf{A} \times \mathbf{B}\right).$$

Substituting this expression into the given equation,

$$k\mathbf{x} + \mathbf{x} \times \mathbf{A} = \mathbf{B},$$

and using the known identity

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B} \left(\mathbf{A} \cdot \mathbf{C}\right) - \mathbf{A} \left(\mathbf{B} \cdot \mathbf{C}\right), \tag{52}$$

we find

$$\mathbf{A} \left(k\alpha - \mathbf{A} \cdot \mathbf{B}\right) + \mathbf{B} \left(k\beta + \mathbf{A} \cdot \mathbf{A} - 1\right) + \left(\mathbf{A} \times \mathbf{B}\right) \left(k\gamma - \beta\right) = 0.$$

On purpose, we write this equation in the form of linear combination of the three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A} \times \mathbf{B}$. Since we are considering the generic case, these three vectors are independent and so each of the coefficients above must be zero:

$$k\alpha - \mathbf{A} \cdot \mathbf{B} = 0, \quad k\beta + \mathbf{A} \cdot \mathbf{A} - 1 = 0, \quad k\gamma - \beta = 0.$$

Solving this system of equations, we find (assuming $k \neq 0$ in the generic case)

$$\alpha = \frac{\mathbf{A} \cdot \mathbf{B}}{k}, \quad \beta = \frac{1 - \mathbf{A} \cdot \mathbf{A}}{k}, \quad \gamma = \frac{1 - \mathbf{A} \cdot \mathbf{A}}{k^2}.$$

b) We have in vector notation

$$\mathbf{x} \times \mathbf{A} = \mathbf{B}, \quad \mathbf{x} \cdot \mathbf{C} = k.$$

Multiply $\times \mathbf{C}$:

$$(\mathbf{x} \times \mathbf{A}) \times \mathbf{C} = \mathbf{B} \times \mathbf{C}.$$

Simplify using the identity (52),

$$\mathbf{A} \left(\mathbf{x} \cdot \mathbf{C}\right) - \mathbf{x} \left(\mathbf{A} \cdot \mathbf{C}\right) = k\mathbf{A} - \mathbf{x} \left(\mathbf{A} \cdot \mathbf{C}\right) = \mathbf{B} \times \mathbf{C}.$$

Therefore

$$\mathbf{x} = \frac{k\mathbf{A} - \mathbf{B} \times \mathbf{C}}{\mathbf{A} \cdot \mathbf{C}}.$$

c), d) The equations have the form $x_\alpha M^\alpha_\beta = A_\beta$, where M^α_β is a matrix and A_β is a known vector. The solution is $\mathbf{x} = M^{-1}\mathbf{A}$, where M^{-1} is the inverse matrix (it exists in the generic case).

2B Tensor equations

a) Since the vectors A_α and B_α are a basis in two-dimensional space (they are nonzero and orthogonal), then the symmetric tensor $X^{\alpha\beta}$ can be written generally as

$$X^{\alpha\beta} = fA^\alpha A^\beta + g\left(A^\alpha B^\beta + A^\beta A^\alpha\right) + hB^\alpha B^\beta,$$

where the coefficients f, g, h are unknown. It remains to determine these coefficients. Using $A_\alpha B^\alpha = 0$ and denoting $A^\alpha A_\alpha \equiv |A|^2$, we get the system of equations

$$\begin{aligned} X^{\alpha\beta}A_\alpha = B^\beta \quad &\Rightarrow \quad f\,|A|^2 A^\beta + g\,|A|^2 B^\beta = B^\beta, \\ X^{\alpha\beta}\delta_{\alpha\beta} = 0 \quad &\Rightarrow \quad f\,|A|^2 + h\,|B|^2 = 0. \end{aligned}$$

The result is $g = |A|^{-2}$, $f = h = 0$, so

$$X^{\alpha\beta} = \frac{A^\alpha B^\beta + B^\alpha A^\beta}{|A|^2}.$$

b) An antisymmetric tensor $X^{\alpha\beta}$ in three dimensions can be always expressed as

$$X^{\alpha\beta} = \varepsilon^{\alpha\beta\gamma} u_\gamma,$$

where u_γ is an unknown vector that we need to determine. We can now rewrite the conditions on $X^{\alpha\beta}$ in a vector form,

$$\begin{aligned} X^{\alpha\beta}A_\alpha = B^\beta \quad &\Rightarrow \quad \mathbf{u}\times\mathbf{A} = \mathbf{B} \\ X^{\alpha\beta}B_\alpha = 0 \quad &\Rightarrow \quad \mathbf{u}\times\mathbf{B} = 0 \end{aligned}$$

It follows that $\mathbf{u}$ is parallel to $\mathbf{B}$ and then the condition $\mathbf{u}\times\mathbf{A} = \mathbf{B}$ leaves the only solution $\mathbf{u} = 0$, and therefore $X^{\alpha\beta} = 0$ is the only admissible solution.

2C Degeneracy of the metric

a) The metric can be written in the basis $\{dx, dy\}$ as the matrix

$$g = \begin{pmatrix} y^2 & 1+x^2 \\ 1+x^2 & 0 \end{pmatrix}.$$

The determinant of this matrix is $-\left(1+x^2\right)^2$ which is always nonzero.

b) In the basis where A_α is parallel to the first basis vector, the vector A_α has components $(A, 0, 0, 0, ...)$ and therefore the metric $g_{\alpha\beta}$ has the form

$$\begin{pmatrix} -r^2 & 0 & 0 & \dots \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ \vdots & & & \ddots \end{pmatrix}.$$

The metric is degenerate if $r = 0$ (i.e. at the origin).

3 The Christoffel symbol

3.1 Transformations 1

We are considering a flat space where Euclidean coordinates exist. Suppose $\{x^\alpha\}$ and $\{\tilde{x}^\alpha\}$ are two coordinate systems, while $\{\xi^\alpha\}$ is the standard Euclidean coordinate system. The Christoffel symbols are defined as

$$\begin{aligned} \Gamma^\mu_{\alpha\beta} &= \frac{\partial^2\xi^\nu}{\partial x^\alpha \partial x^\beta}\frac{\partial x^\mu}{\partial \xi^\nu}, \\ \tilde{\Gamma}^\mu_{\alpha\beta} &= \frac{\partial^2\xi^\nu}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta}\frac{\partial \tilde{x}^\mu}{\partial \xi^\nu}. \end{aligned}$$

The relationship between $\tilde{\Gamma}$ and Γ can be found as follows. Assuming that the functions $x^\alpha(\tilde{x})$ and also $\tilde{x}^\alpha(x)$ are known, we may express the partial derivative operators using the chain rule,

$$\begin{aligned} \frac{\partial}{\partial x^\mu} &= \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial}{\partial \tilde{x}^\alpha}, \\ \frac{\partial}{\partial \tilde{x}^\mu} &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial}{\partial x^\alpha}. \end{aligned}$$

Also we can express

$$\frac{\partial \tilde{x}^\mu}{\partial \xi^\nu} = \frac{\partial \tilde{x}^\mu}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial \xi^\nu}.$$

Therefore we can calculate $\tilde{\Gamma}^\mu_{\alpha\beta}$ (when $\Gamma^\mu_{\alpha\beta}$ is known) as follows,

$$\begin{aligned}
\tilde{\Gamma}^\mu_{\alpha\beta} &= \frac{\partial \tilde{x}^\mu}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial \xi^\nu}\left(\frac{\partial x^\gamma}{\partial \tilde{x}^\alpha}\frac{\partial}{\partial x^\gamma}\right)\left(\frac{\partial x^\delta}{\partial \tilde{x}^\beta}\frac{\partial}{\partial x^\delta}\right)\xi^\nu \\
&= \frac{\partial \tilde{x}^\mu}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial \xi^\nu}\left[\frac{\partial x^\gamma}{\partial \tilde{x}^\alpha}\frac{\partial x^\delta}{\partial \tilde{x}^\beta}\frac{\partial^2 \xi^\nu}{\partial x^\gamma \partial x^\delta} + \frac{\partial^2 x^\delta}{\partial \tilde{x}^\beta \partial \tilde{x}^\gamma}\frac{\partial \xi^\nu}{\partial x^\delta}\right] \\
&= \frac{\partial \tilde{x}^\mu}{\partial x^\lambda}\frac{\partial x^\gamma}{\partial \tilde{x}^\alpha}\frac{\partial x^\delta}{\partial \tilde{x}^\beta}\Gamma^\lambda_{\gamma\delta} + \frac{\partial^2 x^\delta}{\partial \tilde{x}^\beta \partial \tilde{x}^\gamma}\frac{\partial \tilde{x}^\mu}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial \xi^\nu}\frac{\partial \xi^\nu}{\partial x^\delta} \\
&= \frac{\partial \tilde{x}^\mu}{\partial x^\lambda}\frac{\partial x^\gamma}{\partial \tilde{x}^\alpha}\frac{\partial x^\delta}{\partial \tilde{x}^\beta}\Gamma^\lambda_{\gamma\delta} + \frac{\partial^2 x^\lambda}{\partial \tilde{x}^\beta \partial \tilde{x}^\gamma}\frac{\partial \tilde{x}^\mu}{\partial x^\lambda}.
\end{aligned} \tag{53}$$

Note that the Euclidean coordinate system ξ^ν is not needed to determine the transformation of Γ.

3.2 Transformations 2

Consider a vector field u^μ. Assume that $\nabla_\nu u^\mu$ obeys the correct transformation law for rank (1,1) tensors,

$$\nabla_\nu u^\mu = \left(\tilde{\nabla}_\alpha \tilde{u}^\beta\right)\frac{\partial x^\mu}{\partial \tilde{x}^\beta}\frac{\partial \tilde{x}^\alpha}{\partial x^\nu},$$

and substitute

$$\begin{aligned}
\nabla_\nu u^\mu &\equiv \frac{\partial}{\partial x^\nu}u^\mu + \Gamma^\mu_{\alpha\nu}u^\alpha, \\
\tilde{\nabla}_\nu \tilde{u}^\mu &\equiv \frac{\partial}{\partial \tilde{x}^\nu}\tilde{u}^\mu + \tilde{\Gamma}^\mu_{\alpha\nu}\tilde{u}^\alpha.
\end{aligned}$$

We can now express Γ through $\tilde{\Gamma}$. Note that

$$\frac{\partial \tilde{x}^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial \tilde{x}^\gamma} = \delta^\alpha_\gamma$$

because the matrices are $\partial \tilde{x}^\alpha/\partial x^\beta$ and $\partial x^\beta/\partial \tilde{x}^\gamma$ are inverse to each other. The result is

$$\Gamma^\lambda_{\alpha\beta} = \tilde{\Gamma}^\mu_{\nu\rho}\frac{\partial \tilde{x}^\nu}{\partial x^\alpha}\frac{\partial \tilde{x}^\rho}{\partial x^\beta}\frac{\partial x^\lambda}{\partial \tilde{x}^\mu} + \frac{\partial^2 x^\lambda}{\partial \tilde{x}^\nu \partial \tilde{x}^\rho}\frac{\partial \tilde{x}^\nu}{\partial x^\alpha}\frac{\partial \tilde{x}^\rho}{\partial x^\beta}.$$

3.3 Covariant derivatives

The rule is that every upper index gets a $+\Gamma$ and every lower index gets a $-\Gamma$. Each term with Γ replaces one index in the original tensor by one of the indices in Γ. Therefore we can write the answer as

$$\begin{aligned}
T^{\alpha\beta}{}_{\gamma\delta\mu;\nu} &= \partial_\nu T^{\alpha\beta}{}_{\gamma\delta\mu} + \Gamma^\alpha_{\lambda\nu}T^{\lambda\beta}{}_{\gamma\delta\mu} + \Gamma^\beta_{\lambda\nu}T^{\alpha\lambda}{}_{\gamma\delta\mu} \\
&\quad - \Gamma^\lambda_{\gamma\nu}T^{\alpha\beta}{}_{\lambda\delta\mu} - \Gamma^\lambda_{\delta\nu}T^{\alpha\beta}{}_{\gamma\lambda\mu} - \Gamma^\lambda_{\mu\nu}T^{\alpha\beta}{}_{\gamma\delta\lambda}
\end{aligned}$$

3.4 The Leibnitz rule

Perform an explicit calculation,

$$\begin{aligned}
A_{\alpha;\gamma}B^\beta + A_\alpha B^\beta_{;\gamma} &= B^\beta\left(\partial_\gamma A_\alpha - \Gamma^\lambda_{\alpha\gamma}A_\lambda\right) + A_\alpha\left(\partial_\gamma B^\beta + \Gamma^\beta_{\lambda\gamma}B^\lambda\right); \\
\left(A_\alpha B^\beta\right)_{;\gamma} &= \partial_\gamma\left(A_\alpha B^\beta\right) - \Gamma^\lambda_{\alpha\gamma}A_\alpha B^\beta + \Gamma^\beta_{\lambda\gamma}A_\alpha B^\lambda.
\end{aligned}$$

This proves the required property.

3.5 Locally inertial reference frame

In this problem (unlike problem 3.1) the coordinate system $\{\xi^\alpha\}$ is not a flat Euclidean coordinate system, but it is just a coordinate system which is like Euclidean at one point. Now we want to use the formula (53), which will enable us to compute the Christoffel symbol $\tilde{\Gamma}$ in the coordinate system ξ, given the Christoffel symbol Γ in the original coordinate system $\{x^\alpha\}$. To use that formula, we need to compute some derivatives. Denoting $\{\tilde{x}^\alpha\} \equiv \{\xi^\alpha\}$, we find

$$\frac{\partial \tilde{x}^\alpha}{\partial x^\beta} = \delta^\alpha_\beta + \left(x^\mu - x^\mu_{(0)}\right)\delta^\nu_\beta\Gamma^\alpha_{(0)\mu\nu} = \delta^\alpha_\beta + \left(x^\mu - x^\mu_{(0)}\right)\Gamma^\alpha_{(0)\mu\beta}.$$

The inverse derivative, $\partial x^\alpha/\partial\tilde{x}^\beta$, can be found by inverting this matrix; the result can be found simply by assuming that

$$\frac{\partial x^\alpha}{\partial\tilde{x}^\beta} = \delta^\alpha_\beta + \left(x^\mu - x^\mu_{(0)}\right) A^\alpha_{\mu\beta} + O\left((x - x_{(0)})^2\right),$$

where $A^\alpha_{\mu\beta}$ is an unknown matrix. So up to quadratic terms we find

$$\frac{\partial x^\alpha}{\partial\tilde{x}^\beta} = \delta^\alpha_\beta - \left(x^\mu - x^\mu_{(0)}\right) \Gamma^\alpha_{(0)\mu\beta} + O\left((x - x_{(0)})^2\right).$$

Therefore

$$\frac{\partial^2 x^\lambda}{\partial\tilde{x}^\beta\partial\tilde{x}^\gamma} = -\Gamma^\lambda_{(0)\beta\gamma} + O\left(x - x_{(0)}\right).$$

Finally, we find

$$\begin{aligned}\tilde{\Gamma}^\mu_{\alpha\beta} &= \frac{\partial\tilde{x}^\mu}{\partial x^\lambda}\frac{\partial x^\gamma}{\partial\tilde{x}^\alpha}\frac{\partial x^\delta}{\partial\tilde{x}^\beta}\Gamma^\lambda_{\gamma\delta} + \frac{\partial^2 x^\lambda}{\partial\tilde{x}^\beta\partial\tilde{x}^\gamma}\frac{\partial\tilde{x}^\mu}{\partial x^\lambda} \\ &= \Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{(0)\alpha\beta} + O\left(x - x_{(0)}\right).\end{aligned}$$

At $x = x_0$ we have $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{(0)\alpha\beta}$. It follows that the new Christoffel symbol is equal to zero at $x = x_0$.

Alternatively, one can use the transformation law for the Christoffel symbol in the inverse direction, $\Gamma = \tilde{\Gamma}... + ...$, i.e. one denotes $\{x^\alpha\} \equiv \{\xi^\alpha\}$, $\{\tilde{x}^\alpha\} \equiv \{x^\alpha\}$. This has the advantage that only derivatives $\partial^2\xi/\partial x\partial x$ need to be computed, and not the derivatives $\partial^2 x/\partial\xi\partial\xi$. Since all derivatives only need to be evaluated at $x = x_0$, the first-order derivatives $\partial x/\partial\xi$ at $x = x_0$ can be found as the inverse matrix to $\partial\xi^\alpha/\partial x^\beta = \delta^\alpha_\beta$, i.e. $\partial x^\alpha/\partial\xi^\beta = \delta^\alpha_\beta$. This considerably simplifies the calculations.

4 Geodesics and curvature

4.1 Geodesics

(a) Note that d/ds is the ordinary (not "covariant") derivative in the direction of u^α. The geodesic equation can be rewritten for the 1-form u_α as

$$u^\gamma u_{\alpha;\gamma} = 0 = \frac{du_\alpha}{ds} - \Gamma^\beta_{\alpha\gamma} u_\beta u^\gamma.$$

An explicit formula for $\Gamma^\beta_{\alpha\gamma}$ yields

$$\Gamma^\beta_{\alpha\gamma} u_\beta u^\gamma = \Gamma_{\beta\,\alpha\gamma} u^\beta u^\gamma = \frac{1}{2}\left(g_{\beta\alpha,\gamma} + g_{\beta\gamma,\alpha} - g_{\alpha\gamma,\beta}\right) u^\beta u^\gamma.$$

Note that $g_{\beta\alpha,\gamma} - g_{\alpha\gamma,\beta}$ is antisymmetric in $(\beta \leftrightarrow \gamma)$. Therefore these terms will cancel after a contraction with $u^\beta u^\gamma$. The remaining term yields

$$\Gamma^\beta_{\alpha\gamma} u_\beta u^\gamma = \frac{1}{2} g_{\alpha\beta,\gamma} u^\beta u^\gamma.$$

b) We give two derivations; the first one is direct and the second one uses the property (a).

4.1.1 First derivation

Note that

$$\frac{d}{ds}\left(g_{\alpha\beta}u^\alpha u^\beta\right) = u^\gamma\left(g_{\alpha\beta}u^\alpha u^\beta\right)_{,\gamma}$$

where we must use an ordinary derivative instead of the covariant derivative (according to the definition of d/ds). So we find

$$u^\gamma\left(g_{\alpha\beta}u^\alpha u^\beta\right)_{,\gamma} = g_{\alpha\beta,\gamma}u^\alpha u^\beta u^\gamma + 2g_{\alpha\beta}u^\alpha u^\gamma u^\beta_{,\gamma}.$$

Now we need to simplify an expression containing $u^\beta_{,\gamma}$. By assumption, the derivative of the vector field u^α satisfies

$$\begin{aligned}\frac{d}{ds}u^\alpha + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma &= u^\gamma u^\alpha_{,\gamma} + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma = 0 \\ &= u^\gamma u^\alpha_{,\gamma} + \frac{1}{2}g^{\alpha\lambda}\left(g_{\lambda\mu,\gamma} + g_{\lambda\gamma,\mu} - g_{\mu\gamma,\lambda}\right)u^\gamma u^\mu.\end{aligned}$$

Therefore

$$\begin{aligned}u_\alpha u^\gamma u^\alpha_{,\gamma} &= -\frac{1}{2}u_\alpha g^{\alpha\lambda}\left(g_{\lambda\mu,\gamma} + g_{\lambda\gamma,\mu} - g_{\mu\gamma,\lambda}\right)u^\gamma u^\mu \\ &= -\frac{1}{2}u^\lambda\left(g_{\lambda\mu,\gamma} + g_{\lambda\gamma,\mu} - g_{\mu\gamma,\lambda}\right)u^\gamma u^\mu = -\frac{1}{2}g_{\lambda\mu,\gamma}u^\lambda u^\mu u^\gamma.\end{aligned}$$

What remains is a straightforward computation:

$$\begin{aligned}u^\gamma\left(g_{\alpha\beta}u^\alpha u^\beta\right)_{,\gamma} &= g_{\alpha\beta,\gamma}u^\alpha u^\beta u^\gamma + 2g_{\alpha\beta}u^\alpha u^\gamma u^\beta_{,\gamma} \\ &= g_{\alpha\beta,\gamma}u^\alpha u^\beta u^\gamma - g_{\lambda\mu,\gamma}u^\lambda u^\mu u^\gamma \\ &= 0.\end{aligned}$$

4.1.2 Second derivation

We write

$$\frac{d}{ds}\left(g_{\alpha\beta}u^\alpha u^\beta\right) = \frac{d}{ds}\left(g^{\alpha\beta}u_\alpha u_\beta\right) = u^\gamma g^{\alpha\beta}{}_{,\gamma}u_\alpha u_\beta + 2g^{\alpha\beta}u_{\alpha,\gamma}u_\beta u^\gamma.$$

Now we need to express $u_{\alpha,\gamma}u^\gamma \equiv du_\alpha/ds$. To do that, we use the property derived in (a),

$$\frac{du_\alpha}{ds} = \frac{1}{2}g_{\beta\gamma,\alpha}u^\beta u^\gamma,$$

and find

$$\frac{d}{ds}\left(g^{\alpha\beta}u_\alpha u_\beta\right) = \frac{dg^{\alpha\beta}}{ds}u_\alpha u_\beta + g^{\alpha\beta}g_{\lambda\gamma,\alpha}u^\lambda u_\beta u^\gamma = \frac{dg^{\alpha\beta}}{ds}u_\alpha u_\beta + \frac{dg_{\lambda\gamma}}{ds}u^\lambda u^\gamma.$$

It remains to express the derivative of $g^{\alpha\beta}$ through the derivative of $g_{\alpha\beta}$. We use the identity

$$\frac{d}{ds}\left(g^{\alpha\beta}g_{\beta\gamma}\right) = \frac{d}{ds}\left(\delta^\alpha_\gamma\right) = 0,$$

thus

$$\frac{dg^{\alpha\beta}}{ds}g_{\beta\gamma} = -\frac{dg_{\beta\gamma}}{ds}g^{\alpha\beta} \;\Rightarrow\; \frac{dg^{\alpha\beta}}{ds} = -\frac{dg_{\lambda\mu}}{ds}g^{\alpha\lambda}g^{\beta\mu}.$$

Therefore

$$\frac{dg^{\alpha\beta}}{ds}u_\alpha u_\beta = -\frac{dg_{\lambda\mu}}{ds}g^{\alpha\lambda}g^{\beta\mu}u_\alpha u_\beta = -\frac{dg_{\lambda\mu}}{ds}u^\lambda u^\mu,$$

and thus

$$\frac{d}{ds}\left(g^{\alpha\beta}u_\alpha u_\beta\right) = 0.$$

4.2 Commutator of covariant derivatives

First compute

$$\begin{aligned}
u^\alpha_{;\beta} &= u^\alpha_{,\beta} + \Gamma^\alpha_{\beta\lambda}u^\lambda,\\
u^\alpha_{;\beta\gamma} &= \left(u^\alpha_{,\beta} + \Gamma^\alpha_{\beta\lambda}u^\lambda\right)_{,\gamma} + \Gamma^\alpha_{\mu\gamma}\left(u^\mu_{,\beta} + \Gamma^\mu_{\beta\lambda}u^\lambda\right) - \Gamma^\mu_{\beta\gamma}\left(u^\alpha_{,\mu} + \Gamma^\alpha_{\mu\lambda}u^\lambda\right)\\
&= u^\alpha_{,\beta\gamma} + \Gamma^\alpha_{\beta\lambda,\gamma}u^\lambda + \Gamma^\alpha_{\beta\lambda}u^\lambda_{,\gamma} + \Gamma^\alpha_{\mu\gamma}u^\mu_{,\beta} + \Gamma^\alpha_{\mu\gamma}\Gamma^\mu_{\beta\lambda}u^\lambda - \Gamma^\mu_{\beta\gamma}u^\alpha_{;\mu}.
\end{aligned}$$

Since we want to compute the commutator $u^\alpha_{;\beta\gamma} - u^\alpha_{;\gamma\beta}$, we can omit the terms that are symmetric in $(\beta \leftrightarrow \gamma)$. These terms are the following:

$$u^\alpha_{,\beta\gamma}, \qquad \Gamma^\alpha_{\beta\lambda}u^\lambda_{,\gamma} + \Gamma^\alpha_{\mu\gamma}u^\mu_{,\beta}, \qquad \Gamma^\mu_{\beta\gamma}u^\alpha_{;\mu}.$$

The remaining terms are

$$u^\alpha_{;\beta\gamma} = \Gamma^\alpha_{\beta\lambda,\gamma}u^\lambda + \Gamma^\alpha_{\mu\gamma}\Gamma^\mu_{\beta\lambda}u^\lambda + (\text{symmetric in } \beta \leftrightarrow \gamma),$$

which yields

$$u^\alpha_{;\beta\gamma} - u^\alpha_{;\gamma\beta} = u^\lambda\left(\Gamma^\alpha_{\lambda\beta,\gamma} - \Gamma^\alpha_{\lambda\gamma,\beta} + \Gamma^\mu_{\lambda\beta}\Gamma^\alpha_{\gamma\mu} - \Gamma^\mu_{\lambda\gamma}\Gamma^\alpha_{\beta\mu}\right) = u^\lambda R^\alpha_{\lambda\gamma\beta}.$$

4.3 Parallel transport

The parallel-transported vector can be represented by a 1-form $A_\alpha(s)$ such that

$$\frac{dA_\alpha}{ds} - \Gamma^\beta_{\alpha g}A_\beta u^\gamma = 0, \quad u^\gamma \equiv \frac{dx^\gamma}{ds}.$$

However, the closed curve is assumed to cover only a very small neighborhood of one point x_0, so we can approximate A_α by a constant, $A_\alpha(x_0)$, along the curve. Therefore

$$\delta A_\alpha = \oint \frac{dA_\alpha}{ds}ds = \oint \Gamma^\beta_{\alpha\gamma}(x)A_\beta(x)dx^\gamma \approx A_\beta(x_0)\oint \Gamma^\beta_{\alpha\gamma}(x)dx^\gamma.$$

Now, in a locally inertial system at x_0 we have $\Gamma^\beta_{\alpha\gamma}(x_0) = 0$. Therefore we can Taylor expand $\Gamma^\beta_{\alpha\gamma}(x)$ near x_0 as

$$\Gamma^\beta_{\alpha\gamma}(x) = \left(x^\lambda - x^\lambda_0\right)\Gamma^\beta_{\alpha\gamma,\lambda} + O\left((x-x_0)^2\right).$$

Therefore

$$\delta A_\alpha \approx A_\beta(x_0)\oint \Gamma^\beta_{\alpha\gamma,\lambda}\left(x^\lambda - x^\lambda_0\right)dx^\gamma \approx A_\beta(x_0)\Gamma^\beta_{\alpha\gamma,\lambda}(x_0)\oint x^\lambda dx^\gamma,$$

where we have again approximated $\Gamma^\beta_{\alpha\gamma,\lambda}(x)$ by its value at $x = x_0$, and also used the identity $\oint dx^\gamma = 0$. Further,

$$\oint d\left(x^\gamma x^\lambda\right) = 0 = \oint x^\gamma dx^\lambda + \oint x^\lambda dx^\gamma.$$

Therefore we may rewrite

$$\delta A_\alpha \approx A_\beta \Gamma^\beta_{\alpha\gamma,\lambda} \oint x^\lambda dx^\gamma = \frac{1}{2} A_\beta \left(\Gamma^\beta_{\alpha\gamma,\lambda} - \Gamma^\beta_{\alpha\lambda,\gamma} \right) \oint x^\lambda dx^\gamma$$
$$= \frac{1}{2} A_\beta R^\beta_{\alpha\lambda\gamma} \oint x^\lambda dx^\gamma.$$

4.4 Riemann tensor

a) It is more convenient to consider the fully covariant tensor $R_{\alpha\beta\gamma\delta}$. This tensor has the following symmetries,

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma}, \tag{54}$$
$$R_{\alpha\beta\gamma\delta} + R_{\beta\gamma\alpha\delta} + R_{\gamma\alpha\beta\delta} = 0, \tag{55}$$
$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}. \tag{56}$$

However, it is known that the property (56) follows from (54)-(55), therefore it is sufficient to consider these two properties [note that (55) does not follow from (54), (56)]. Let us first consider the property (54). For fixed γ, δ, we have that $R_{\alpha\beta\gamma\delta}$ is an antisymmetric $n \times n$ matrix (indices α, β). This matrix has $\frac{1}{2}n(n-1)$ independent components. Likewise for fixed α, β. Therefore, the number of independent components of $R_{\alpha\beta\gamma\delta}$ is reduced to

$$N_1 = \left[\frac{1}{2} n(n-1) \right]^2.$$

Now we use the property (55). Let us see whether the property (55) is nontrivial at fixed δ. If $\alpha = \gamma$, then the property (55) becomes

$$R_{\alpha\beta\alpha\delta} + R_{\beta\alpha\alpha\delta} + R_{\alpha\alpha\beta\delta} = 0 \quad \text{(no summation)},$$

which is already a consequence of (54). Likewise for $\beta = \gamma$ or for $\alpha = \beta$. Therefore, the property (55) is a new constraint only if all three indices α, β, γ are different (i.e. $\alpha \neq \beta$, $\alpha \neq \gamma$, $\beta \neq \gamma$). Suppose that α, β, γ are different. There are

$$N_2 = \frac{1}{6} n(n-1)(n-2)$$

choices of such α, β, γ. Therefore, for each $\delta = 1, ..., n$ we obtain N_2 additional constraints. Finally, let us check that every such constraint is nontrivial for every δ (even if δ is equal to one of α, β, γ). Suppose $\delta = \alpha$, then (55) becomes

$$R_{\alpha\beta\gamma\alpha} + R_{\beta\gamma\alpha\alpha} + R_{\gamma\alpha\beta\alpha} = 0 \quad \text{(no summation)}.$$

This is a nontrivial constraint (equivalent to $R_{\alpha\beta\alpha\gamma} = R_{\alpha\gamma\alpha\beta}$). Therefore, the number of constraints is nN_2, and thus the total number of independent components of $R_{\alpha\beta\gamma\delta}$ is

$$N = N_1 - nN_2 = \frac{n^2 \left(n^2 - 1\right)}{12}.$$

b) Weinberg, Chapter 6, §8

c) There is only one independent component of $R_{\alpha\beta\gamma\delta}$ in two dimensions. For instance, we can choose R_{1212} as the independent parameter. Then we can express the Ricci tensor as

$$R_{\alpha\beta} = g^{\lambda\mu} R_{\lambda\alpha\mu\beta}.$$

Calculating component by component, we find

$$R_{11} = g^{22} R_{1212}, \quad R_{12} = -g^{12} R_{1212}, \quad R_{22} = g^{11} R_{1212},$$
$$R = g^{\alpha\beta} R_{\alpha\beta} = \left(2g^{11} g^{22} - 2g^{12} g^{21}\right) R_{1212} = 2g R_{1212}.$$

Note that the matrix

$$\begin{pmatrix} g^{22} & -g^{12} \\ -g^{12} & g^{11} \end{pmatrix}$$

is equal to the inverse matrix to $g^{\alpha\beta}$ (which is $g_{\alpha\beta}$), multiplied by the determinant $\det g^{\alpha\beta}$; since $\det g^{\alpha\beta} = 1/g$, we have

$$\begin{pmatrix} g^{22} & -g^{12} \\ -g^{12} & g^{11} \end{pmatrix} = g g_{\alpha\beta}.$$

Therefore

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = g g_{\alpha\beta} R_{1212} - \frac{1}{2} g_{\alpha\beta} 2g R_{1212} = 0.$$

4.5 Lorentz transformations

A Lorentz transformation is represented by a matrix Λ^α_β such that

$$\Lambda^\alpha_\lambda \Lambda^\beta_\mu g_{\alpha\beta} = g_{\lambda\mu}.$$

Consider an infinitesimal Lorentz transformation,

$$\Lambda^\alpha_\lambda = \delta^\alpha_\lambda + \varepsilon H^\alpha_\lambda.$$

The number of parameters in Lorentz transformations is the same as the number of parameters in H^α_β. The condition for H^α_β is

$$\left(\delta^\alpha_\lambda + \varepsilon H^\alpha_\lambda\right)\left(\delta^\beta_\mu + \varepsilon H^\beta_\mu\right) g_{\alpha\beta} = g_{\lambda\mu}.$$

Disregarding terms of order ε^2, we find

$$0 = \delta^\beta_\mu H^\alpha_\lambda g_{\alpha\beta} + \delta^\alpha_\lambda H^\beta_\mu g_{\alpha\beta} = H_{\mu\lambda} + H_{\lambda\mu}.$$

Therefore, $H_{\lambda\mu}$ is an antisymmetric $n \times n$ matrix, which has $\frac{1}{2}n(n-1)$ independent components. For $n = 4$ we get 6 components. These can be interpreted as three spatial rotations and three Lorentz rotations (boosts).

5 Gravitation theory applied

5.1 Redshift

In the weak field limit, the Newtonian gravitational potential near a mass M is

$$\Phi = \frac{GM}{r},$$

while the component g_{00} of the metric is

$$g_{00} = 1 + \frac{2\Phi}{c^2}.$$

(We write the units explicitly.) Therefore the redshift factor $z(r)$ at distance r from the center of the Earth is

$$z(r) = \sqrt{1 + \frac{2GM}{c^2 r}} \approx 1 + \frac{GM}{c^2 r}.$$

To compare the redshift factors at the surface of the Earth, denote by R_E the radius of the Earth. We know that the gravitational acceleration at the surface is

$$g_E = \frac{GM}{R_E^2} \approx 9.81 \frac{\mathrm{m}}{\mathrm{s}^2}.$$

Therefore, it is convenient to express $GM = g_E R_E^2$. For a vertical distance L between sender and receiver, we find

$$\frac{z(R_E)}{z(R_E + L)} = \frac{1 + g_E R_E c^{-2}}{1 + g_E R_E c^{-2} \frac{R_E}{R_E + L}} \approx 1 + \frac{g_E R_E}{c^2}\left(1 - \frac{R_E}{R_E + L}\right) = 1 + \frac{g_E R_E L}{c^2 \left(R_E + L\right)}.$$

Since in our problem $L \ll R_E$, we may approximate

$$\frac{z(R_E)}{z(R_E + L)} \approx 1 + \frac{g_E L}{c^2} \approx 1 + 1.1 \frac{L}{10^{16}\mathrm{m}}.$$

5.2 Energy-momentum tensor 1

In the nonrelativistic limit, we may disregard gravitation; $g_{\alpha\beta} = \eta_{\alpha\beta}$. The EMT of an ideal fluid is

$$T^{\alpha\beta} = -p\eta^{\alpha\beta} + (p + \rho)\, u^\alpha u^\beta,$$

where u^α is the 4-velocity vector of the fluid motion. In the nonrelativistic limit, $u^\alpha \approx (1, \vec{v})$, where $\vec{v}$ is the 3-vector of velocity and $|\vec{v}| \ll 1$ in the units where $c = 1$.

The conservation law is

$$0 = T^{\alpha\beta}{}_{,\beta} = -p^{,\alpha} + (p + \rho)_{,\beta}\, u^\alpha u^\beta + (p + \rho)\, u^\alpha u^\beta{}_{,\beta} + (p + \rho)\, u^\alpha{}_{,\beta} u^\beta.$$

Let us simplify this expression by introducing the time derivative along the fluid flow,

$$\frac{d}{dt} \equiv u^\alpha \partial_\alpha.$$

Then we find

$$0 = -p^{,\alpha} + u^\alpha \left(\dot{p} + \dot{\rho} + (p+\rho)\, u^\beta{}_{,\beta}\right) + (p+\rho)\, \dot{u}^\alpha. \tag{57}$$

Contracting with u_α and using $u_\alpha \dot{u}^\alpha = 0$, we find

$$\dot{\rho} + (p+\rho)\, u^\beta{}_{,\beta} = 0. \tag{58}$$

This is the relativistic continuity equation. Using this equation, we find from Eq. (57) that

$$0 = -p^{,\alpha} + u^\alpha \dot{p} + (p+\rho)\, \dot{u}^\alpha. \tag{59}$$

Now let us apply the nonrelativistic limit, $u^\alpha \approx (1, \vec{v})$, to Eqs. (58) and (59). In our notation, for any quantity X we have

$$\dot{X} \equiv \frac{d}{dt} X \equiv \frac{\partial}{\partial t} X + (\vec{v} \cdot \nabla)\, X.$$

The continuity equation (58) gives

$$\frac{d\rho}{dt} + (p+\rho)\, \mathrm{div}\vec{v} = 0.$$

This is the ordinary, nonrelativistic continuity equation.[1] Finally, Eq. (59) gives

$$\vec{\nabla} p + \vec{v}\dot{p} + (p+\rho)\, \dot{\vec{v}} = 0.$$

(Note that $p^{,j} = -\nabla^j p$.) This is the Euler equation,

$$\frac{d\vec{v}}{dt} = \frac{1}{p+\rho}\left(-\vec{\nabla} p - \vec{v}\frac{dp}{dt}\right).$$

5.3 Energy-momentum tensor 2

Compute the covariant derivative,

$$\begin{aligned} T^{\alpha\beta}{}_{;\alpha} &= \left[\Phi^{;\alpha}\Phi^{;\beta} - \frac{1}{2} g^{\alpha\beta}\Phi^{;\lambda}\Phi_{;\lambda}\right]_{;\alpha} \\ &= \Phi^{;\alpha}{}_{;\alpha}\Phi^{;\beta} + \Phi^{;\alpha}\Phi^{;\beta}{}_{;\alpha} - g^{\alpha\beta}\Phi^{;\lambda}\Phi_{;\lambda\alpha} \\ &= \Phi^{;\alpha}{}_{;\alpha}\Phi^{;\beta}. \end{aligned}$$

Here we used $\Phi_{;\alpha\beta} = \Phi_{;\beta\alpha}$ which follows from $\Gamma^\lambda_{\alpha\beta} = \Gamma^\lambda_{\beta\alpha}$ (note that Φ is a scalar; covariant derivatives do not commute when applied to vectors!) and also the property

$$\Phi^{;\alpha}{}_{;\beta} X^\alpha = \Phi_{;\alpha\beta} X^\alpha,$$

which is due to $g_{\alpha\beta;\mu} = 0$. Therefore, we get

$$\Phi^{;\alpha}{}_{;\alpha}\Phi^{;\beta} = 0;$$

this entails $\Phi^{;\alpha}{}_{;\alpha} = 0$ (since $\Phi^{;\beta} = 0$ is a weaker condition than $\Phi^{;\alpha}_{;\alpha} = 0$, i.e. if $\Phi^{;\beta} = 0$ then also $\Phi^{;\alpha}{}_{;\alpha} = 0$, so it is sufficient to write the latter).

5.4 Weak gravity

A very short solution is to write R_{00} directly through Γ and note that only $\Gamma^\alpha_{00,\alpha}$ comes in (if we disregard terms of second order). Then compute Γ^α_{00} explicitly through Φ. (Assume that $g_{\alpha\beta,0} = 0$.) We may disregard terms of order $\Gamma\Gamma$ because Γ is of order Φ, and also we may raise and lower indices using $\eta_{\mu\nu}$ instead of $g_{\mu\nu}$. (This is somewhat heuristic; see below.) The calculation goes like this:

$$\begin{aligned} R_{00} &= \Gamma^\alpha_{00,\alpha} - \Gamma^\alpha_{0\alpha,0} + O(\,\Gamma\,\Gamma\,), \\ \Gamma^\alpha_{00} &= \frac{1}{2}\eta^{\alpha\beta}\left(g_{0\beta,0} + g_{\beta 0,0} - g_{00,\beta}\right) = -\eta^{\alpha\beta}\Phi_{,\beta}, \end{aligned}$$

therefore (using $\Phi_{,0} = 0$)

$$R_{00} = -\eta^{\alpha\beta}\Phi_{,\alpha\beta} = -\Box\Phi = -\Phi_{,00} + \Phi_{,11} + \Phi_{,22} + \Phi_{,33} = \Delta\Phi.$$

Here is another, somewhat more comprehensive solution. In the weak field limit, we write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$

[1] Note that in the usual, nonrelativistic continuity equations as they are written in most books, there is no $p + \rho$ - just ρ. This is so because in most cases the matter is nonrelativistic, so $p \ll \rho$ and $p + \rho \approx \rho$. This is, however, not true for relativistic matter, such as photons (electromagnetic radiation) for which $p = \frac{1}{3}\rho$.

Then we only compute everything up to first order in $h_{\mu\nu}$. Therefore, we may raise and lower indices using the Minkowski metric $\eta_{\alpha\beta}$ instead of $g_{\alpha\beta}$.

Note: the Newtonian limit does not determine the components of $g_{\mu\nu}$ except for $g_{00} = 1 + 2\Phi$. The actual metric $g_{\mu\nu}$ is equal to $\eta_{\mu\nu}$ plus a small first-order deviation, $h_{\mu\nu}$, but this deviation cannot be expressed just through the Newtonian potential $\Phi \equiv \frac{1}{2}h_{00}$. In principle, one needs to solve the full Einstein equations to find $h_{\mu\nu}$; in other words, one needs to determine other, "post-Newtonian potentials" and not just the Newtonian potential Φ. However, when one only wants to compute effects of gravitation on motion of slow bodies, only g_{00} is necessary. So it is sufficient to compute just the Newtonian potential Φ. But e.g. trajectories of light rays cannot be computed accurately in the Newtonian limit (because light does not move slowly). To compute trajectories of light rays, one needs all components of $h_{\mu\nu}$, not just h_{00}.

Let us do the computation through $h_{\mu\nu}$ in a more general way. First we compute the Christoffel symbol and the Ricci tensor:

$$\Gamma^\lambda_{\alpha\nu} = \frac{1}{2}\eta^{\lambda\gamma}\left(h_{\alpha\gamma,\nu} + h_{\nu\gamma,\alpha} - h_{\alpha\nu,\gamma}\right); \quad \Rightarrow \Gamma \sim O(\Phi);$$
$$R_{\alpha\beta} = R^\lambda_{\alpha\lambda\beta} = \Gamma^\lambda_{\alpha\beta,\lambda} - \Gamma^\lambda_{\alpha\lambda,\beta} + \Gamma\Gamma - \Gamma\Gamma \approx \Gamma^\lambda_{\alpha\beta,\lambda} - \Gamma^\lambda_{\alpha\lambda,\beta}.$$

(We may disregard the $\Gamma\Gamma$ terms since they are second order in Φ.) Now we compute (again up to first order in Φ)

$$\Gamma^\lambda_{\alpha\lambda,\beta} = \frac{1}{2}\eta^{\lambda\gamma}\left(h_{\alpha\gamma,\lambda\beta} + h_{\lambda\gamma,\alpha\beta} - h_{\alpha\lambda,\gamma\beta}\right) = \frac{1}{2}\eta^{\lambda\gamma}h_{\lambda\gamma,\alpha\beta},$$
$$R_{\alpha\beta} = \frac{1}{2}\eta^{\lambda\gamma}\left(h_{\alpha\gamma,\beta\lambda} + h_{\beta\gamma,\alpha\lambda} - h_{\alpha\beta,\gamma\lambda} - h_{\lambda\gamma,\alpha\beta}\right).$$

Let us now compute just the component R_{00}, recalling that $h_{\mu\nu}$ is time-independent (so $h_{\mu\nu,0} = 0$) and $h_{00} = 2\Phi$:

$$R_{00} = \frac{1}{2}\eta^{\lambda\gamma}\left(h_{0\gamma,0\lambda} + h_{0\gamma,0\lambda} - h_{00,\gamma\lambda} - h_{\lambda\gamma,00}\right) = -\frac{1}{2}\eta^{\lambda\gamma}h_{00,\gamma\lambda}$$
$$= -\eta^{\lambda\gamma}\Phi_{,\lambda\gamma} = \Delta\Phi.$$

5.5 Equations of motion from conservation law

We would like to rewrite the covariant conservation law $T^{\mu\nu}{}_{;\nu} = 0$ through ordinary derivatives. The given relations are useful; let's derive them first.

$$\frac{\partial}{\partial x^\nu}\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\left(\frac{\partial}{\partial x^\nu}g\right) = -\frac{1}{2\sqrt{-g}}\left(gg^{\alpha\beta}g_{\alpha\beta,\nu}\right)$$
$$= \frac{1}{2}\sqrt{-g}g^{\alpha\beta}g_{\alpha\beta,\nu};$$
$$\Gamma^\mu_{\mu\nu} = \frac{1}{2}g^{\mu\alpha}\left(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\nu\mu,\alpha}\right) = \frac{1}{2}g^{\mu\alpha}g_{\mu\alpha,\nu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\sqrt{-g}.$$

Now rewrite the covariant derivative of $T^{\mu\nu}$ explicitly:

$$T^{\mu\nu}{}_{;\mu} = T^{\mu\nu}_{,\mu} + \Gamma^\mu_{\alpha\mu}T^{\alpha\nu} + \Gamma^\nu_{\alpha\mu}T^{\mu\alpha} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}T^{\mu\nu}\right)_{,\nu} + \Gamma^\nu_{\alpha\beta}T^{\alpha\beta}.$$

Apply this to the given $T^{\mu\nu}$:

$$0 = T^{\mu\nu}{}_{;\mu} = \frac{m_0}{\sqrt{-g}}\left[\int ds\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\delta^{(4)}(x^\sigma - x^\sigma(s))\right]_{,\mu}$$
$$+ \frac{m_0}{\sqrt{-g}}\int ds\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\Gamma^\nu_{\alpha\beta}(x)\delta^{(4)}(x^\sigma - x^\sigma(s)).$$

Since in the first term the dependence on x^σ is only through $\delta^{(4)}$, we can rewrite

$$\frac{dx^\mu}{ds}\frac{\partial}{\partial x^\mu}\left[\delta^{(4)}(x^\alpha - x^\alpha(s))\right] = -\frac{d}{ds}\left[\delta^{(4)}(x^\alpha - x^\alpha(s))\right]$$

(this is easily understood if read from right to left) and then integrate by parts,

$$\int ds\frac{dx^\nu}{ds}\frac{d}{ds}\delta^{(4)}(x^\sigma - x^\sigma(s)) = -\int ds\frac{d^2x^\nu}{ds^2}\delta^{(4)}(x^\sigma - x^\sigma(s)).$$

Finally,

$$0 = \frac{\sqrt{-g}}{m_0}T^{\mu\nu}{}_{;\mu} = \int ds\left[\frac{d^2x^\nu}{ds^2} + \Gamma^\nu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right]\delta^{(4)}(x^\sigma - x^\sigma(s)).$$

This is a function of x^σ which should equal zero everywhere. Therefore, the integrand should vanish for every value of s,

$$\frac{d^2x^\nu}{ds^2} + \Gamma^\nu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0.$$

Remark: in general, equations of motion do not follow from conservation law, but they do follow if there is only one field. (e.g. one fluid, or one scalar field, or some number of point particles). The situation in ordinary mechanics is similar: e.g. the equation of motion for a particle follow from the conservation of energy only if the motion is in one dimension:

$$E = \frac{mv^2}{2} + V(x) = \text{const},$$
$$0 = \frac{dE}{dt} = (m\dot{v} + V'(x))\, v \quad \Rightarrow m\dot{v} = -V'(x).$$

However, equations of motion do not follow from conservation of energy if there is more than one degree of freedom. Similarly, equations of motion for say *two* scalar fields Φ, Ψ do not follow from the conservation of their combined $T_{\mu\nu}$. These fields have two different equations of motion, and one cannot hope to derive them from a single conservation law.

6 The gravitational field

6.1 Degrees of freedom

The electromagnetic field is described by a 4-vector potential $A_\mu(x)$. This would give 4 degrees of freedom. However, there is also a gauge symmetry,

$$A_\mu \to A_\mu + \phi_{,\mu},$$

where $\phi(x)$ is an arbitrary function of spacetime. Using this gauge symmetry, we may e.g. set the component $A_0(x) = 0$. Then only three functions of spacetime (A_1, A_2, A_3) are left. Hence the electromagnetic field has 3 degrees of freedom. There are additional gauge symmetries involving functions $\phi(x)$ that do not depend on time. Since these functions $\phi(x)$ are functions only of three arguments, they do not change the number of degrees of freedom.

6.2 Spherically symmetric spacetime

6.2.1 Straightforward solution

A direct computation listing all the possible Christoffel symbols and components of the Ricci tensor is certainly straightforward but very long. Here is a way to compute the curvature tensor without writing individual components. Since the metric has a diagonal form, let us denote

$$g_{\alpha\beta} = \eta_{\alpha\beta} A_\alpha, \quad g^{\alpha\beta} = \eta^{\alpha\beta} \frac{1}{A_\alpha} \quad \text{(no summation!)}, \tag{60}$$

where

$$A_\alpha \equiv \left\{ e^N, e^L, r^2, r^2 \sin^2\theta \right\} \tag{61}$$

is a fixed array of four functions. For this calculation, we do not use the Einstein summation convention any more; every summation will be written explicitly. However, we make heavy use of the fact that $\eta_{\alpha\beta} \neq 0$ only if $\alpha = \beta$, and that $\eta_{\lambda\lambda} = \eta^{\lambda\lambda}$. At the end of the calculation of the Ricci tensor $R_{\alpha\beta}$, we shall substitute the known functions A_α and use the resulting simplifications.

We begin with the calculation of the Christoffel symbols,

$$\begin{aligned}\Gamma^\lambda_{\alpha\beta} &= \sum_\mu \frac{1}{2}\eta^{\lambda\mu} \frac{1}{A_\lambda} \left[\eta_{\alpha\mu} A_{\mu,\beta} + \eta_{\beta\mu} A_{\mu,\alpha} - \eta_{\alpha\beta} A_{\alpha,\mu}\right] \\ &= \frac{1}{2}\delta^\lambda_\alpha \frac{A_{\lambda,\beta}}{A_\lambda} + \frac{1}{2}\delta^\lambda_\beta \frac{A_{\lambda,\alpha}}{A_\lambda} - \frac{1}{2}\eta_{\lambda\lambda}\eta_{\alpha\beta} \frac{A_{\alpha,\lambda}}{A_\lambda}.\end{aligned} \tag{62}$$

Note that the summation over μ results in setting $\lambda = \mu$ due to $\eta^{\lambda\mu}$, and that we have relations such as $\eta_{\lambda\lambda}\eta_{\alpha\lambda} = \delta^\lambda_\alpha$ and $\delta^\beta_\alpha \eta_{\alpha\alpha} = \eta_{\alpha\beta}$, which hold *without summation*. For convenience, we rewrite Eq. (62) as

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2}\left[\delta^\lambda_\alpha B_{\lambda,\beta} + \delta^\lambda_\beta B_{\lambda,\alpha} - \eta_{\lambda\lambda}\eta_{\alpha\beta} \frac{A_\alpha}{A_\lambda} B_{\alpha,\lambda}\right],$$

where we defined the auxiliary function

$$B_\alpha \equiv \ln A_\alpha.$$

As a check, we compute the "trace" of the Christoffel symbols and compare with the known formula,

$$\sum_\lambda \Gamma^\lambda_{\alpha\lambda} = \frac{1}{2}\left[B_{\alpha,\alpha} + \sum_\lambda B_{\lambda,\alpha} - \frac{A_\alpha}{A_\alpha} B_{\alpha,\alpha}\right] = \frac{1}{2}\sum_\lambda B_{\lambda,\alpha} = \left(\ln\sqrt{-g}\right)_{,\alpha}.$$

Let us also denote for brevity

$$C \equiv \ln\sqrt{-g} = \frac{1}{2}\sum_\lambda B_\lambda; \quad \sum_\lambda \Gamma^\lambda_{\alpha\lambda} = C_{,\alpha}.$$

We proceed to the computation of the Ricci tensor. We use the formula (with Landau-Lifshitz sign conventions)

$$R_{\alpha\beta} = \sum_\lambda \left(\Gamma^\lambda_{\alpha\beta,\lambda} - \Gamma^\lambda_{\alpha\lambda,\beta}\right) + \sum_{\lambda,\mu} \left(\Gamma^\mu_{\lambda\mu}\Gamma^\lambda_{\alpha\beta} - \Gamma^\lambda_{\alpha\mu}\Gamma^\mu_{\beta\lambda}\right).$$

We now compute the necessary terms:

$$\begin{aligned}
\sum_\lambda \Gamma^\lambda_{\alpha\beta,\lambda} &= \frac{1}{2}B_{\alpha,\alpha\beta} + \frac{1}{2}B_{\beta,\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\sum_\lambda \eta_{\lambda\lambda}\left(\frac{A_\alpha}{A_\lambda}B_{\alpha,\lambda}\right)_{,\lambda},\\
\sum_\lambda \Gamma^\lambda_{\alpha\lambda,\beta} &= C_{,\alpha\beta},\\
\sum_{\lambda,\mu} \Gamma^\mu_{\lambda\mu}\Gamma^\lambda_{\alpha\beta} &= \sum_\lambda C_{,\lambda}\frac{1}{2}\left[\delta^\lambda_\alpha B_{\lambda,\beta} + \frac{1}{2}\delta^\lambda_\beta B_{\lambda,\alpha} - \frac{1}{2}\eta_{\lambda\lambda}\eta_{\alpha\beta}\frac{A_\alpha}{A_\lambda}B_{\alpha,\lambda}\right]\\
&= \frac{1}{2}\left[C_{,\alpha}B_{\alpha,\beta} + C_{,\beta}B_{\beta,\alpha} - \eta_{\alpha\beta}\sum_\lambda C_{,\lambda}\eta_{\lambda\lambda}\frac{A_\alpha}{A_\lambda}B_{\alpha,\lambda}\right],
\end{aligned}$$

$$\begin{aligned}
\sum_{\lambda,\mu}\Gamma^\lambda_{\alpha\mu}\Gamma^\mu_{\lambda\beta} &= \frac{1}{4}\sum_{\lambda,\mu}\left[\delta^\lambda_\alpha B_{\lambda,\mu} + \delta^\lambda_\mu B_{\lambda,\alpha} - \eta_{\lambda\lambda}\eta_{\alpha\mu}\frac{A_\alpha}{A_\lambda}B_{\alpha,\lambda}\right]\left[\delta^\mu_\lambda B_{\mu,\beta} + \delta^\mu_\beta B_{\mu,\lambda} - \eta_{\mu\mu}\eta_{\lambda\beta}\frac{A_\lambda}{A_\mu}B_{\lambda,\mu}\right]\\
&= \frac{1}{4}\sum_\mu B_{\alpha,\mu}\left[\delta^\mu_\alpha B_{\mu,\beta} + \delta^\mu_\beta B_{\mu,\alpha} - \eta_{\mu\mu}\eta_{\alpha\beta}\frac{A_\alpha}{A_\mu}B_{\alpha,\mu}\right] \quad \text{(here set } \lambda = \alpha)\\
&+ \frac{1}{4}\sum_\mu B_{\mu,\alpha}\left[B_{\mu,\beta} + \delta^\mu_\beta B_{\mu,\mu} - \eta_{\mu\mu}\eta_{\mu\beta}\frac{A_\mu}{A_\mu}B_{\mu,\mu}\right] \quad \text{(here set } \lambda = \mu)\\
&- \frac{1}{4}\sum_\lambda \eta_{\lambda\lambda}\eta_{\alpha\alpha}\frac{A_\alpha}{A_\lambda}B_{\alpha,\lambda}\left[\delta^\alpha_\lambda B_{\alpha,\beta} + \delta^\alpha_\beta B_{\alpha,\lambda} - \eta_{\alpha\alpha}\eta_{\lambda\beta}\frac{A_\lambda}{A_\alpha}B_{\lambda,\alpha}\right] \quad \text{(here set } \mu = \alpha)\\
&= \frac{1}{4}B_{\alpha,\alpha}B_{\alpha,\beta} + \frac{1}{4}B_{\alpha,\beta}B_{\beta,\alpha} - \frac{1}{4}\eta_{\alpha\beta}\sum_\mu \eta_{\mu\mu}\frac{A_\alpha}{A_\mu}B_{\alpha,\mu}B_{\alpha,\mu}\\
&+ \frac{1}{4}\sum_\mu B_{\mu,\alpha}B_{\mu,\beta} + \frac{1}{4}B_{\beta,\alpha}B_{\beta,\beta} - \frac{1}{4}B_{\beta,\alpha}B_{\beta,\beta}\\
&- \frac{1}{4}B_{\alpha,\alpha}B_{\alpha,\beta} - \frac{1}{4}\eta_{\alpha\beta}\sum_\lambda \eta_{\lambda\lambda}\frac{A_\alpha}{A_\lambda}B_{\alpha,\lambda}B_{\alpha,\lambda} + \frac{1}{4}B_{\alpha,\beta}B_{\beta,\alpha}\\
&= \frac{1}{2}B_{\alpha,\beta}B_{\beta,\alpha} - \frac{1}{2}\eta_{\alpha\beta}\sum_\mu \eta_{\mu\mu}\frac{A_\alpha}{A_\mu}B_{\alpha,\mu}B_{\alpha,\mu} + \frac{1}{4}\sum_\mu B_{\mu,\alpha}B_{\mu,\beta}.
\end{aligned}$$

Finally, we put all the terms together:

$$\begin{aligned}
R_{\alpha\beta} = &\left(\frac{1}{2}B_\alpha + \frac{1}{2}B_\beta - C\right)_{,\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}A_\alpha\sum_\lambda \eta_{\lambda\lambda}\left[\left(\frac{B_{\alpha,\lambda}}{A_\lambda}\right)_{,\lambda} + C_{,\lambda}\frac{B_{\alpha,\lambda}}{A_\lambda}\right]\\
&+ \frac{1}{2}C_{,\alpha}B_{\alpha,\beta} + \frac{1}{2}C_{,\beta}B_{\beta,\alpha} - \frac{1}{2}B_{\alpha,\beta}B_{\beta,\alpha} - \frac{1}{4}\sum_\mu B_{\mu,\alpha}B_{\mu,\beta}.
\end{aligned}$$

We can simplify this expression by considering separately diagonal and off-diagonal components:

$$\begin{aligned}
R_{\alpha\alpha} &= (B_\alpha - C)_{,\alpha\alpha} - \frac{1}{2}\eta_{\alpha\alpha}A_\alpha\sum_\lambda \eta_{\lambda\lambda}\left[\left(\frac{B_{\alpha,\lambda}}{A_\lambda}\right)_{,\lambda} + C_{,\lambda}\frac{B_{\alpha,\lambda}}{A_\lambda}\right]\\
&\quad + \left(C_{,\alpha} - \frac{1}{2}B_{\alpha,\alpha}\right)B_{\alpha,\alpha} - \frac{1}{4}\sum_\mu B_{\mu,\alpha}B_{\mu,\alpha};\\
R_{\alpha\beta} &= \frac{1}{2}\left(B_\alpha + B_\beta - \sum_\lambda B_\lambda\right)_{,\alpha\beta} + \frac{1}{2}\left(C_{,\alpha}B_{\alpha,\beta} + C_{,\beta}B_{\beta,\alpha} - B_{\alpha,\beta}B_{\beta,\alpha}\right)\\
&\quad - \frac{1}{4}\sum_\mu B_{\mu,\alpha}B_{\mu,\beta}. \qquad \text{(only for } \alpha \neq \beta)
\end{aligned}$$

Now we need to simplify this expression further by using the specific form of the metric (60)-(61). We have

$$
\begin{aligned}
&A_t = e^N, \quad A_r = e^L, \quad A_\theta = r^2, \quad A_\phi = r^2 \sin^2\theta; \\
&B_t = N, \quad B_r = L, \quad B_\theta = 2\ln r, \quad B_\phi = 2\ln r + 2\ln\sin\theta; \\
&C = \ln\sqrt{-g} = \frac{N+L}{2} + 2\ln r + \ln\sin\theta; \\
&C_{,t} = \frac{\dot{N}+\dot{L}}{2}, \quad C_{,r} = \frac{N'+L'}{2} + \frac{2}{r}, \quad C_{,\theta} = \cot\theta, \quad C_{,\phi} = 0.
\end{aligned}
$$

Note that the term

$$
\frac{1}{2}\left(B_\alpha + B_\beta - \sum_\lambda B_\lambda\right)_{,\alpha\beta} \tag{63}
$$

always vanishes when $\alpha \neq \beta$. We find (after some omitted algebra):

$$
\begin{aligned}
R_{\phi\phi} &= \frac{1}{2} r^2 \sin^2\theta \sum_\lambda \eta_{\lambda\lambda} \left[\left(\frac{\left[\ln\left(r^2\sin^2\theta\right)\right]_{,\lambda}}{A_\lambda}\right)_{,\lambda} + C_{,\lambda}\frac{\left[\ln\left(r^2\sin^2\theta\right)\right]_{,\lambda}}{A_\lambda}\right] \\
&= -r^2\sin^2\theta\left[\left(\frac{N'-L'}{2r} + \frac{1}{r^2}\right)e^{-L} - \frac{1}{r^2}\right]; \\
R_{\theta\theta} &= 1 - r^2 e^{-L}\left(\frac{N'-L'}{2r} + \frac{1}{r^2}\right); \\
R_{t\phi} &= R_{r\phi} = R_{\theta\phi} = R_{t\theta} = 0; \\
R_{r\theta} &= \frac{1}{2}C_{,\theta}B_{\theta,r} - \frac{1}{4}B_{\phi,r}B_{\phi,\theta} = \frac{1}{2}\cot\theta\frac{2}{r} - \frac{1}{4}\frac{2}{r}2\cot\theta = 0; \\
R_{tr} &= \frac{1}{2}\left(\frac{\dot{N}+\dot{L}}{2}N' + \left(\frac{N'+L'}{2} + \frac{2}{r}\right)\dot{L} - \dot{N}L'\right) - \frac{1}{4}\dot{N}N' - \frac{1}{4}\dot{L}L' = \frac{\dot{L}}{r}; \\
R_{rr} &= \frac{-N''}{2} + \frac{2}{r^2} + N'\frac{L'-N'}{4} + \frac{1}{2}e^{L-N}\left(\ddot{L} + \frac{\dot{L}-\dot{N}}{2}\dot{L}\right) + \frac{1}{r}L'; \\
R_{tt} &= \frac{-\ddot{L}}{2} + \dot{L}\frac{\dot{N}-\dot{L}}{4} + \frac{1}{2}e^{N-L}\left[N'' + \left(\frac{N'-L'}{2} + \frac{2}{r}\right)N'\right].
\end{aligned}
$$

Finally, we compute the Ricci scalar,

$$
\begin{aligned}
R &= \sum_\lambda \eta_{\lambda\lambda}\frac{1}{A_\lambda}R_{\lambda\lambda} \\
&= e^{-N}\left(-\frac{\ddot{L}}{2} + \dot{L}\frac{\dot{N}-\dot{L}}{4}\right) + \frac{1}{2}e^{-L}\left[N'' + \left(\frac{N'-L'}{2} + \frac{2}{r}\right)N'\right] \\
&\quad - e^{-L}\left(\frac{-N''}{2} + \frac{2}{r^2} + N'\frac{L'-N'}{4} + \frac{1}{r}L'\right) - \frac{1}{2}e^{-N}\left(\ddot{L} + \frac{\dot{L}-\dot{N}}{2}\dot{L}\right) \\
&\quad - \frac{1}{r^2} + e^{-L}\left(\frac{N'-L'}{2r} + \frac{1}{r^2}\right) + \left(\frac{N'-L'}{2r} + \frac{1}{r^2}\right)e^{-L} - \frac{1}{r^2} \\
&= e^{-N}\left(-\ddot{L} + \dot{L}\frac{\dot{N}-\dot{L}}{2}\right) - e^{-L}\left(-2\frac{N'-L'}{r} - N'' + N'\frac{L'-N'}{2}\right) - \frac{2}{r^2}.
\end{aligned}
$$

Hence, the nonzero components of the Einstein tensor are

$$
\begin{aligned}
G_{tt} &= e^{N-L}\frac{L'}{r} + \frac{e^N}{r^2}; \\
G_{tr} &= \frac{\dot{L}}{r}; \\
G_{rr} &= \frac{2}{r^2} + \frac{N'}{r} - \frac{e^L}{r^2}; \\
G_{\theta\theta} &= -r^2 e^{-L}\left(-\frac{N'-L'}{2r} + \frac{1}{r^2} + -\frac{N''}{2} + N'\frac{L'-N'}{4}\right) + r^2\frac{e^{-N}}{2}\left(-\ddot{L} + \dot{L}\frac{\dot{N}-\dot{L}}{2}\right); \\
G_{\phi\phi} &= -\frac{1}{2}r^2\sin^2\theta\left[e^{-N}\left(-\ddot{L} + \dot{L}\frac{\dot{N}-\dot{L}}{2}\right) - e^{-L}\left(-\frac{N'-L'}{r} + \frac{2}{r^2} - N'' + N'\frac{L'-N'}{2}\right)\right].
\end{aligned}
$$

6.2.2 Solution using conformal transformation

A more clever way to reduce the amount of computation is to notice that the metric $g_{\mu\nu}$ is simplified after a conformal transformation,

$$g_{\mu\nu} = r^2 h_{\mu\nu} = r^2 \begin{pmatrix} r^{-2}e^L & & & 0 \\ & -r^{-2}e^N & & \\ & & 1 & \\ 0 & & & \sin^2\theta \end{pmatrix}.$$

The metric $h_{\mu\nu}$ separates into the $r-t$ components and the $\theta-\phi$ components. We shall first compute the Ricci tensor for the metric $h_{\mu\nu}$ and then determine how $R_{\alpha\beta}$ changes under a conformal transformation. The calculation of $R_{\alpha\beta}$ for the metric $h_{\mu\nu}$ is much simpler because $h_{\mu\nu}$ is a direct ("block") sum of two metrics defined on 2-dimensional spaces. It is clear that $R_{\alpha\beta}$ will also be a direct sum of the corresponding two-dimensional Ricci tensors.

Let us first compute the Ricci tensor for a diagonal metric $\gamma_{ab} = \operatorname{diag}\left(e^A, e^B\right)$ in two dimensions; set $A \equiv A_1, B \equiv A_2$, and indices $a, b, c, \ldots$ range from 1 to 2. For a two-dimensional metric γ_{ab}, we know that the Ricci tensor is proportional to γ_{ab}, namely (see Problem 4.4c)

$$R_{ab} = \frac{1}{2}\gamma_{ab}R, \quad R \equiv \gamma^{ab}R_{ab} = 2\left(\det\gamma_{ab}\right)R_{1212}.$$

So it is sufficient to compute say R_{11},

$$R_{11} = \Gamma^a_{11,a} - \Gamma^a_{1a,1} + \Gamma^a_{ba}\Gamma^b_{11} - \Gamma^b_{1a}\Gamma^a_{1b},$$

and afterwards we will have

$$R_{ab} = \gamma_{ab}\frac{R_{11}}{\gamma_{11}}; \quad R = \gamma^{ab}R_{ab} = 2\frac{R_{11}}{\gamma_{11}}.$$

The necessary Christoffel symbols are found as (no implicit summation from now on!)

$$\begin{aligned}
\Gamma^a_{1b} &= \sum_c \frac{1}{2}\gamma^{ac}\left(\gamma_{1c,b} + \gamma_{bc,1} - \gamma_{1b,c}\right) = \frac{1}{2}\left(A_{1,b}\delta^a_1 + \delta^a_b A_{b,1} - \delta^1_b e^{A_1 - A_a}A_{1,a}\right); \\
\Gamma^1_{12} &= \frac{1}{2}A_{,2}; \quad \Gamma^2_{12} = \frac{1}{2}B_{,1}; \\
\Gamma^a_{11} &= A_{1,1}\delta^a_1 - \frac{1}{2}e^{A_1 - A_a}A_{1,a}; \quad \Gamma^1_{11} = \frac{1}{2}A_{,1}; \quad \Gamma^2_{11} = -\frac{1}{2}e^{A-B}A_{,2}. \\
\sum_a \Gamma^a_{ba} &= \frac{1}{2}\partial_b \ln\left(\det\gamma_{cd}\right) = \frac{1}{2}\left(A_{1,b} + A_{2,b}\right) = \frac{1}{2}\left(A+B\right)_{,b}.
\end{aligned}$$

Then the component R_{11} of the Ricci tensor is

$$\begin{aligned}
R_{11} &= \Gamma^a_{11,a} - \Gamma^a_{1a,1} + \Gamma^a_{ba}\Gamma^b_{11} - \Gamma^b_{1a}\Gamma^a_{1b} \\
&= \Gamma^1_{11,1} + \Gamma^2_{11,2} - \frac{1}{2}\left(A+B\right)_{,11} + \frac{1}{2}\left(A+B\right)_{,1}\Gamma^1_{11} + \frac{1}{2}\left(A+B\right)_{,2}\Gamma^2_{11} - \Gamma^1_{11}\Gamma^1_{11} - 2\Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12} \\
&= \frac{1}{2}A_{,11} - \frac{1}{2}\left(e^{A-B}A_{,2}\right)_{,2} - \frac{1}{2}\left(A+B\right)_{,11} + \frac{1}{2}\left(A+B\right)_{,1}\frac{1}{2}A_{,1} - \frac{1}{2}\left(A+B\right)_{,2}\frac{1}{2}e^{A-B}A_{,2} - \frac{1}{4}A_{,1}A_{,1} + A_{,2}\frac{1}{2}e^{A-B}A_{,2} - \frac{1}{4}B_{,1}B_{,} \\
&= -\frac{1}{2}B_{,11} - \frac{1}{2}e^{A-B}A_{,22} - \frac{1}{4}A_{,2}\left(A-B\right)_{,2}e^{A-B} + \frac{1}{4}B_{,1}\left(A-B\right)_{,1}.
\end{aligned}$$

We also find (note the symmetry apparent in this formula; this shows that at least this is not obviously wrong)

$$R = 2\frac{R_{11}}{\gamma_{11}} = -e^{-A}B_{,11} - e^{-B}A_{,22} - \frac{1}{2}A_{,2}\left(A-B\right)_{,2}e^{-B} - \frac{1}{2}e^{-A}B_{,1}\left(B-A\right)_{,1}.$$

Well, I am not going to finish this calculation here, anyway. But this is roughly how it goes. Let us at least derive a useful formula below.

The relationship between the curvature tensors under a conformal transformation is found as follows. First we define the conformally transformed metric for convenience as follows,

$$\tilde{g}_{\alpha\beta} = e^{2\Omega}g_{\alpha\beta}.$$

Then the Christoffel symbols receive a correction which is a tensor $B^\lambda_{\alpha\beta}$,

$$\tilde{\Gamma}^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} + B^\lambda_{\alpha\beta}; \quad B^\lambda_{\alpha\beta} \equiv \delta^\lambda_\alpha\Omega_{,\beta} + \delta^\lambda_\beta\Omega_{,\alpha} - g_{\alpha\beta}\Omega^{,\lambda}.$$

The Riemann and the Ricci tensors are defined (in Landau-Lifshitz sign convention) by

$$\begin{aligned}
R^\lambda_{\ \alpha\mu\beta} &= \Gamma^\lambda_{\alpha\beta,\mu} - \Gamma^\lambda_{\alpha\mu,\beta} + \Gamma^\lambda_{\mu\nu}\Gamma^\nu_{\alpha\beta} - \Gamma^\lambda_{\beta\nu}\Gamma^\nu_{\alpha\mu}, \\
R_{\alpha\beta} = R^\lambda_{\ \alpha\lambda\beta} &= \Gamma^\lambda_{\alpha\beta,\lambda} - \Gamma^\lambda_{\lambda\alpha,\beta} + \Gamma^\lambda_{\lambda\nu}\Gamma^\nu_{\alpha\beta} - \Gamma^\lambda_{\beta\nu}\Gamma^\nu_{\alpha\lambda}.
\end{aligned}$$

The same relation holds for $\tilde{R}^{\lambda}{}_{\alpha\mu\beta}$ and $R_{\alpha\beta}$ through $\tilde{\Gamma}^{\lambda}_{\alpha\beta}$ (note that these relations do not involve the metric $g_{\alpha\beta}$ explicitly). Therefore

$$\begin{aligned}\tilde{R}^{\lambda}{}_{\alpha\mu\beta} - R^{\lambda}{}_{\alpha\mu\beta} &= B^{\lambda}_{\alpha\beta,\mu} - B^{\lambda}_{\alpha\mu,\beta} + B^{\lambda}_{\mu\nu}\Gamma^{\nu}_{\alpha\beta} + \Gamma^{\lambda}_{\mu\nu}B^{\nu}_{\alpha\beta} + B^{\lambda}_{\mu\nu}B^{\nu}_{\alpha\beta} - B^{\lambda}_{\beta\nu}\Gamma^{\nu}_{\alpha\mu} - \Gamma^{\lambda}_{\beta\nu}B^{\nu}_{\alpha\mu} - B^{\lambda}_{\beta\nu}B^{\nu}_{\alpha\mu};\\ \tilde{R}_{\alpha\beta} - R_{\alpha\beta} &= B^{\lambda}_{\alpha\beta,\lambda} - B^{\lambda}_{\lambda\alpha,\beta} + B^{\lambda}_{\lambda\nu}\Gamma^{\nu}_{\alpha\beta} + \Gamma^{\lambda}_{\lambda\nu}B^{\nu}_{\alpha\beta} + B^{\lambda}_{\lambda\nu}B^{\nu}_{\alpha\beta} - B^{\lambda}_{\beta\nu}\Gamma^{\nu}_{\alpha\lambda} - \Gamma^{\lambda}_{\beta\nu}B^{\nu}_{\alpha\lambda} - B^{\lambda}_{\beta\nu}B^{\nu}_{\alpha\lambda}.\end{aligned}$$

We shall only compute the expression for the Ricci tensor $R_{\alpha\beta}$. As a preparation, we compute

$$B^{\lambda}_{\lambda\alpha} = \delta^{\lambda}_{\lambda}\Omega_{,\alpha} \equiv N\Omega_{,\alpha},$$

where $N = \delta^{\lambda}_{\lambda} = g_{\alpha\beta}g^{\alpha\beta}$ is the number of spacetime dimensions. We shall always raise and lower indices using the original metric $g_{\alpha\beta}$. So we compute term by term,

$$\begin{aligned}\tilde{R}_{\alpha\beta} - R_{\alpha\beta} &= \left(\delta^{\lambda}_{\alpha}\Omega_{,\beta} + \delta^{\lambda}_{\beta}\Omega_{,\alpha} - g_{\alpha\beta}\Omega^{,\lambda}\right)_{,\lambda} - N\Omega_{,\alpha\beta} + N\Omega_{,\nu}\Gamma^{\nu}_{\alpha\beta} + \Gamma^{\lambda}_{\lambda\nu}\left(\delta^{\nu}_{\alpha}\Omega_{,\beta} + \delta^{\nu}_{\beta}\Omega_{,\alpha} - g_{\alpha\beta}\Omega^{,\nu}\right)\\ &\quad + N\Omega_{,\nu}\left(\delta^{\nu}_{\alpha}\Omega_{,\beta} + \delta^{\nu}_{\beta}\Omega_{,\alpha} - g_{\alpha\beta}\Omega^{,\nu}\right) - \left(\delta^{\lambda}_{\beta}\Omega_{,\nu} + \delta^{\lambda}_{\nu}\Omega_{,\beta} - g_{\beta\nu}\Omega^{,\lambda}\right)\Gamma^{\nu}_{\alpha\lambda}\\ &\quad - \Gamma^{\lambda}_{\beta\nu}\left(\delta^{\nu}_{\alpha}\Omega_{,\lambda} + \delta^{\nu}_{\lambda}\Omega_{,\alpha} - g_{\alpha\lambda}\Omega^{,\nu}\right) - \left(\delta^{\lambda}_{\beta}\Omega_{,\nu} + \delta^{\lambda}_{\nu}\Omega_{,\beta} - g_{\beta\nu}\Omega^{,\lambda}\right)\left(\delta^{\nu}_{\alpha}\Omega_{,\lambda} + \delta^{\nu}_{\lambda}\Omega_{,\alpha} - g_{\alpha\lambda}\Omega^{,\nu}\right)\\ &= 2\Omega_{,\alpha\beta} - g_{\alpha\beta}\Omega^{,\lambda}_{,\lambda} - g_{\alpha\beta,\lambda}\Omega^{,\lambda} - N\Omega_{,\alpha\beta} + N\Omega_{,\nu}\Gamma^{\nu}_{\alpha\beta} + \Gamma^{\lambda}_{\lambda\alpha}\Omega_{,\beta} + \Gamma^{\lambda}_{\lambda\beta}\Omega_{,\alpha} - g_{\alpha\beta}\Gamma^{\lambda}_{\lambda\nu}\Omega^{,\nu}\\ &\quad + 2N\Omega_{,\alpha}\Omega_{,\beta} - Ng_{\alpha\beta}\Omega_{,\nu}\Omega^{,\nu} - \Gamma^{\nu}_{\alpha\beta}\Omega_{,\nu} - \Gamma^{\lambda}_{\lambda\alpha}\Omega_{,\beta} + g_{\beta\nu}\Gamma^{\nu}_{\alpha\lambda}\Omega^{,\lambda} - \Gamma^{\lambda}_{\alpha\beta}\Omega_{,\lambda} - \Gamma^{\lambda}_{\lambda\beta}\Omega_{,\alpha} + g_{\alpha\lambda}\Gamma^{\lambda}_{\beta\nu}\Omega^{,\nu}\\ &\quad - (2+N)\,\Omega_{,\alpha}\Omega_{,\beta} + 2g_{\alpha\beta}\Omega_{,\lambda}\Omega^{,\lambda}\\ &= -g_{\alpha\beta}\Omega^{,\lambda}_{,\lambda} - g_{\alpha\beta}\Gamma^{\lambda}_{\lambda\nu}\Omega^{,\nu} - Ng_{\alpha\beta}\Omega_{,\nu}\Omega^{,\nu} + 2g_{\alpha\beta}\Omega_{,\lambda}\Omega^{,\lambda}\\ &= -(N-2)\left[\Omega_{,\alpha\beta} - \Omega_{,\nu}\Gamma^{\nu}_{\alpha\beta}\right] + (N-2)\,\Omega_{,\alpha}\Omega_{,\beta} - g_{\alpha\beta}\left[(N-2)\,\Omega_{,\lambda}\Omega^{,\lambda} + \Omega^{;\lambda}_{;\lambda} + \Gamma^{\lambda}_{\lambda\nu}\Omega^{,\nu}\right]\\ &\quad + \left[g_{\beta\nu}\Gamma^{\nu}_{\alpha\lambda}\Omega^{,\lambda} + g_{\alpha\nu}\Gamma^{\nu}_{\beta\lambda}\Omega^{,\lambda} - g_{\alpha\beta,\lambda}\Omega^{,\lambda}\right].\end{aligned}$$

Now we note that some of the Γ terms can be absorbed into covariant derivatives, and also that the terms in the last bracket cancel,

$$\left[g_{\beta\nu}\Gamma^{\nu}_{\alpha\lambda}\Omega^{,\lambda} + g_{\alpha\nu}\Gamma^{\nu}_{\beta\lambda}\Omega^{,\lambda} - g_{\alpha\beta,\lambda}\Omega^{,\lambda}\right] = 0,$$

so the resulting formula can be written more concisely as

$$\tilde{R}_{\alpha\beta} - R_{\alpha\beta} = (N-2)\left[\Omega_{,\alpha}\Omega_{,\beta} - \Omega_{;\alpha\beta}\right] - g_{\alpha\beta}\left[(N-2)\,\Omega_{,\lambda}\Omega^{,\lambda} + \Omega^{;\lambda}{}_{;\lambda}\right].$$

The modified Ricci scalar is

$$\begin{aligned}\tilde{R} = \tilde{g}^{\alpha\beta}\tilde{R}_{\alpha\beta} &= e^{-2\Omega}g^{\alpha\beta}R_{\alpha\beta} + e^{-2\Omega}g^{\alpha\beta}\left\{(N-2)\left[\Omega_{,\alpha}\Omega_{,\beta} - \Omega_{;\alpha\beta}\right] - g_{\alpha\beta}\left[(N-2)\,\Omega_{,\lambda}\Omega^{,\lambda} + \Omega^{;\lambda}{}_{;\lambda}\right]\right\}\\ &= e^{-2\Omega}R + e^{-2\Omega}\left\{(N-2)\left[\Omega_{,\alpha}\Omega^{,\alpha} - \Omega_{;\alpha}{}^{;\alpha}\right] - N\left[(N-2)\,\Omega_{,\lambda}\Omega^{,\lambda} + \Omega^{;\lambda}{}_{;\lambda}\right]\right\}\\ &= e^{-2\Omega}\left\{R - (N-2)(N-1)\,\Omega_{,\alpha}\Omega^{,\alpha} - 2(N-1)\,\Omega^{;\alpha}{}_{;\alpha}\right\}.\end{aligned}$$

The Einstein tensor is modified as follows,

$$\begin{aligned}\tilde{G}_{\alpha\beta} = \tilde{R}_{\alpha\beta} - \frac{1}{2}\tilde{g}_{\alpha\beta}\tilde{R} &= R_{\alpha\beta} + (N-2)\left[\Omega_{,\alpha}\Omega_{,\beta} - \Omega_{;\alpha\beta}\right] - g_{\alpha\beta}\left[(N-2)\,\Omega_{,\lambda}\Omega^{,\lambda} + \Omega^{;\lambda}{}_{;\lambda}\right]\\ &\quad - \frac{1}{2}g_{\alpha\beta}\left[R - (N-2)(N-1)\,\Omega_{,\alpha}\Omega^{,\alpha} - 2(N-1)\,\Omega^{;\alpha}{}_{;\alpha}\right]\\ &= G_{\alpha\beta} + (N-2)\left[\Omega_{,\alpha}\Omega_{,\beta} - \Omega_{;\alpha\beta}\right] + g_{\alpha\beta}\left[\frac{(N-2)(N-3)}{2}\Omega_{,\alpha}\Omega^{,\alpha} + (N-2)\,\Omega^{;\alpha}{}_{;\alpha}\right].\end{aligned}$$

Note that there is no change in $G_{\alpha\beta}$ in two dimensions (since the Einstein tensor is always equal to zero).

6.3 Motion in Schwarzschild spacetime

The equation for the covariant component $u_1(s)$ is

$$\frac{du_1}{ds} - \frac{1}{2}u^{\alpha}u^{\beta}\frac{\partial}{\partial r}\left(g_{\alpha\beta}\right) = 0.$$

Using the metric $g_{\alpha\beta} = \mathrm{diag}\left(f, -1/f, -r^2, -r^2\sin^2\theta\right)$, where $f \equiv 1 - r_g/r$, and $u^{\alpha} = \left\{\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}\right\}$, where $\dot{} \equiv d/d\lambda$, we find

$$\frac{d}{d\lambda}\left(-f^{-1}\dot{r}\right) - \frac{1}{2}\left(\frac{df}{dr}\dot{t}^2 - \frac{d}{dr}\left(\frac{1}{f}\right)\dot{r}^2 - 2r\dot{\theta}^2 - 2r\dot{\phi}^2\sin^2\theta\right) = 0. \tag{64}$$

To derive this equation from other equations given in the lecture, we transform in a clever way the expression

$$0 = \frac{d}{d\lambda}\mathcal{K} = \frac{d}{d\lambda}\left[f\dot{t}^2 - f^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\dot{\phi}^2\sin^2\theta\right].$$

Namely, we try to separate terms of the form $\frac{d}{d\lambda}(u_\alpha)$ out of the terms of the form $\frac{d}{d\lambda}(u_\alpha u^\alpha)$ in the following way,

$$\frac{d}{d\lambda}(u_1 u^1) = \frac{d}{d\lambda}(g_{11}u^1u^1) = 2u^1\frac{d}{d\lambda}(g_{11}u^1) - u^1u^1\frac{d}{d\lambda}g_{11} \quad \text{(no summation!)}.$$

For example,

$$\frac{d}{d\lambda}(f\dot{t}^2) = 2\dot{t}\frac{d}{d\lambda}(f\dot{t}) - \dot{t}^2\frac{d}{d\lambda}f, \qquad \text{etc.}$$

We find

$$\begin{aligned}
0 = \frac{d}{d\lambda}\mathcal{K} &= \frac{d}{d\lambda}\left[f\dot{t}^2 - f^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\right] \\
&= 2\dot{t}\frac{d}{d\lambda}(f\dot{t}) - \dot{t}^2\frac{df}{d\lambda} - 2\dot{r}\frac{d}{d\lambda}(f^{-1}\dot{r}) + \dot{r}^2\frac{d}{d\lambda}f^{-1} - 2\dot{\theta}\frac{d}{d\lambda}\left(r^2\dot{\theta}\right) + \dot{\theta}^2\frac{d}{d\lambda}(r^2) \\
&\quad - 2\dot{\phi}\frac{d}{d\lambda}\left(r^2\sin^2\theta\dot{\phi}\right) + \dot{\phi}^2\frac{d}{d\lambda}(r^2\sin^2\theta).
\end{aligned}$$

Now we substitute the given equations (2)-(4), and also evaluate derivatives of the metric, e.g. $df/d\lambda = f'\dot{r}$, so

$$\begin{aligned}
0 &= -\dot{t}^2 f'\dot{r} - 2\dot{r}\frac{d}{d\lambda}(f^{-1}\dot{r}) - \dot{r}^2\frac{f'}{f^2}\dot{r} - 2\dot{\theta}r^2\dot{\phi}^2\sin\theta\cos\theta + \dot{\theta}^2 2r\dot{r} + \dot{\phi}^2 2r\sin^2\theta\dot{r} + 2\dot{\phi}^2 r^2\dot{\theta}\sin\theta\cos\theta \\
&= \dot{r}\left\{\frac{d}{d\lambda}(-2f^{-1}\dot{r}) - f'\dot{t}^2 - \frac{f'}{f^2}\dot{r}^2 + 2r\dot{\theta}^2 + 2r\dot{\phi}^2\sin^2\theta\right\}.
\end{aligned}$$

This is obviously equivalent to Eq. (64).

Note: the reason one of the equations follows from other equations is that the equation $u_\alpha u^\alpha = \text{const}$ is a consequence of the four geodesic equations, $u^\beta u^\alpha{}_{;\beta} = 0$, and the fact that $g_{\alpha\beta;\mu} = 0$. Therefore, when we consider the four geodesic equations and the equation $u_\alpha u^\alpha = \text{const}$, any one of these five equations is a consequence of four others.

6.4 Equations of motion

I didn't write a solution to this.

7 Weak gravitational fields

7.1 Gravitational bending of light

In the lecture it was shown that the trajectory of a light ray in polar coordinates satisfies the equation

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{3}{2}\frac{r_g}{r^2}, \quad r_g \equiv \frac{2GM}{c^2} \approx 3\text{km},$$

where M is the mass of the Sun. Introduce an auxiliary variable $v(\phi) \equiv r^{-1}$ and solve the equation

$$v'' + v = \frac{3}{2}r_g v^2$$

perturbatively, assuming that v is small,

$$v(\phi) = v_0(\phi) + v_1(\phi) + ...$$

The unperturbed solution is

$$v_0(\phi) = \frac{1}{R_0}\cos\phi,$$

where R_0 is the distance of closest approach to the Sun. Then

$$v_1'' + v_1 = \frac{3}{2}\frac{r_g}{R_0^2}\cos^2\phi = \frac{3}{4}\frac{r_g}{R_0^2}(1 + \cos 2\phi).$$

The solution is found with undetermined coefficients,

$$v_1(\phi) = A + B\cos 2\phi, \quad A = \frac{3}{4}\frac{r_g}{R_0^2}, \quad B = -\frac{1}{4}\frac{r_g}{R_0^2}.$$

The total deflection angle is found as $\delta = \phi_1 - \phi_2 - \pi$, where $\phi_{1,2}$ are fixed by the condition $v(\phi) = 0$. We find a quadratic equation

$$\cos^2\phi - \frac{2R_0}{r_g}\cos\phi - 2 = 0, \quad \cos\phi = \frac{R_0}{r_g} \pm \sqrt{\frac{R_0^2}{r_g^2} + 2}.$$

Only the solution with the minus sign is meaningful ($\cos\phi < 1$). Since $r_g \ll R_0$, we may expand this in Taylor series and find

$$\cos\phi \approx -\frac{r_g}{R_0} + O(r_g^3/R_0^3).$$

Therefore, the angle ϕ is very close to $\pi/2$,

$$\phi_{1,2} = \pm\left(\frac{\pi}{2} + \varepsilon\right), \quad \varepsilon \approx \frac{r_g}{R_0}, \quad \Rightarrow \quad \delta = 2\varepsilon = \frac{2r_g}{R_0}.$$

This formula can be rewritten as

$$\delta = \frac{2r_g/R_\odot}{R_0/R_\odot} = \left[\frac{2r_g}{R_\odot}\right]\frac{R_\odot}{R_0}.$$

For the Sun we have $R_\odot = 6,96 \times 10^5$km and $r_g = 2,954$ km, therefore

$$\begin{aligned} 2r_g/R_\odot &= 8,489 \times 10^{-6} = \left[\left[8,489 \times 10^{-6} \times 360^\circ/2\pi\right] \times 3600\right] \text{[arc seconds]} \\ &= 1,751" \end{aligned}$$

(see R. Oloff "Geometrie der Raumzeit," 2nd German edition, page 151).

7.2 Einstein tensor for weak field

For this problem Chapter 4 from the book Norbert Straumann "General Relativity and Relativistic Astrophysics" is useful. We have $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and

$$R_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu,\lambda} - \Gamma^\lambda{}_{\lambda\mu,\nu},$$

where $(...)_{,\mu}$denotes a derivative $\partial_\mu(...)$. Here one can ask the students about the symmetry of this tensor.

Furthermore

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}\left[h_{\mu\beta,\nu} + h_{\nu\beta,\mu} - h_{\mu\nu,\beta}\right] = \frac{1}{2}\left[h^\alpha_{\mu,\nu} + h^\alpha_{\nu,\mu} - h_{\mu\nu}{}^{,\alpha}\right], \tag{65}$$

where as usual we use the convention that indices are raised or lowered with $\eta^{\mu\nu}$; thus e.g. $h^\alpha{}_\beta \equiv \eta^{\alpha\lambda}h_{\lambda\beta}$. Using Eq. (65) we have

$$R_{\mu\nu} = \frac{1}{2}\left[h^\lambda_{\mu,\nu\lambda} + h^\lambda_{\nu,\mu\lambda} - \Box h_{\mu\nu} - h_{,\mu\nu}\right],$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and $h = h^\lambda_\lambda = \eta^{\lambda\alpha}h_{\alpha\lambda}$. And for the Ricci scalar we obtain

$$R = \eta^{\mu\nu}R_{\mu\nu} = h^{\lambda\nu}{}_{,\nu\lambda} - \Box h.$$

Thus in the linear approximation we have

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = \frac{1}{2}\left[h^\lambda_{\mu,\nu\lambda} + h^\lambda_{\nu,\mu\lambda} - \Box h_{\mu\nu} - h_{,\mu\nu} - \eta_{\mu\nu}h^{\lambda\beta}{}_{,\beta\lambda} + \eta_{\mu\nu}\Box h\right].$$

Let us introduce a new variable $\gamma_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$. The traces of two tensors h and γ are related by $\gamma = -h$, thus $h_{\mu\nu} \equiv \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma$. Inserting the last expression for $h_{\mu\nu}$ in $G_{\mu\nu}$, we have

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{2}\left[\gamma^\lambda_{\mu,\nu\lambda} + \gamma^\lambda_{\nu,\mu\lambda} - \Box\gamma_{\mu\nu} - \eta_{\mu\nu}\gamma^{\lambda\beta}{}_{,\beta\lambda}\right] = \\ &= \frac{1}{2}\left[\gamma_{\mu\lambda,\nu}{}^{,\lambda} + \gamma_{\nu\lambda,\mu}{}^{,\lambda} - \Box\gamma_{\mu\nu} - \eta_{\mu\nu}\gamma_{\beta\lambda}{}^{,\lambda\beta}\right], \end{aligned}$$

or finally

$$G^\mu_\nu = \frac{1}{2}\left[\gamma^{\mu,\lambda}_{\lambda,\nu} + \gamma^{\lambda,\mu}_{\nu,\lambda} - \Box\gamma^\mu_\nu - \delta^\mu_\nu\gamma^{\beta,\lambda}_{\lambda,\beta}\right].$$

7.3 Gravitational perturbations I

The metric is written as $g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}$, i.e.

$$g_{00} = 1 + 2\Phi, \; g_{0i} = B_{,i} + S_i, \quad g_{ij} = -\delta_{ij} + 2\Psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij}, \tag{66}$$

where $S_i{}^{,i} = F_i{}^{,i} = h_{ij}^{,i} = h_{ij}\eta^{ij} = 0$, $h_{ij} = h_{ji}$. We shall use the formula for G^μ_ν derived in Problem 7.2. All 3-dimensional indices are raised and lowered using δ_{ij}, so we can write these indices in any position, as convenient:

$$\delta g^0_j = \delta g_{0j} = -\delta g^j_0, \quad \delta g^j_i = -\delta g_{ij}.$$

Also note that for any quantity X we have

$$X^{,j} = \left(\dot{X}, -X_{,j}\right).$$

We need to write the components of

$$\gamma^\mu_\nu = \delta g^\mu_\nu - \frac{1}{2}\delta^\mu_\nu \bar{h}, \quad \bar{h} \equiv \delta g^\mu_\mu,$$

using the 3+1 decomposition:

$$\begin{aligned}
\bar{h} &= \delta g^\mu_\mu = \eta^{\mu\nu}\delta g_{\mu\nu} = 2\left(\Phi - 3\Psi - \Delta E\right), \\
\gamma^0_0 &= \Phi + 3\Psi + \Delta E, \quad \gamma^0_j = B_{,j} + S_j = -\gamma^j_0 = \gamma_{0j}, \\
\gamma^i_j &= -\left(\Phi - \Psi - \Delta E\right)\delta_{ij} - 2E_{,ij} - F_{i,j} - F_{j,i} - h_{ij} = \gamma^j_i.
\end{aligned}$$

Now we compute

$$\begin{aligned}
\gamma^{0,\lambda}_\lambda &= \dot{\gamma}^0_0 - \gamma^0_{j,j} = \dot{\Phi} + 3\dot{\Psi} + \Delta\left(\dot{E} - B\right); \\
\gamma^{j,\lambda}_\lambda &= -\gamma^\lambda_{j,\lambda} = -\dot{\gamma}_{0j} - \gamma^j_{i,i} = -\dot{B}_{,j} - \dot{S}_j - \left(-\left(\Phi - \Psi - \Delta E\right)\delta_{ij} - 2E_{,ij} - F_{i,j} - F_{j,i} - h_{ij}\right)_{,i} \\
&= -\dot{B}_{,j} - \dot{S}_j + \left(\Phi - \Psi + \Delta E\right)_{,j} + \Delta F_j; \\
\gamma^{\beta,\lambda}_{\lambda,\beta} &= \left(\gamma^{0,\lambda}_\lambda\right)_{,0} + \left(\gamma^{j,\lambda}_\lambda\right)_{,j} = \ddot{\Phi} + 3\ddot{\Psi} + \Delta\left(\ddot{E} - \dot{B}\right) + \left[-\dot{B}_{,j} - \dot{S}_j + \left(\Phi - \Psi + \Delta E\right)_{,j} + \Delta F_j\right]_{,j} \\
&= \ddot{\Phi} + 3\ddot{\Psi} + \Delta\ddot{E} - 2\Delta\dot{B} + \Delta\left(\Phi - \Psi + \Delta E\right).
\end{aligned}$$

Then we compute each component of G^μ_ν separately:

$$\begin{aligned}
2G^0_0 &= 2\gamma^{0,\lambda}_{\lambda,0} - \Box\gamma^0_0 - \delta^0_0\gamma^{\beta,\lambda}_{\lambda,\beta} = 2\left(\ddot{\Phi} + 3\ddot{\Psi} + \Delta\left(\ddot{E} - \dot{B}\right)\right) - \partial_0\partial_0\left(\Phi + 3\Psi + \Delta E\right) + \Delta\left(\Phi + 3\Psi + \Delta E\right) \\
&\quad - \left(\ddot{\Phi} + 3\ddot{\Psi} + \Delta\ddot{E}\right) + 2\Delta\dot{B} - \Delta\left(\Phi - \Psi + \Delta E\right) \\
&= 4\Delta\Psi; \\
2G^0_j &= \gamma^{0,\lambda}_{\lambda,j} + \gamma^{\lambda,0}_{j,\lambda} - \Box\gamma^0_j - \delta^0_j\gamma^{\beta,\lambda}_{\lambda,\beta} = \left(\dot{\Phi} + 3\dot{\Psi} + \Delta\left(\dot{E} - B\right)\right)_{,j} -^* \left[-\ddot{B}_{,j} - \ddot{S}_j + \left(\dot{\Phi} - \dot{\Psi} + \Delta\dot{E}\right)_{,j} + \Delta\dot{F}_j\right] \\
&\quad - \left(\ddot{B}_{,j} + \ddot{S}_j\right) + \Delta\left(B_{,j} + S_j\right) = 4\dot{\Psi}_{,j} + \Delta S_j - \Delta\dot{F}_j, \\
2G^i_j &= \gamma^{i,\lambda}_{\lambda,j} + \gamma^{j,\lambda}_{\lambda,i} - \Box\gamma^i_j - \delta^i_j\gamma^{\beta,\lambda}_{\lambda,\beta} = \left[-\dot{B}_{,j} - \dot{S}_j + \left(\Phi - \Psi + \Delta E\right)_{,j} + \Delta F_j\right]_{,i} + \left[-\dot{B}_{,i} - \dot{S}_i + \left(\Phi - \Psi + \Delta E\right)_{,i} + \Delta F_i\right]_{,j} \\
&\quad + \Box\left(\left(\Phi - \Psi - \Delta E\right)\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij}\right) - \delta_{ij}\left[\ddot{\Phi} + 3\ddot{\Psi} + \Delta\ddot{E} - 2\Delta\dot{B} + \Delta\left(\Phi - \Psi + \Delta E\right)\right] \\
&= 2\left(\Phi - \Psi - \dot{B} + \ddot{E}\right)_{,ij} + \ddot{F}_{i,j} + \ddot{F}_{j,i} - \dot{S}_{i,j} - \dot{S}_{j,i} + \Box h_{ij} - 2\delta_{ij}\left[2\ddot{\Psi} + \Delta\left(\Phi - \Psi - \dot{B} + \ddot{E}\right)\right].
\end{aligned}$$

* - the origin of the minus sign here is $\gamma^{\lambda,0}_{j,\lambda} = -\gamma^{j,\lambda}_{\lambda,0}$.

7.4 Gravitational perturbations II

Under an infinitesimal transformation $x^\mu \to x^\mu + \xi^\mu$, the metric changes as

$$g_{\alpha\beta} \to g_{\alpha\beta} - g_{\alpha\gamma}\xi^\gamma_{,\beta} - g_{\beta\gamma}\xi^\gamma_{,\alpha} = g_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}. \tag{67}$$

(This can be easily found from the standard formula for the change of coordinages, involving $\partial\tilde{x}^\mu/\partial x^\nu$.) Now let us write Eq. (67) in full, using the perturbation variables (66), the covariant components ξ_μ, and the decomposition $\xi_\mu = \left(\xi^0, \xi_{\perp i} + \zeta_{,i}\right)$. We can write the transformation of $g_{\alpha\beta}$ component by component using the 3+1 decomposition, and we use the fact that the background metric is diagonal,

$$g_{00} \to g_{00} - 2\xi^0_{,0}; \quad g_{0i} \to g_{0i} - \xi^0_{,i} - \xi_{i,0}; \quad g_{ij} \to g_{ij} - \xi_{i,j} - \xi_{j,i}.$$

To simplify calculations, we adopt the convention of raising and lowering the spatial indices $i, j, ...$ by the *Euclidean* spatial metric δ_{ij} rather than by η_{ij}. This will get rid of some minus signs. We also denote $\partial_0 \equiv \partial_t$ by the overdot. Thus we have

$$g_{00} \to g_{00} - 2\dot{\xi}^0; \quad g_{0i} \to g_{0i} - \xi^0_{,i} - \dot{\xi}_i; \quad g_{ij} \to g_{ij} - \xi_{i,j} - \xi_{j,i}.$$

Substituting the perturbation variables from Eq. (66), we get

$$\Phi \to \Phi - \dot{\xi}^0, \tag{68}$$

$$B_{,i} + S_i \to B_{,i} + S_i - \xi^0_{,i} - \dot{\xi}_{\perp i} - \dot{\zeta}_{,i}, \tag{69}$$

$$2\Psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij} \to 2\Psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij} - \xi_{\perp i,j} - \xi_{\perp j,i} - 2\zeta_{,ij}. \tag{70}$$

Now we need to separate these equations and derive the transformation laws for the individual perturbation variables. This is easy to do if we perform a Fourier transform of Eqs. (68)-(70) and pass to the Fourier space (where every variable is a function of a 3-vector k). A vector V_i is decomposed into scalar and vector components as follows,

$$V_j = \mathrm{i}k_j V^{(S)} + V_j^{(V)}; \quad V^{(S)} \equiv \frac{V_l k_l}{k^2}, \quad V_j^{(V)} \equiv V_j - \mathrm{i}k_j \frac{V_l k_l}{k^2} = V_j - \mathrm{i}k_j V^{(S)}. \tag{71}$$

The idea is first, to project the given vector $V_i(k)$ onto the direction of k_i, and second, to subtract the projection from V_i and to obtain the component of V_i which is transversal to k_i. The imaginary unit factors are added as coefficients at k_j for convenience: with these factors, the decomposition (71) translates to real space as

$$V_j = \partial_j V^{(S)} + V_j^{(V)}.$$

The same procedure applied to a symmetric tensor T_{ij} leads to a decomposition into scalar, vector, and tensor components. Let us go through this procedure in more detail. First, we subtract the trace and obtain the traceless part $T^{(1)}$ of the tensor T,

$$T_{ij}^{(1)} \equiv T_{ij} - \frac{1}{3}\delta_{ij} T_{ll}; \quad T_{ii}^{(1)} = 0.$$

Note the coefficient $\frac{1}{3}$ that depends on the number of spatial dimensions (three). Now we project $T_{ij}^{(1)}$ onto $k_i k_j$ and obtain the scalar component $T^{(S)}$ proportional to $k_i k_j$ and the tensor $T_{ij}^{(2)}$ orthogonal to $k_i k_j$:

$$T_{ij}^{(1)} \equiv \left(-k_i k_j + \frac{1}{3}\delta_{ij} k^2\right) T^{(S)} + T_{ij}^{(2)}; \quad T^{(S)} \equiv -\frac{3}{2}\frac{k_i k_j T_{ij}^{(1)}}{k^4}; \quad T_{ij}^{(2)} k_i k_j = 0.$$

Note that $T_{ij}^{(2)}$ is again a trace-free tensor, $T_{ii}^{(2)} = 0$, due to the subtraction of $\frac{1}{3}k^2\delta_{ij}$ in the first term. Finally, we project $T_{ij}^{(2)}$ onto k_i and k_j separately, to obtain a "vector" part $T_j^{(V)}$ such that $T_j^{(V)} k_j = 0$, and a completely traceless ("tensor") part $T_{ij}^{(T)}$ such that $T_{ij}^{(T)} k_j = 0$ and $T_{ii}^{(T)} = 0$:

$$T_{ij}^{(2)} = \mathrm{i}k_i T_j^{(V)} + \mathrm{i}k_j T_i^{(V)} + T_{ij}^{(T)}; \quad T_j^{(V)} \equiv \mathrm{i}\frac{k_l}{k^2} T_{jl}^{(2)}, \quad T_{ij}^{(T)} \equiv T_{ij}^{(2)} - \mathrm{i}\left(k_i T_j^{(V)} + k_j T_i^{(V)}\right).$$

In real space, the full decomposition is

$$\begin{aligned}
T_{ij} &= \frac{1}{3}T_{ll}\delta_{ij} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta\right) T^{(S)} + \partial_i T_j^{(V)} + \partial_j T_i^{(V)} + T_{ij}^{(T)}. \\
T^{(S)} &\equiv \frac{3}{2}\frac{1}{\Delta^2}\partial_i\partial_j\left(T_{ij} - \frac{1}{3}T_{ll}\delta_{ij}\right); \quad T_{ij}^{(2)} \equiv \left(T_{ij} - \frac{1}{3}T_{ll}\delta_{ij}\right) - \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta\right) T^{(S)}; \\
T_j^{(V)} &= \frac{1}{\Delta}\partial_i T_{ij}^{(2)}, \quad T_{ij}^{(T)} = T_{ij}^{(2)} - \partial_i T_j^{(V)} - \partial_j T_i^{(V)}.
\end{aligned}$$

It may be convenient to gather the "trace" terms (the terms containing δ_{ij}) as one term,

$$T_{ij} = T^{(tr)}\delta_{ij} + \partial_i\partial_j T^{(S)} + \partial_i T_j^{(V)} + \partial_j T_i^{(V)} + T_{ij}^{(T)}, \quad T^{(tr)} \equiv \frac{1}{3}T_{ll} - \frac{1}{3}\Delta T^{(S)}.$$

Note that the perturbation variables Ψ, E, F_i, h_{ij} are obtained by this decomposition method, starting from the symmetric perturbation tensor δg_{ij}, with slight modifications: there are some cosmetic factors of 2 and some minus signs.

Applying the decomposition method to Eqs. (68)-(70), we get the following transformation laws for the perturbation variables,

$$\begin{aligned}
&\Phi \to \Phi - \dot{\xi}^0, \quad B \to B - \xi^0 - \dot{\zeta}, \quad S_i \to S_i - \dot{\xi}_{\perp i}, \\
&E \to E - \zeta, \quad \Psi \to \Psi, \quad F_i \to F_i - \xi_{\perp i}, \quad h_{ij} \to h_{ij}.
\end{aligned}$$

Remarks:

1. It is clear that one can set $F_i = 0$, $B = E = 0$ with a coordinate transformation. Other components will then show whether the geometry is really perturbed or it's just a coordinate transformation of a flat space. In general, there will remain 6 independent components of perturbations $(\Phi, \Psi, S_i, h_{ij})$.

2. These considerations depend *rather crucially* on the silently made assumption that all the metric perturbations vanish, $\delta g_{\mu\nu} \to 0$, at spatial infinity. These boundary conditions are implicitly used when defining the Fourier transforms necessary for the tensor/vector/scalar decompositions (a Fourier transform is undefined without this boundary condition). Alternatively, one may do without Fourier transforms but then one still needs boundary conditions to solve the relevant Poisson equations for components. Without boundary conditions, there is no unique decomposition of the form

$$X_i = A_{,i} + B_i, \quad A = \frac{1}{\Delta}X_{i,i},$$

because the function A is defined up to solutions of $\Delta A = 0$. So the tensor/vector/scalar decomposition is actually *undefined* without a fixed assumption about the boundary conditions. The boundary conditions $\delta g_{\mu\nu} \to 0$ at spatial infinity is a natural, physically motivated set of boundary conditions. An explicit counterexample where these boundary conditions are not satisfied: $g_{\mu\nu} = \mathrm{diag}\,(A, -B, -B, -B)$, where $A \neq 1$, $B \neq 1$ are constants. This metric is flat but one cannot see this by using the perturbation formalism! (The component $\Psi \neq 0$ cannot be removed by a gauge transformation.) The reason is that this $g_{\mu\nu}$ is a "perturbation" of flat metric with Φ and Ψ that do not decay to zero at spatial infinity. So a coordinate transformation with ξ^μ decaying to zero cannot bring this metric to $\eta_{\mu\nu}$.

8 Gravitational radiation I

8.1 Gauge invariant variables

Using the equations derived in Problem 7.4, it is very easy to verify that $D = \Phi - \Psi - \dot{B} + \ddot{E}$ and $S_i - \dot{F}_i$ are invariant under infinitesimal changes of coordinates (i.e. invariant under infinitesimal gauge transformations).

8.2 Detecting gravitational waves

8.2.1 Using distances between particles

(This solution follows Hobson-Efstathiou-Lasenby [2006], §18.4.)

Consider a plane wave moving in the z direction, (all other components of $h_{\mu\nu}$ are zero)

$$h_{xx} = -h_{yy} = A_+ \mathrm{e}^{-\mathrm{i}\omega(t-z)}, \quad h_{xy} = h_{yx} = A_\times \mathrm{e}^{-\mathrm{i}\omega(t-z)}. \tag{72}$$

To detect the presence of this gravitational wave, let us imagine a cloud of particles initially at rest at different positions. The 4-vectors describing the particles are $u^\mu = (1, 0, 0, 0)$, so one can easily see that these particles move along geodesics:

$$\begin{aligned} u^\nu u^\mu{}_{;\nu} &= u^\nu u^\mu{}_{,\nu} + \Gamma^\mu_{\nu\alpha} u^\nu u^\alpha = \Gamma^\mu_{00}, \\ \Gamma^\mu_{\nu\alpha} &= \frac{1}{2}\eta^{\lambda\mu}\left(h_{\lambda\nu,\alpha} + h_{\lambda\alpha,\nu} - h_{\alpha\nu,\lambda}\right), \\ \Gamma^\mu_{00} &= \frac{1}{2}\eta^{\lambda\mu}\left(h_{\lambda 0,0} + h_{\lambda 0,0} - h_{00,\lambda}\right) = 0. \end{aligned}$$

Therefore the coordinates x^μ of each particle remain constant with time. However, the distance between each pair of particles is determined through the spacelike vector $\Delta x^\mu \equiv x^\mu_{(1)} - x^\mu_{(2)}$ as

$$\Delta L^2 \approx \left(\eta_{\mu\nu} + h_{\mu\nu}\right) \Delta x^\mu \Delta x^\nu$$

and will change with time because of the dependence on $h_{\mu\nu}$. Since the only nonzero components of $h_{\mu\nu}$ are the x, y components, it is clear that only changing lengths are between particles that have some separation in the x, y directions. Therefore it is sufficient to consider a ring of particles situated in the $x - y$ plane. The physically measured distances between the particles in the ring will change with time, i.e. the ring will experience a deformation.

To visualize the deformation, it is convenient to make a local coordinate transformation (local in the neighborhood of the ring) such that the metric becomes flat, $g_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} \tilde{x}^\mu \tilde{x}^\nu$ (up to second-order terms). The trick that performs this transformation is the following,

$$\tilde{x}^\mu = x^\mu + \frac{1}{2} h^\mu_\lambda x^\lambda \equiv x^\mu + \frac{1}{2} h_{\alpha\lambda} x^\lambda \eta^{\alpha\mu}.$$

It is easy to check that

$$g_{\mu\nu} x^\mu x^\nu \equiv \left(\eta_{\mu\nu} + h_{\mu\nu}\right) x^\mu x^\nu = \eta_{\mu\nu} \tilde{x}^\mu \tilde{x}^\nu + O(h^2).$$

Therefore, $\tilde{x}^\mu$ can be understood as the (approximate) Cartesian coordinates where the length is given by the usual Pythagorean formula. Now if we compute the shape of the ring in these coordinates, it will be easy to interpret this shape in a straightforward way.

Consider a particle with constant 3-coordinates (x, y, z). After the coordinate transformation, we have

$$\begin{aligned} \tilde{x} &= x + \frac{1}{2}\left(A_+ x + A_\times y\right) \mathrm{e}^{-\mathrm{i}\omega(t-z)}, \\ \tilde{y} &= y + \frac{1}{2}\left(A_\times x - A_+ y\right) \mathrm{e}^{-\mathrm{i}\omega(t-z)}, \\ \tilde{z} &= z. \end{aligned}$$

To visualize the deformation, it is convenient to consider first the case $A_+ \neq 0$, $A_\times = 0$ and then the opposite case. The deformation of the ring is squeezing in one direction and expansion in the orthogonal direction. It follows that A_+ describes a deformation in the two vertical directions, while $A_\times$ describes a deformation in the directions at 45°.

Note that the deformations change the shape of the ring in the same way, except for the rotated orientation. This can be verified by performing a rotation by $\frac{\pi}{4}$,

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix},$$

and then it is straightforward to see that this will exchange A_+ and $A_\times$.

8.2.2 Using geodesic deviation equation

PLEASE NOTE: The commonly found arguments that use the geodesic deviation equation are suspect because the geodesic deviation equation uses *coordinates* ξ^μ rather than gauge-invariant quantities. A cloud of particles at rest in the gravitational field $h_{\mu\nu}$ described by Eq. (72) will stay indefinitely at rest in the coordinate system ($\xi^\mu = \text{const}$) even though the distances between particles will change with time. See arxiv:gr-qc/0605033 for nice explanations. The solution given above is simple and straightforward. The argument using the geodesic deviation (see Carroll, Chapter 6, p. 152-154) goes like this:

The geodesic deviation equation can be simplified for a deviation vector S^σ corresponding to nonrelativistic (almost stationary) particles moving with 4-velocity approximately equal to $(1, 0, 0, 0)$,

$$\frac{d^2 S^\sigma}{dt^2} = R^\sigma_{\ 00\lambda} S^\lambda.$$

The Riemann tensor to first order in h can be expressed as

$$R^\sigma_{\ 00\lambda} = \ddot{h}^\sigma_\lambda$$

(note that $h_{\mu 0} = 0$). Therefore, the geodesic deviation equation becomes

$$\begin{aligned}
\ddot{S}^\sigma &= \ddot{h}^\sigma_\lambda S^\lambda, \\
\ddot{S}^x &= \omega^2 \left(h_{xx} S^x + h_{xy} S^y\right) = \omega^2 \left(A_+ S^x + A_\times S^y\right) \mathrm{e}^{-\mathrm{i}\omega(t-z)}, \\
\ddot{S}^y &= \omega^2 \left(h_{yx} S^x + h_{yy} S^y\right) = \omega^2 \left(A_\times S^x - A_+ S^y\right) \mathrm{e}^{-\mathrm{i}\omega(t-z)},
\end{aligned}$$

and there is no change in the z direction.

8.3 Poisson equation

The general solution of the Poisson equation,

$$\Delta\phi = 4\pi\rho,$$

with boundary conditions $\phi \to 0$ at infinity, is easy to find using the Fourier transform:

$$\begin{aligned}
-k^2 \phi(\mathbf{k}) &= 4\pi\rho(\mathbf{k}), \\
\phi(\mathbf{x}) &= -\int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{4\pi\rho(\mathbf{k})}{k^2} = -\int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{4\pi}{k^2} \int d^3\mathbf{y} e^{-i\mathbf{k}\cdot\mathbf{y}} \rho(\mathbf{y}) = \int d^3\mathbf{y}\, \rho(\mathbf{y}) G(\mathbf{x}-\mathbf{y}),
\end{aligned}$$

where $G(\mathbf{x})$ is the Green's function,

$$G(\mathbf{x}) = -\int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{4\pi}{k^2} = -\frac{1}{\pi} \int_0^\infty dk \int_0^\pi d\theta \sin\theta\, e^{ik|\mathbf{x}|\cos\theta} = -\frac{2}{\pi\,|\mathbf{x}|} \int_0^\infty \frac{dk}{k} \sin kx = -\frac{1}{|\mathbf{x}|}.$$

Here we used the known integral

$$\int_0^\infty \frac{\sin z}{z} dz = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\sin z}{z} dz = \frac{1}{2}\pi.$$

Therefore

$$\phi(\mathbf{x}) = -\int \frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \rho(\mathbf{y}). \tag{73}$$

One can denote this integral more concisely,

$$\phi = 4\pi \frac{1}{\Delta} \rho,$$

where the operator $\frac{1}{\Delta}$ is just a shorthand notation for the integral in Eq. (73).

Note that the function ρ must fall off sufficiently rapidly as $|\mathbf{x}| \to \infty$ or else the integral (73) will not converge. It is sufficient that $|\rho(\mathbf{x})| \sim |\mathbf{x}|^{-2-\varepsilon}$ at large $|\mathbf{x}|$ (where $\varepsilon > 0$).

8.4 Metric perturbations 1

An arbitrary 3-vector X_i (such as $T^0{}_i$) is decomposed into scalar and vector parts as follows,

$$X_i = a_{,i} + b_i, \quad b_{i,i} = 0.$$

To determine an explicit expression for a, let us compute the divergence of X_i,

$$X_{i,i} = a_{,ii} = \Delta a.$$

Therefore

$$a(\mathbf{x}) = -\frac{1}{4\pi} \int \frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} X_{i,i}(\mathbf{y}).$$

One can write more concisely

$$a = \frac{1}{\Delta} X_{i,i}.$$

8.5 Metric perturbations 2

The energy-momentum tensor $T_{\mu\nu}$ is decomposed as

$$T^0_i = \alpha_{,i} + \beta_i, \quad \alpha \equiv \frac{1}{\Delta} T^0_{k,k}, \quad \beta_i \equiv T^0_i - \left[\frac{1}{\Delta} T^0_{k,k}\right]_{,i};$$
$$T^i_k = \mu\delta_{ik} + \lambda_{,ik} + \sigma_{i,k} + \sigma_{k,i} + T^{(T)i}_k,$$
$$\beta_{i,i} = \sigma_{i,i} = 0, \quad T^{(T)i}_i = 0, \quad T^{(T)i}_{k,i} = 0.$$

We need to verify that the equation

$$-\frac{1}{16\pi G}(\dot{S}_i - \ddot{F}_i) = \sigma_i, \tag{74}$$

which represents the vector part of the spatial Einstein equation (here σ_i is the vector part of the spatial T_{ij}), also follows from the conservation of $T_{\mu\nu}$ and from the other Einstein equations.

To calculate the components λ, μ of the EMT, we compute

$$T^i_i = \Delta\lambda + 3\mu,$$
$$T^i_{k,i} = \Delta\lambda_{,k} + \mu_{,k} + \Delta\sigma_k,$$
$$T^i_{k,ik} = \Delta\Delta\lambda + \Delta\mu.$$

Now we solve this system of equations and find

$$\mu = \frac{1}{2}\left(T^i_i - \frac{1}{\Delta} T^i_{k,ik}\right), \quad \lambda = \frac{3}{2}\frac{1}{\Delta} T^i_{k,ik} - \frac{1}{2} T^i_i,$$
$$\sigma_j = \frac{1}{\Delta} T^i_{j,i} - \frac{1}{\Delta}\left[\frac{1}{\Delta} T^i_{k,ik}\right]_{,j}, \quad {}^{(T)}T^i_k = T^i_k - \mu\delta^i_k - \lambda_{,ik} - \sigma_{i,k} - \sigma_{k,i}.$$

Note that the operator $\frac{1}{\Delta^2}$ applied to a function $f(\mathbf{x})$ is defined only if the function f has a sufficiently fast decay at $|\mathbf{x}| \to \infty$. It is sufficient that $|f(\mathbf{x})| \sim |\mathbf{x}|^{-3-\varepsilon}$ with $\varepsilon > 0$ at large $|\mathbf{x}|$. This is a faster decay than that required by the operator $\frac{1}{\Delta}$.

The Einstein equations are

$$2\Delta\Psi = 8\pi G T^0_0,$$
$$2\dot{\Psi}_{,i} + \frac{1}{2}\Delta\tilde{S}_i = 8\pi G T^0_i = 8\pi G\left(\alpha_{,i} + \beta_i\right),$$
$$D_{,ij} - \delta_{ij}\left(\Delta D + 2\ddot{\Psi}\right) - \frac{1}{2}\left[\dot{\tilde{S}}_{i,j} + \dot{\tilde{S}}_{j,i}\right] + \frac{1}{2}\Box h_{ij} = 8\pi G T^i_j = 8\pi G\left(\mu\delta_{ik} + \lambda_{,ik} + \sigma_{i,k} + \sigma_{k,i} + T^{(T)i}_k\right),$$

where we have denoted for brevity

$$\tilde{S}_i \equiv S_i - \dot{F}_i, \quad D \equiv \Phi - \Psi + B - \dot{E},$$

which are gauge-invariant variables. In the 3+1 decomposition, the Einstein equations become

$$\Delta\Psi = 4\pi G T^0_0, \tag{75}$$
$$\dot{\Psi} = 4\pi G\alpha, \tag{76}$$
$$\Delta\tilde{S}_i = 16\pi G\beta_i, \tag{77}$$
$$D = 8\pi G\lambda, \tag{78}$$
$$\Delta D + 2\ddot{\Psi} = -8\pi G\mu, \tag{79}$$
$$\dot{\tilde{S}}_i = -16\pi G\sigma_i, \tag{80}$$
$$\Box h_{ij} = 16\pi G T^{(T)i}_j. \tag{81}$$

The conservation law of the EMT in 3+1 decomposition looks like this,

$$T^0_{0,0} + T^j_{0,j} = 0, \quad T^0_{i,0} + T^j_{i,j} = 0. \tag{82}$$

This gives

$$\dot{T}^0_0 = \Delta\alpha, \quad \dot{\alpha}_{,i} + \dot{\beta}_i + \Delta\lambda_{,i} + \mu_{,i} + \Delta\sigma_i = 0,$$

therefore

$$\dot{T}^0_0 = \Delta\alpha, \quad \dot{\alpha} + \Delta\lambda + \mu = 0, \quad \dot{\beta}_i + \Delta\sigma_i = 0. \tag{83}$$

Then it is easy to see that Eqs. (76), (79), and (80) are consequences of Eqs. (75), (77), (78), and the conservation laws (83). In particular,

$$\dot{\tilde{S}}_i = \partial_t \frac{1}{\Delta} 16\pi G\beta_i = -\frac{1}{\Delta} 16\pi G\Delta\sigma_i = -16\pi G\sigma_i.$$

9 Gravitational radiation II

9.1 Projection of the matter tensor

a) First note that P_{ab} is a projector,

$$P_{ab}P_{bc} = P_{ac},$$

and its image has dimension 2, that is, the trace of P_{ab} is 2,

$$P_{ii} = 3 - n_i n_i = 2.$$

Therefore, for any X_{ab} we have

$$^{(T)}X_{ii} = P_{ia}X_{ab}P_{bi} - \frac{1}{2}P_{ii}P_{ab}X_{ab} = P_{ab}X_{ab} - \frac{1}{2}2P_{ab}X_{ab} = 0.$$

b) We compute

$$^{(T)}X_{ik,i} = \left(P_{ia}X_{ab}P_{bk} - \frac{1}{2}P_{ik}P_{ab}X_{ab}\right)_{,i} = \left(P_{ia}P_{bk} - \frac{1}{2}P_{ik}P_{ab}\right)_{,i} X_{ab} + \left(P_{ia}P_{bk} - \frac{1}{2}P_{ik}P_{ab}\right) X_{ab,i}. \tag{84}$$

Note that the projection kills any component proportional to R_i because $P_{ia}R_i = 0$. At the same time, $X_{ab,i}$ is proportional to R_i because

$$X_{ab,i} = \left[X(t - |\vec{R}|)\right]_{,i} = -\frac{R_i}{R}X'.$$

Therefore the second term in Eq. (84) vanishes:

$$\left(P_{ia}P_{bk} - \frac{1}{2}P_{ik}P_{ab}\right) R_i = 0.$$

So only the first term remains,

$$^{(T)}X_{ik,i} = \left(P_{ia}P_{bk} - \frac{1}{2}P_{ik}P_{ab}\right)_{,i} X_{ab}.$$

However, this term contains derivatives of P_{ab}, which are also sometimes proportional to R_i. We compute

$$\begin{aligned}
P_{ik,a} &= -n_{i,a}n_k - n_i n_{k,a}; \quad n_{i,a} = \left(\frac{R_i}{R}\right)_{,a} = \frac{R_{i,a}}{R} - \frac{R_i}{R^2}\left|R\right|_{,a} = \frac{\delta_{ia}}{R} - \frac{R_i R_a}{R^3} = \frac{1}{R}P_{ia}, \\
P_{ik,a} &= -\frac{1}{R}\left(P_{ai}n_k + P_{ak}n_i\right), \quad P_{ik,i} = -\frac{2}{R}n_k, \qquad \text{(note that } P_{ak}n_k = 0) \\
\left(P_{ia}P_{bk} - \frac{1}{2}P_{ik}P_{ab}\right)_{,i} &= P_{ia,i}P_{bk} - \frac{1}{2}P_{ik,i}P_{ab} + P_{ia}P_{bk,i} - \frac{1}{2}P_{ik}P_{ab,i} \\
&= -\frac{2}{R}n_a P_{bk} + \frac{1}{R}n_k P_{ab} - \frac{1}{R}P_{ia}\left(P_{bi}n_k + P_{ik}n_b\right) + \frac{1}{2R}P_{ik}\left(P_{ai}n_b + P_{bi}n_a\right) \\
&= \frac{1}{R}\left(-2n_a P_{bk} + n_k P_{ab} - P_{ab}n_k - P_{ak}n_b + \frac{1}{2}P_{ak}n_b + \frac{1}{2}P_{bk}n_a\right) \\
&= -\frac{1}{R}\left(\frac{3}{2}P_{bk}n_a + \frac{1}{2}P_{ak}n_b\right).
\end{aligned}$$

This is higher-order in $1/|\vec{R}|$ than P_{ab}, as required.

9.2 Matter sources

The question is to verify the following property,

$$^{(T)}X_{ik} = {}^{(T)}Q_{ik},$$

where

$$Q_{ik} = X_{ik} - \int \frac{1}{3}\delta_{ik}\, r^2\, T^0{}_0\, d^3r.$$

It is easy to see that X_{ik} differs from Q_{ik} only by a term of the form $A(t,R)\delta_{ij}$. The transverse-traceless part of δ_{ij} is zero,

$$\left(P_{ia}P_{bk} - \frac{1}{2}P_{ik}P_{ab}\right)\delta_{ab} = 0.$$

Therefore the transverse-traceless parts of X_{ik} and Q_{ik} are the same.

9.3 Energy-momentum tensor of gravitational waves

See Hobson-Efstathiou-Lasenby [2006], §17.11.

We need to compute the second-order terms in the Einstein tensor. The idea is to separate the second-order terms already in the Ricci tensor. We will also try to simplify things by using the fact that $h_{\mu\nu}$ is purely transverse-traceless; $h_{0\alpha} = 0$, $h_{ii} = 0$, $h_{ik,i} = 0$. It follows that

$$\eta^{\mu\nu} h_{\mu\nu} = 0, \quad h_{\mu\nu}{}^{,\mu} = 0.$$

Also, it is given that the EMT of matter vanishes, $T_{\mu\nu} = 0$, which we will use below.

First we decompose the metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu};$$

note that now indices are *always* raised and lowered using $\eta_{\mu\nu}$. We need to compute the Ricci tensor to second order. The Christoffel symbol up to second order is

$$\begin{aligned}
\Gamma^{\lambda}_{\alpha\beta} &= \frac{1}{2}\left(\eta^{\lambda\mu} - h^{\lambda\mu}\right)\left(h_{\mu\alpha,\beta} + h_{\mu\beta,\alpha} - h_{\alpha\beta,\mu}\right) = \Gamma^{(1)\lambda}_{\alpha\beta} + \Gamma^{(2)\lambda}_{\alpha\beta}, \\
\Gamma^{(1)\lambda}_{\alpha\beta} &= \frac{1}{2}\eta^{\lambda\mu}\left(h_{\alpha\mu,\beta} + h_{\beta\mu,\alpha} - h_{\alpha\beta,\mu}\right), \\
\Gamma^{(2)\lambda}_{\alpha\beta} &= -\frac{1}{2}h^{\lambda\mu}\left(h_{\alpha\mu,\beta} + h_{\beta\mu,\alpha} - h_{\alpha\beta,\mu}\right).
\end{aligned}$$

The Ricci tensor is

$$\begin{aligned}
R_{\alpha\beta} &= \Gamma^{\lambda}_{\alpha\beta,\lambda} - \Gamma^{\lambda}_{\lambda\alpha,\beta} + \Gamma^{\lambda}_{\lambda\nu}\Gamma^{\nu}_{\alpha\beta} - \Gamma^{\lambda}_{\beta\nu}\Gamma^{\nu}_{\alpha\lambda} = R^{(1)}_{\alpha\beta} + R^{(2)}_{\alpha\beta}, \\
R^{(1)}_{\alpha\beta} &= \Gamma^{(1)\lambda}_{\alpha\beta,\lambda} - \Gamma^{(1)\lambda}_{\lambda\alpha,\beta}, \\
R^{(2)}_{\alpha\beta} &= \Gamma^{(2)\lambda}_{\alpha\beta,\lambda} - \Gamma^{(2)\lambda}_{\lambda\alpha,\beta} + \Gamma^{(1)\lambda}_{\lambda\nu}\Gamma^{(1)\nu}_{\alpha\beta} - \Gamma^{(1)\lambda}_{\beta\nu}\Gamma^{(1)\nu}_{\alpha\lambda}.
\end{aligned}$$

Let us now evaluate these expressions and simplify as much as possible, as early as possible:

$$\begin{aligned}
\Gamma^{(1)\lambda}_{\alpha\lambda} &= \frac{1}{2}\eta^{\lambda\mu}h_{\lambda\mu,\alpha} = \frac{1}{2}\left(\eta^{\lambda\mu}h_{\lambda\mu}\right)_{,\alpha} = 0, \\
R^{(1)}_{\alpha\beta} &= \Gamma^{(1)\lambda}_{\alpha\beta,\lambda} - \Gamma^{(1)\lambda}_{\lambda\alpha,\beta} = \frac{1}{2}\eta^{\lambda\mu}\left(h_{\alpha\mu,\beta\lambda} + h_{\beta\mu,\alpha\lambda} - h_{\alpha\beta,\mu\lambda}\right) = -\frac{1}{2}\Box h_{\alpha\beta},
\end{aligned}$$

because of the transverse traceless property of $h_{\mu\nu}$. Now, since $R^{(1)}_{\alpha\beta}$ is found from the first-order Einstein equation

$$R^{(1)}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}R^{(1)} = 8\pi G T_{\alpha\beta},$$

and it is given that $T_{\alpha\beta} = 0$. Hence, we have $\Box h_{\alpha\beta} = 0$.

Let us now evaluate derivatives of the second-order terms in the Christoffel symbols:

$$\begin{aligned}
\Gamma^{(2)\lambda}_{\alpha\lambda} &= -\frac{1}{2}h^{\lambda\mu}\left(h_{\alpha\mu,\lambda} + h_{\lambda\mu,\alpha} - h_{\alpha\lambda,\mu}\right) = -\frac{1}{2}h^{\lambda\mu}h_{\lambda\mu,\alpha}, \\
-\Gamma^{(2)\lambda}_{\alpha\lambda,\beta} &= \frac{1}{2}h^{\lambda\mu}h_{\lambda\mu,\alpha\beta} + \frac{1}{2}h^{\lambda\mu}_{,\alpha}h_{\lambda\mu,\beta}, \\
\Gamma^{(2)\lambda}_{\alpha\beta,\lambda} &= -\frac{1}{2}h^{\lambda\mu}\left(h_{\alpha\mu,\beta\lambda} + h_{\beta\mu,\alpha\lambda} - h_{\alpha\beta,\mu\lambda}\right)
\end{aligned}$$

(in the last line we used $h^{\lambda\mu}_{,\lambda} = 0$). Finally, we tackle the term $\Gamma^{(1)\lambda}_{\beta\nu}\Gamma^{(1)\nu}_{\alpha\lambda}$. In this term, it helps to write

$$\Gamma^{(1)\lambda}_{\alpha\beta} = \frac{1}{2}\left(h^{\lambda}_{\alpha,\beta} + h^{\lambda}_{\beta,\alpha} - h^{,\lambda}_{\alpha\beta}\right),$$

where again the indices are raised via $\eta^{\mu\nu}$ since we only need this term to first order. Then we can simplify this expression by grouping together terms where α, β appear in similar positions:

$$\begin{aligned}
4\Gamma^{(1)\lambda}_{\beta\nu}\Gamma^{(1)\nu}_{\alpha\lambda} &= \left(h^{\lambda}_{\beta,\nu} + h^{\lambda}_{\nu,\beta} - h^{,\lambda}_{\beta\nu}\right)\left(h^{\nu}_{\alpha,\lambda} + h^{\nu}_{\lambda,\alpha} - h^{,\nu}_{\alpha\lambda}\right) \\
\text{(expand brackets)} &= h^{\lambda}_{\beta,\nu}h^{\nu}_{\alpha,\lambda} + h^{\lambda}_{\beta,\nu}h^{\nu}_{\lambda,\alpha} - h^{\lambda}_{\beta,\nu}h^{,\nu}_{\alpha\lambda} + h^{\lambda}_{\nu,\beta}h^{\nu}_{\alpha,\lambda} + h^{\lambda}_{\nu,\beta}h^{\nu}_{\lambda,\alpha} - h^{\lambda}_{\nu,\beta}h^{,\nu}_{\alpha\lambda} - h^{,\lambda}_{\beta\nu}h^{\nu}_{\alpha,\lambda} - h^{,\lambda}_{\beta\nu}h^{\nu}_{\lambda,\alpha} + h^{,\lambda}_{\beta\nu}h^{,\nu}_{\alpha\lambda} \\
\text{(move, rename } \lambda,\nu) &= h^{,\nu}_{\beta\lambda}h^{,\lambda}_{\alpha\nu} + h_{\beta\lambda,\nu}h^{\lambda\nu}_{,\alpha} - h^{\lambda}_{\beta,\nu}h^{,\nu}_{\alpha\lambda} + h^{\lambda\nu}_{,\beta}h_{\alpha\nu,\lambda} + h^{,\lambda}_{,\beta}h_{\lambda\nu,\alpha} - h^{\lambda\nu}_{,\beta}h_{\alpha\lambda,\nu} - h^{\lambda}_{\beta,\nu}h^{,\nu}_{\lambda\alpha} - h_{\beta\nu,\lambda}h^{\lambda\nu}_{,\alpha} + h^{,\nu}_{\beta\lambda}h^{,\lambda}_{\alpha\nu} \\
\text{(gather terms)} &= 2h^{,\nu}_{\beta\lambda}h^{,\lambda}_{\alpha\nu} + \left(h_{\beta\lambda,\nu} - h_{\beta\nu,\lambda}\right)h^{\lambda\nu}_{,\alpha} - 2h^{\lambda}_{\beta,\nu}h^{,\nu}_{\alpha\lambda} + h^{\lambda\nu}_{,\beta}\left(h_{\alpha\nu,\lambda} - h_{\alpha\lambda,\nu}\right) + h^{\lambda\nu}_{,\beta}h_{\lambda\nu,\alpha} \\
\text{(symmetry of } h_{\mu\nu}) &= h_{\lambda\nu,\alpha}h^{\lambda\nu}_{,\beta} + 2h^{,\nu}_{\beta\lambda}h^{,\lambda}_{\alpha\nu} - 2h^{\lambda}_{\beta,\nu}h^{,\nu}_{\alpha\lambda}.
\end{aligned}$$

Finally, we put together the expression for $R^{(2)}_{\alpha\beta}$:

$$\begin{aligned}
R^{(2)}_{\alpha\beta} &= \frac{1}{2}h^{\lambda\mu}\left(-h_{\alpha\mu,\beta\lambda} - h_{\beta\mu,\alpha\lambda} + h_{\alpha\beta,\mu\lambda} + h_{\lambda\mu,\alpha\beta}\right) + \frac{1}{2}h^{\lambda\mu}_{,\alpha}h_{\lambda\mu,\beta} - \frac{1}{4}\left(h_{\lambda\nu,\alpha}h^{\lambda\nu}_{,\beta} + 2h^{,\nu}_{\beta\lambda}h^{,\lambda}_{\alpha\nu} - 2h^{\lambda}_{\beta,\nu}h^{,\nu}_{\alpha\lambda}\right) \\
&= \frac{1}{2}h^{\lambda\mu}\left(-h_{\alpha\mu,\beta\lambda} - h_{\beta\mu,\alpha\lambda} + h_{\alpha\beta,\mu\lambda} + h_{\lambda\mu,\alpha\beta}\right) + \frac{1}{4}h^{\lambda\mu}_{,\alpha}h_{\lambda\mu,\beta} + \frac{1}{2}\left(h^{,\nu}_{\alpha\lambda}h^{\lambda}_{\beta,\nu} - h^{,\nu}_{\beta\lambda}h^{,\lambda}_{\alpha\nu}\right).
\end{aligned} \tag{85}$$

The Ricci scalar is

$$R^{(2)} = \eta^{\alpha\beta} R^{(2)}_{\alpha\beta} = \frac{1}{2} h^{\lambda\mu} \Box h_{\lambda\mu} + \frac{1}{4} h^{\lambda\mu}_{,\alpha} h^{,\alpha}_{\lambda\mu} + \frac{1}{2} \left(h^{,\nu}_{\alpha\lambda} h^{\alpha\lambda}_{,\nu} - h^{\alpha\lambda,\nu} h_{\alpha\nu,\lambda} \right) = \frac{3}{4} h^{\lambda\mu,\alpha} h_{\lambda\mu,\alpha} - \frac{1}{2} h^{\alpha\lambda,\nu} h_{\alpha\nu,\lambda}, \tag{86}$$

where we again used the transverse traceless property of $h_{\mu\nu}$ and also $\Box h_{\alpha\beta} = 0$. Note that the first-order Ricci scalar is zero, $R^{(1)} = 0$, since $T_{\alpha\beta} = 0$. For this reason we may use $\eta^{\alpha\beta}$ in Eq. (86), otherwise we would have to write $(\eta + h)\left(R^{(1)} + R^{(2)}\right)$ and pick up a second-order term $hR^{(1)}$.

Again, since $R^{(1)} = 0$, we may use $\eta_{\mu\nu}$ rather than $g_{\mu\nu}$ to compute the Einstein tensor:

$$G^{(2)}_{\alpha\beta} = R^{(2)}_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} R^{(2)}.$$

We do not write the answer explicitly since it is a combination of Eqs. (85) and (86).

Now let us perform an averaging of the quantity $G^{(2)}_{\alpha\beta}$ over both space and time. In other words, we integrate $G^{(2)}_{\alpha\beta}$ over a 4-dimensional region such that $h_{\mu\nu} = 0$ and $h_{\mu\nu,\alpha} = 0$ on the boundary of that region. Then $\langle \partial_\mu (...) \rangle = 0$ and so we may integrate by parts, for example

$$\langle A_\mu B_{\nu,\alpha} \rangle = - \langle A_{\mu,\alpha} B_\nu \rangle ,$$

as long as $A_\mu B_\nu$ contains first powers of $h_{\alpha\beta}$ or $h_{\alpha\beta,\gamma}$, so that boundary terms vanish. Then, for example,

$$\begin{aligned} \left\langle h^{\lambda\mu} h_{\lambda\mu,\alpha\beta} \right\rangle &= - \left\langle h^{\lambda\mu}_{,\alpha} h_{\lambda\mu,\beta} \right\rangle , \\ \left\langle h^{\lambda\mu} h_{\alpha\nu,\beta\lambda} \right\rangle &= - \left\langle h^{\lambda\mu}_{,\lambda} h_{\alpha\nu,\beta} \right\rangle = 0, \\ \left\langle h^{,\nu}_{\alpha\lambda} h_{\beta\mu,\nu} \right\rangle &= - \left\langle h_{\alpha\lambda} \Box h_{\beta\mu} \right\rangle = 0, \end{aligned} \tag{87}$$

by $\Box h_{\alpha\beta} = 0$ and by the transverse traceless property of $h_{\mu\nu}$. Many terms cancel in this way; for instance, $\left\langle R^{(2)} \right\rangle = 0$. Finally, we get

$$\begin{aligned} \left\langle G^{(2)}_{\alpha\beta} \right\rangle &= \left\langle \frac{1}{2} h^{\lambda\mu} \left(-h_{\alpha\mu,\beta\lambda} - h_{\beta\mu,\alpha\lambda} + h_{\alpha\beta,\mu\lambda} + h_{\lambda\mu,\alpha\beta} \right) + \frac{1}{4} h^{\lambda\mu}_{,\alpha} h_{\lambda\mu,\beta} + \frac{1}{2} \left(h^{,\nu}_{\alpha\lambda} h^{\lambda}_{\beta,\nu} - h^{,\nu}_{\beta\lambda} h^{,\lambda}_{\alpha\nu} \right) \right\rangle \\ &= \frac{1}{2} \left\langle h^{\lambda\mu} h_{\lambda\mu,\alpha\beta} \right\rangle + \frac{1}{4} \left\langle h^{\lambda\mu}_{,\alpha} h_{\lambda\mu,\beta} \right\rangle = -\frac{1}{4} \left\langle h^{\lambda\mu}_{,\alpha} h_{\lambda\mu,\beta} \right\rangle \end{aligned}$$

using Eq. (87). Finally, we obtain the required equation,

$$^{(\mathrm{GW})}T_{\mu\nu} = -\frac{1}{8\pi G} \left\langle G^{(2)}_{\alpha\beta} \right\rangle = \frac{1}{32\pi G} \left\langle h^{\lambda\mu}_{,\alpha} h_{\lambda\mu,\beta} \right\rangle = \frac{1}{32\pi G} \left\langle h^{ij}_{,\alpha} h_{ij,\beta} \right\rangle .$$

9.4 Power of emitted radiation

To derive the relations

$$\begin{aligned} \int n^l n^m \frac{d\Omega}{4\pi} &= \frac{1}{3} \delta^{lm}, \\ \int n^l n^m n^k n^r \frac{d\Omega}{4\pi} &= \frac{1}{15} (\delta^{lm} \delta^{kr} + \delta^{lk} \delta^{mr} + \delta^{lk} \delta^{mr}), \end{aligned}$$

let us consider the generating function

$$g_\Omega(q_l) \equiv \int \frac{d\Omega}{4\pi} \exp \left[-\mathrm{i} n^l q_l \right] ,$$

which is a function of a vector argument q_l. After computing $g_\Omega(q_l)$ it will be easy to obtain integrals such as the above:

$$\int n^l n^m \frac{d\Omega}{4\pi} = \mathrm{i} \frac{\partial}{\partial q_l} \mathrm{i} \frac{\partial}{q_m} g_\Omega(q_j) \bigg|_{q_j = 0} , \quad \text{etc.}$$

The computation is easy if we introduce spherical coordinates with the z axis parallel to the vector q_l, then $n^l q_l = |q| \cos\theta$, where $|q| \equiv \sqrt{q_l q_l}$, and then we have

$$\begin{aligned} g_\Omega(q_l) &= \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \exp \left[-\mathrm{i} |q| \cos\theta \right] = \frac{1}{4\pi} 2\pi \frac{-2\mathrm{i} \sin|q|}{-\mathrm{i} |q|} \\ &= \frac{\sin|q|}{|q|} = 1 - \frac{1}{3!} q_l q_l + \frac{1}{5!} (q_l q_l)^2 - \frac{1}{7!} (q_l q_l)^3 + ... \end{aligned}$$

We have used the Taylor expansion for convenience of evaluating derivatives at $|q| = 0$. These derivatives can be found as follows,

$$\begin{aligned}
\frac{\partial g_\Omega}{\partial q_l} &= -\frac{2}{3!}q_l + \frac{4}{5!}q_l\,|q|^2 - ... = \left(-\frac{1}{3} + \frac{1}{30}\,|q|^2\right) q_l, \\
\frac{\partial^2 g_\Omega}{\partial q_k \partial q_l} &= \left(-\frac{1}{3} + \frac{1}{30}\,|q|^2\right)\delta_{kl} + \frac{1}{15}q_l q_k, \\
\frac{\partial^3 g_\Omega}{\partial q_j \partial q_k \partial q_l} &= \frac{1}{15}\left(q_j\delta_{kl} + q_k\delta_{jl} + q_l\delta_{jk}\right), \\
\frac{\partial^4 g_\Omega}{\partial q_j \partial q_k \partial q_l \partial q_m} &= \frac{1}{15}\left(\delta_{jm}\delta_{kl} + \delta_{km}\delta_{jl} + \delta_{lm}\delta_{jk}\right).
\end{aligned}$$

Now we compute the intensity of radiation. The flux of radiation in the direction n_k is ${}^{(GW)}T_{0k}n^k$, and we need to integrate this flux through a sphere of radius R:

$$\frac{dE}{dt} = R^2 \int d^2\Omega\, {}^{(GW)}T_{0k}n^k = \frac{R^2}{32\pi G}\int d^2\Omega \left\langle h^{ij}_{,0}h_{ij,k}\right\rangle n^k.$$

The perturbation h_{ij} is found from the Einstein equation. It was derived in the lecture that, in the leading order in $1/R$, we have

$$\begin{aligned}
h_{ij} &= 2G\frac{{}^{(TT)}\ddot{Q}_{ij}(t - |\vec{R}|)}{|\vec{R}|}, \\
{}^{(TT)}Q_{ij} &\equiv \left(P_{ai}P_{bj} - \frac{1}{2}P_{ab}P_{ij}\right)Q_{ab}.
\end{aligned}$$

The projection tensor P_{ij} is defined in Problem 9.1. The tensor Q_{ij} is defined by

$$Q_{ij}(t) \equiv \int d^3\mathbf{x}\left(x_i x_j - \frac{1}{3}\,|\mathbf{x}|^2\,\delta_{ij}\right)T_{00}(\mathbf{x}, t)$$

and is by definition trace-free, $Q_{ii} = 0$. Thus we have

$$\begin{aligned}
h_{ij,k} &= \frac{16\pi G}{R}{}^{(TT)}\ddot{Q}_{ij}(t - |\mathbf{R}|)\frac{R_k}{R}, \\
\frac{dE}{dt} &= \frac{G}{8\pi}\int d^2\Omega \left\langle {}^{(TT)}\dddot{Q}_{ij}{}^{(TT)}\dddot{Q}_{ij}\right\rangle.
\end{aligned}$$

It remains to compute the average over the sphere of

$${}^{(TT)}\dddot{Q}_{ij}{}^{(TT)}\dddot{Q}_{ij}.$$

Consider any symmetric, trace-free tensor A_{ij} instead of $\dddot{Q}_{ij}$; the transverse-traceless part of A_{ij} is defined by

$${}^{(TT)}A_{ij} \equiv \left(P_{ai}P_{bj} - \frac{1}{2}P_{ab}P_{ij}\right)A_{ab}.$$

Since $A_{ii} = 0$, we have $A_{ab}P_{ab} = -A_{ab}n_a n_b$ and so

$$\begin{aligned}
{}^{(TT)}A_{ij}{}^{(TT)}A_{ij} &= \left(P_{ai}P_{bj} - \frac{1}{2}P_{ab}P_{ij}\right)\left(P_{ci}P_{dj} - \frac{1}{2}P_{cd}P_{ij}\right)A_{ab}A_{cd} \\
&= \left(P_{ac}P_{bd} - \frac{1}{2}P_{ab}P_{cd}\right)A_{ab}A_{cd} = \left(P_{ac}P_{bd} - \frac{1}{2}n_a n_b n_c n_d\right)A_{ab}A_{cd} \\
&= A_{ab}A_{ab} - 2A_{ac}A_{bc}n_a n_b + \frac{1}{2}A_{ab}A_{cd}n_a n_b n_c n_d.
\end{aligned}$$

After integration over the sphere, according to formulas derived above, we have (again note that $\delta_{ab}A_{ab} = 0$ and $A_{ab} = A_{ba}$)

$$\begin{aligned}
\frac{1}{4\pi}\int d^2\Omega\, A_{ac}A_{bc}n_a n_b &= \frac{1}{3}A_{ac}A_{bc}\delta_{ab} = \frac{1}{3}A_{ab}A_{ab}, \\
\frac{1}{4\pi}\int d^2\Omega\, A_{ab}A_{cd}n_a n_b n_c n_d &= \frac{1}{15}A_{ab}A_{cd}\left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right) = \frac{2}{15}A_{ab}A_{ab},
\end{aligned}$$

and thus

$$\frac{1}{4\pi}\int d^2\Omega\, {}^{(TT)}A_{ij}{}^{(TT)}A_{ij} = A_{ab}A_{ab}\left(1 - \frac{2}{3} + \frac{1}{2}\frac{2}{15}\right) = \frac{2}{5}A_{ab}A_{ab}.$$

Finally, substituting $\dddot{Q}_{ij}$ instead of A_{ij}, we find

$$\frac{dE}{dt} = \frac{G}{2}\frac{1}{4\pi}\int d^2\Omega \left\langle {}^{(TT)}\dddot{Q}_{ij}\,{}^{(TT)}\dddot{Q}_{ij}\right\rangle = \frac{G}{5}\left\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\right\rangle . \tag{88}$$

The angular brackets $\langle ... \rangle$ indicate that we must perform an averaging over spacetime domains. This means, for us, that we need to average over time (since Q_{ij} is a function only of time). Averaging is performed over timescales larger than the typical timescale of change in the source. For instance, if the source is a rotating body, then averaging must be performed over several periods of rotation.

10 Sample exam problems

10.1 Metric and curvature

1. The answer to the torus: $ds^2 = a^2 d\phi^2 + (b + a\sin\phi)^2\, d\theta^2$.
2. The form $\omega^r = dr$.
3. This spacetime is flat and (u, v) are the Rindler coordinates.

10.2 Geodesics

(a) This is a metric of de Sitter spacetime.

(b) Yes, it is a geodesic. $u^\mu = (1, 0, 0, 0)$;

$$u^\nu u^\mu{}_{;\nu} = u^\mu_{,0} + \Gamma^\mu_{00} = \frac{1}{2} g^{\lambda\mu}\left(g_{\mu 0,0} + g_{0\mu,0} - g_{00,\mu}\right) = 0. \tag{89}$$

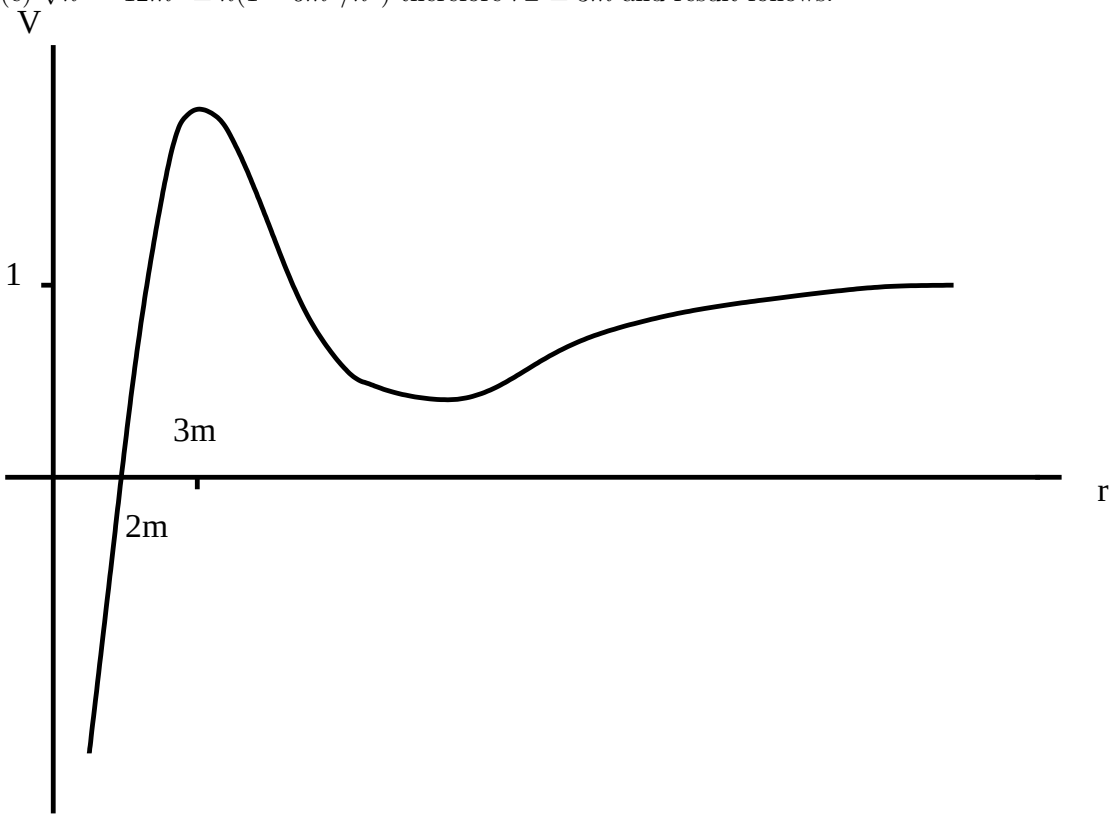
10.3 Motion in central field

(a) $V'(r) = 0$ implies $mr^2/h^2 - r + 3m = 0$. This has solutions when $1 - 12m^2/h^2 \geq 0$, in other words $h^2 \geq 12m^2$. One also has $r = 3m + mr^2/h^2 > 3m$. The actual solutions are

$$r_\pm = \frac{h\left(h \pm \sqrt{h^2 - 12m^2}\right)}{2m}.$$

(b) $V''(r) > 0$ implies $2mr/h^2 - 3r + 12m < 0$. Since $V'(r) = 0$, this becomes $r - 6m > 0$. Now $r_+ > h^2/2m > 6m$ so it is stable.

(c) $\sqrt{h^2 - 12m^2} \simeq h(1 - 6m^2/h^2)$ therefore $r_- \simeq 3m$ and result follows.



(d) The particle will be captured. There is no infinite centrifugal barrier like in Newtonian gravity.

10.4 Gravitational radiation

It is sufficient to compute only the time-dependent components of the quadrupole tensor, so we disregard the star and set $\rho(x) = m\delta(\mathbf{x} - \mathbf{x}_0(t))$, where the trajectory of the planet is $\mathbf{x}_0(t) = (R\cos\omega t, R\sin\omega t, 0)$ in the $x - y$ plane. The period $T \equiv 2\pi/\omega$ is found from the Newtonian calculation,

$$T = 2\pi\sqrt{\frac{R^3}{GM}}, \quad \omega = \sqrt{\frac{GM}{R^3}}.$$

Then we compute (omitting constant terms)

$$\begin{aligned}
Q_{xx} &= mR^2\cos^2\omega t + \text{const} = \frac{mR^2}{2}\cos 2\omega t + \text{const}.\\
Q_{xy} &= \frac{mR^2}{2}\sin 2\omega t + \text{const},\\
Q_{yy} &= \frac{mR^2}{2}\cos 2\omega t + \text{const}, \qquad Q_{zz} = Q_{xz} = Q_{yz} = 0,\\
\sum_{ij}\dddot{Q}_{ij}\dddot{Q}_{ij} &= \left(8\omega^3\right)^2\left(\frac{mR^2}{2}\right)^2\left(2\cos^2 2\omega t + 2\sin^2 2\omega t\right) = 32\omega^6 m^2 R^4,\\
L_{GW} &= \frac{32G}{5c^5}\omega^6 m^2 R^4.
\end{aligned}$$

The initial kinetic energy of the planet is

$$E_0 = \frac{mv^2}{2} = \frac{1}{2}\frac{GMm}{R} = \frac{1}{2}m\omega^2 R^2,$$

and this energy will be radiated during the time ΔT,

$$\Delta T = \frac{E_0}{L_{GW}}.$$

The dimensionless ratio of ΔT to the period T is

$$\frac{\Delta T}{T} = \frac{\omega}{2\pi}\frac{5c^5}{64G}\frac{m\omega^2 R^2}{R^4 m^2\omega^6} = \frac{5}{128\pi}\left(\frac{R}{R_s}\right)^{5/2}\frac{M}{m}.$$

For the Earth-Sun system, a calculation gives

$$\frac{\Delta T}{T} \sim 7\cdot 10^{23}. \tag{90}$$

Part III
Addendum

1 Derivation: gravitational waves in flat spacetime

This is not a solution to any exercise, but a more detailed derivation of the formula for the energy radiated by the gravitational waves due to a small matter source.

The metric is assumed to be of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the Minkowski metric for flat space and $h_{\mu\nu}$ is a small perturbation which is assumed to fall off to zero quickly at infinity. We start with a 3+1 decomposition of the metric perturbation $h_{\mu\nu}$ and compute the Einstein tensor (see Problems 7.2, 7.3, 7.4) in terms of the perturbation variables Φ, Ψ, etc. We also decompose the matter energy-momentum tensor $T_{\mu\nu}$ and obtain the Einstein equations separately for each component (8.5). The result is that (a) the variables E, B, F_i can be set to zero by choosing a coordinate system; (b) if there is no matter (vacuum) the scalar and vector components of the metric perturbation are equal to zero; (c) the tensor component h_{ij} satisfies the wave equation (81).

Solutions of the wave equation in four dimensions with retarded boundary condition can be written using the known Green's function. For instance, if

$$\Box f(t, \mathbf{r}) = A(t, \mathbf{r}) \quad \Rightarrow \quad f(t, \mathbf{R}) = -\frac{1}{4\pi}\int d^3\mathbf{r}\frac{A(t - |\mathbf{r} - \mathbf{R}|, \mathbf{r})}{|\mathbf{r} - \mathbf{R}|}.$$

We will use this formula for $f \equiv {}^{(T)}h_{ij}$ and $A \equiv 16\pi G\,{}^{(T)}T^i_j$. Now, we are interested in describing the radiation sent far away by a matter distribution, so we take the limit $|\mathbf{R}| \gg |\mathbf{r}|$, and then we can approximately set

$${}^{(T)}h_{ij} \approx -\frac{4G}{|\mathbf{R}|}\int d^3\mathbf{r}\,{}^{(T)}T^i_j(t - |\mathbf{r} - \mathbf{R}|, \mathbf{r}).$$

Now we use a trick (See Hobson-Efstathiou-Lasenby, §17.9) to express the components T^j_i through T^0_0; it is much easier to compute with T^0_0 because this is just the energy density of matter. Consider first the tensor T^i_j rather than its transverse-traceless part ${}^{(T)}T^i_j$. The trick is to write the integral (out of sheer luck)

$$\int d^3\mathbf{r}\,\partial_a\partial_b\left(r^i r^j\right)T^{ab} = 2\int d^3\mathbf{r}\,T^{ij}.$$

Then we integrate by parts and use the conservation laws (82),

$$T^i_{j,i} = -T^0_{j,0}, \quad T^i_{j,ij} = -T^0_{j,0j} = T^0_{0,00} \equiv \ddot{T}^0_0; \quad T^{ij}{}_{,ij} = -\ddot{T}^0_0.$$
$$2\int d^3\mathbf{r}\,T^{ij} = \int d^3\mathbf{r}\,T^{ab}{}_{,ab}r^i r^j = -\int d^3\mathbf{r}\,r^i r^j \ddot{T}^0_0.$$

Now, we need to obtain the transverse-traceless part of the tensor. In principle, we have the formulas for this (see Problem 8.5). But they are very complicated. A shortcut is to notice that the projection operator P_{ab} does the job (Problems 9.1 and 9.2), at least in the leading order in $1/|\mathbf{R}|$. (We are only interested in everything to leading order in $1/|\mathbf{R}|$ since all smaller terms will not give any flux of radiated energy.) The result is

$$\begin{aligned} {}^{(T)}h_{ik}(\mathbf{R},t) &= \frac{2G}{|\mathbf{R}|}{}^{(TT)}\ddot{Q}_{ik}(t-|\mathbf{R}|), \\ Q_{ik}(t) &\equiv \int d^3\mathbf{r}\,T^0_0(\mathbf{r},t)\left(r_i r_k - \frac{1}{3}r^2\delta_{ik}\right). \end{aligned} \tag{91}$$

The tensor Q_{ik} is the quadrupole moment of energy distribution; it is a traceless and symmetric tensor. In principle, we could just use the integral

$$\int d^3\mathbf{r}\,T^0_0(\mathbf{r},\mathbf{t})r_i r_k, \tag{92}$$

because the transverse-traceless parts of (92) and of Q_{ik} are the same, but it is more convenient to use Q_{ik}.

Since we found the tensor perturbation ${}^{(T)}h_{ij}$, now we would like to compute the energy radiated in the gravitational waves. For this we need the energy-momentum tensor of gravitational waves. This is a rather nontrivial object, since in general the gravitational field does not have any energy-momentum tensor. In the case of gravitational waves in flat background spacetime, one can define some quantity ${}^{(GW)}T_{\mu\nu}$ which looks like the energy-momentum tensor of gravitational waves (but actually is not even a generally covariant tensor). We will compute this quantity below. This quantity is useful because it gives the correct value of the energy after one integrates over a large region of spacetime. The real justification for using this procedure is complicated and is beyond the scope of this introductory course of General Relativity. We will only show a heuristic justification, which is the following. Gravitation is sensitive to every kind of energy, because the energy-momentum tensor acts as a "source" for gravity (it is on the right-hand side of the Einstein equation). So gravitation should be also sensitive to the energy in gravitational waves. One expects that the energy-momentum tensor for gravitational waves, ${}^{(GW)}T_{\mu\nu}$ (if we know how to compute it), will act as an additional source for gravity, like every other energy-momentum tensor for other kinds of matter. We will guess the formula for ${}^{(GW)}T_{\mu\nu}$ as follows. We can write the Einstein equation and expand it in powers of the perturbation $h_{\mu\nu}$:

$$G^\alpha_\beta\left[\eta_{\mu\nu}+h_{\mu\nu}\right] = G^{(1)\alpha}_\beta[h] + G^{(2)\alpha}_\beta[h] + ... = 8\pi G T^\alpha_\beta. \tag{93}$$

Here $G^{(1)}$ is the first-order Einstein tensor, $G^{(2)}$ is the second-order etc. First we solve only to first-order in h (this is what we have been doing so far) and then we will get an *approximate* solution $h^{(1)}_{\mu\nu}$:

$$G^{(1)\alpha}_\beta[h^{(1)}] = 8\pi G T^\alpha_\beta. \tag{94}$$

This solution disregards the effect of gravitational waves and only takes into account the effect of matter T^α_β. We can try to get a more precise solution by using the second-order terms in Eq. (93). Then we will get a correction $h^{(2)}$ to the solution; the solution $g = \eta + h^{(1)} + h^{(2)}$ will be more precise. From Eq. (93) we find

$$G^{(1)\alpha}_\beta[h^{(1)}+h^{(2)}] + G^{(2)\alpha}_\beta[h^{(1)}] = 8\pi G T^\alpha_\beta.$$

Now this is similar to Eq. (94), but it looks as if there is an additional term in the energy-momentum tensor, which we may rewrite as

$$\begin{aligned} G^{(1)\alpha}_\beta[h^{(1)}+h^{(2)}] &= 8\pi G\left\{T^\alpha_\beta + {}^{(\mathrm{GW})}T^\alpha_\beta\right\}, \\ {}^{(\mathrm{GW})}T^\alpha_\beta &\equiv -\frac{1}{8\pi G}G^{(2)\alpha}_\beta[h^{(1)}]. \end{aligned}$$

This motivates us to say that the EMT for gravitational waves is given by this formula. But of course this is not a real derivation because this does not show why the quantity ${}^{(\mathrm{GW})}T_{\mu\nu}$ has anything to do with the energy carried by waves.

The second-order terms $G_\beta^{(2)\alpha}$ are computed in Problem 9.3. The result is used to compute the power radiated in gravitational waves (Problem 9.4). Note that the calculation of $G_\beta^{(2)\alpha}$ uses averaging over spacetime in an essential way. Thus, the result is an averaged power radiated during a long time—much longer than the typical time scale of change in the sources—and averaged over large distances, much larger than the typical length scale of the sources. This kind of averaging is assumed in Eq. (88). It remains unclear exactly how one performs averaging over space and time; this is not well explained in any books at the undergraduate level.

The result is that we can use the formula (88) to compute the gravitational radiation emitted by nonrelativistic matter far away from those places where the matter is contained. The distribution of the energy density, $T_0^0(\mathbf{r}, t)$, should be given. Then we compute the tensor Q_{ij} according to Eq. (91), by integrating over space where the matter is contained. Finally, we compute the third derivative $\dddot{Q}_{ij}$, the trace, and averages over long times, as indicated in Eq. (88). If we want to insert factors of c, we replace G by Gc^{-9}.

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