

RESOURCE LETTER

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Resource Letter TMGR-1: Teaching the mathematics of general relativity

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This Resource Letter provides some guidance on issues that arise in teaching general relativity at both the undergraduate and graduate levels. Particular emphasis is placed on strategies for presenting the mathematical material needed for the formulation of general relativity. © 2006 American Association of Physics Teachers.

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I. INTRODUCTION

General Relativity is the theory of space, time, and gravity formulated by Einstein in 1915. It is widely regarded as a very abstruse, mathematical theory and, indeed, until recently it has not generally been regarded as a suitable subject for an undergraduate course. In actual fact, the mathematical material (namely, differential geometry) needed to attain a deep understanding of general relativity is not particularly difficult and requires a background no greater than that provided by standard courses in advanced calculus and linear algebra. (By contrast, considerably more mathematical sophistication is needed to provide a rigorous formulation of quantum theory.) Nevertheless, this mathematical material is unfamiliar to most physics students and its application to general relativity goes against what students have been taught since high school (or earlier): namely, that “space” has the natural structure of a vector space. Thus, the mathematical material poses a major challenge to teaching general relativity—particularly for a one-semester course. If one takes the time to teach the mathematical material properly, one runs the risk of turning the course into a course on differential geometry and doing very little physics. On the other hand, if one does not teach it properly, then one is greatly handicapped in one’s ability to explain the major conceptual differences between general relativity and the prerelativistic and special-relativistic notions of spacetime structure.

The purpose of this Resource Letter is to provide a brief guide to the issues and pitfalls involved in teaching general relativity at both the undergraduate and graduate levels. The main focus will be on how to introduce the mathematical material necessary for the formulation of general relativity. By contrast, I shall not devote much attention to how to teach the various topics that normally would be included in a general-relativity course after one has formulated the theory, such as the “weak-field” limit, tests of general relativity,

gravitational radiation, cosmology, and black holes. This Resource Letter also will be relatively light on the enumeration of published “resources.”

I will begin by briefly outlining the major new conceptual ideas introduced by general relativity. I will then describe the mathematical concepts that are needed to formulate the theory in a precise manner. Finally, I will discuss strategies for dealing with this mathematical material in courses on general relativity.

II. GENERAL RELATIVITY

Prior to 1905, it was taken for granted that the causal structure of space-time defines a notion of *simultaneity*. For a given event A (i.e., a “point of space at an instant of time”), we can define the *future of A* to consist of all events that, in principle, could be reached by a particle starting from event A . Similarly, the *past of A* consists of all events such that, in principle, a particle starting from that event could arrive at A . The events that lie neither to the future nor the past of event A were assumed to comprise a three-dimensional set, called the events *simultaneous* with A . This notion of simultaneity defines a notion of “all of space at an instant of time,” which, in essence, allows one to decompose the study of space-time into separate studies of “space” and “time.” It is important to emphasize to students the key role of this assumption in prerelativistic notions of space-time structure.

The major revolution introduced by special relativity rests largely on the premise that the assertions of the preceding paragraph concerning the causal structure of space-time are wrong. Most strikingly, the set of events that fail to be causally connected to an event A comprise much more than a three-dimensional region. In a space-time diagram, the future of an event A looks like the interior of a “cone” with vertex A , where the boundary of this cone corresponds to the trajectories of light rays emitted at event A . Thus, in special

relativity, the causal structure of space-time defines a notion of a “light cone” of an event, but it does not define a notion of simultaneity.

It is important to focus on the “invariant structure” of space-time, i.e., the aspects of space-time structure that are well defined, independently of which observer makes the measurements. In prerelativity physics, the time interval between any pair of events is such an invariant; the space interval between simultaneous events is also an invariant. However, in special relativity neither time intervals nor space intervals are invariants. In special relativity, the only invariant quantity related to a pair of events, A and B , is their *space-time interval*, given in any global inertial coordinate system by the formula

$$I(A,B) = -(\Delta t)^2 + \frac{1}{c^2}[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]. \quad (1)$$

All features of space-time structure in special relativity can be derived from the space-time interval.

It is remarkable that—except for the key minus sign in front of $(\Delta t)^2$ —the space-time interval has exactly the same mathematical form as the Pythagorean formula for the square of the distance between two points in Euclidean geometry. This was first realized by Minkowski in 1908, but its deep significance was not appreciated by Einstein until several years later, as he began to develop general relativity. It enables one to understand special relativity as a theory of *flat Lorentzian geometry*. In special relativity, space-time is described in a manner that is mathematically identical to Euclidean geometry, except for the changes that result from the presence of a term with a minus sign on the right-hand side of Eq. (1). In particular, the global inertial coordinates of special relativity are direct analogs of Cartesian coordinates in Euclidean geometry, and the worldlines of inertial observers are direct analogs of the straight lines (geodesics) of Euclidean geometry.

This understanding of special relativity as a theory of flat Lorentzian geometry is a key step in the progression toward general relativity. General relativity arose from the attempt to formulate a theory of gravity that is compatible with the basic ideas of special relativity and also fundamentally builds in the *equivalence principle*: All bodies are affected by gravity and, indeed, all bodies fall the same way in a gravitational field. The equivalence principle strongly suggests that freely falling motion in a gravitational field should be viewed as analogous to inertial motion in prerelativity physics and special relativity. Gravity is not a “force” at all, but rather a change in space-time structure that allows inertial observers to accelerate relative to each other. Remarkably, after many years of effort, Einstein discovered that this idea could be implemented by simply generalizing the flat Lorentzian geometry of special relativity to a curved Lorentzian geometry—in exactly the same way as flat Euclidean geometry can be generalized to curved Riemannian geometry. General relativity is thereby a theory of the structure of space and time that accounts for all of the physical effects of gravitation in terms of the curved geometry of space-time.

In addition to the replacement of a flat space-time geometry by a curved space-time geometry, general relativity differs radically from special relativity in that the space-time geometry is not fixed in advance but rather evolves dynamically. The dynamical evolution equation for the metric—known as *Einstein’s equation*—equates part of the curvature

of space-time to the stress-energy-momentum tensor of matter.

III. DIFFERENTIAL GEOMETRY

The geometry required for an understanding of general relativity is simply the generalization of Riemannian geometry to metrics that are not positive-definite. Fortunately, there are few significant mathematical changes that result from this generalization. Consequently, much of the intuition that most people have for understanding the Riemannian geometry of two-dimensional surfaces encountered in everyday life—such as the surface of a potato—can usually be extended to general relativity in a reliable manner. However, two significant cautions should be kept in mind: (1) Much of the intuition that most people have about the curvature of two-dimensional surfaces concerns the manner in which the surface *bends* within the three-dimensional Euclidean space in which it lies. This *extrinsic* notion of curvature must be carefully distinguished from the purely *intrinsic* notion of curvature that concerns, e.g., the failure of initially parallel geodesics within the surface itself to remain parallel. It is the intrinsic notion of curvature that is relevant to the formulation of general relativity. (2) A new feature that arises for non-positive-definite metrics is the presence of *null vectors*, i.e., nonzero vectors whose “length” is zero. Attempts to apply intuition from Riemannian geometry to null vectors and null surfaces (i.e., surfaces that are everywhere orthogonal to a null vector) often result in serious errors.

When I teach general relativity at either the undergraduate or graduate level, I emphasize to the students that one of their main challenges is to “unlearn” some of the fundamental falsehoods about the nature of space and time that they have been taught to assume since high school (if not earlier). I have already discussed above one such key falsehood, namely the notion of absolute simultaneity. Normally, students taking general relativity have had some prior exposure to special relativity, and thus they are aware—at least at some level—of the lack of a notion of absolute simultaneity in special relativity. However, very few students have any inkling that, in nature, the points of space and the events in space-time fail to have any natural vector-space structure. Indeed, the concept of a “vector” is normally introduced to students early in their physics education through the concept of “position vectors” representing the points of space. Students are taught that, given the choice of a point to serve as an “origin,” it makes sense to add and scalar multiply points of space. The only significant change introduced by special relativity is the generalization of this vector-space structure from space to space-time: In special relativity, the position vector \vec{x} representing a point of space is replaced by the “4-vector” x^μ representing an event in space-time. One can add and scalar multiply 4-vectors in special relativity in exactly the same way as one adds and scalar multiplies ordinary position vectors in prerelativity physics.

This situation changes dramatically in general relativity, since the vector-space character of space and space-time depends crucially on having a flat geometry. In general relativity, it does not make any more sense to “add” two events in space-time than it would make sense to try to define a notion of addition of points on the surface of a potato.

How does one go about giving a precise mathematical description of the geometry of a space-time in general relativity—or, for that matter, of the geometry of a surface of

a potato? The notion of a “distance function” between (finitely separated) points can be defined for the surface of a potato, and, similarly, the notion of a “space-time interval” could be defined for (finitely separated, but sufficiently close) events in general relativity, but it would be extremely cumbersome to base a geometrical description of these entities on such a notion. A much better idea is to work infinitesimally, using the idea that, on sufficiently small scales, a curved geometry looks very nearly flat. These departures from flatness can then be described by differential calculus. To do so, one begins by introducing the notion of a tangent vector to describe an infinitesimal displacement about a point p . The collection of all tangent vectors at p can be given the natural structure of a vector space, but in a curved geometry, a tangent vector at p cannot naturally be identified with a tangent vector at a different point q . One then uses basic constructions of linear algebra to define the more general notion of tensors at p . A particularly important example of a tensor field (i.e., a tensor defined at all points p) is a *metric*, which is simply a (not necessarily positive-definite) inner product on tangent vectors (see below). When a metric (of any type) is present, it gives rise to a natural notion of differentiation of tensor fields. This notion of differentiation allows one to define the notion of a geodesic (as a curve that is “as straight as possible”) and curvature—which can be defined in terms of the failure of initially parallel geodesics to remain parallel, or, more directly, in terms of the failure of successive derivatives of tensor fields to commute.

Let me now explain in more detail what is actually needed in order to introduce the above basic concepts of differential geometry in a mathematically precise manner. First, one needs a mathematically precise notion of the “set of points” that constitute space-time (or that constitute a surface in ordinary geometry). The appropriate notion is that of a *manifold*, which is a set that locally “looks like” \mathbf{R}^n with respect to differentiability properties, but has no metrical or other structure. The points of an n -dimensional manifold can thereby be labeled locally by coordinates (x^1, \dots, x^n) , but these coordinate labels are arbitrary and could equally well be replaced by any other coordinate labels (x'^1, \dots, x'^n) that are related to (x^1, \dots, x^n) in a smooth, nonsingular manner. A precise definition of an n -dimensional manifold can be given as a set that can be covered by local coordinate systems that satisfy suitable compatibility conditions in the overlap regions.

Unfortunately, it is not as easy as one might think to give a mathematically precise notion of a “tangent vector.” The most elegant and mathematically clear way of proceeding is to define a tangent vector to be a “derivation” (i.e., directional derivative operator) acting on functions; derivations can be defined axiomatically in a simple manner. This definition has the virtue of stating clearly what a tangent vector is, without introducing extraneous concepts like coordinate bases. Essentially all modern mathematics books define tangent vectors in this way. However, most students do not find this definition to be particularly intuitive.

A more intuitive way of proceeding is to consider a curve, which can be locally described by giving the coordinates $x^\mu(t)$ of the point on the curve as a function of the curve parameter t . One can identify the tangent to the curve at the point $x^\mu(t)$ with the collection of n numbers, $(dx^1/dt, \dots, dx^n/dt)$, at the point on the curve labeled by t . The coordinate lines themselves are curves, and the tangent

to the μ th coordinate line would be identified with the numbers $(0, \dots, 0, 1, 0, \dots, 0)$, where the “1” is in the μ th place. One may therefore view the tangents to the coordinate lines at each point as comprising a basis for the “tangent vectors” at that point. For an arbitrary curve $x^\mu(t)$, one then may view $(dx^1/dt, \dots, dx^n/dt)$ as the components of the tangent to this curve in this coordinate basis. Of course, if we chose a different coordinate system, the components of the tangent to this curve would “transform” by a formula known as the “vector-transformation law,” which is easily derived from the chain rule.

A somewhat more direct way of proceeding in accord with the preceding paragraph is to define a tangent vector at a point to be a collection of n numbers associated with a coordinate system that transforms by the vector-transformation law under a change of coordinates. This approach allows one to define a tangent vector in one sentence and thereby move on quickly to other topics. This definition can be found in most mathematics books written prior to the mid-20th century as well as in most treatments of general relativity written by physicists. However, it is not particularly intuitive. Furthermore, by tying the notion of a tangent vector to the presence of a coordinate system, it makes it extremely difficult for students to think about tangent vectors (and tensors—see below) in a geometrical, coordinate-independent way.

After tangent vectors have been introduced, the next step is to define tensors of arbitrary rank. This is done by a standard construction in linear algebra. Linear algebra is quite “easy” compared with many other mathematical topics, and students taking a general-relativity class will normally have had a course in linear algebra or considerable exposure to it. Unfortunately, however, the way students are normally taught linear algebra does not mesh properly with what is needed for general relativity. The problem is that in the context in which students have been exposed to linear algebra, a (positive-definite) inner product is normally present. One then normally works with the components of tensors in an orthonormal basis. One thereby effectively “hides” the role played by the inner product in various constructions. One also hides the major distinction between vectors and dual vectors (see below). In general relativity, the key “unknown variable” that one wishes to solve for is the metric of space-time, which, as mentioned above, is simply a (non-positive-definite) inner product on tangent vectors. It is therefore essential that all of the basic linear-algebra constructions be done without assuming an inner product, so that the role of the metric in all subsequent constructions is completely explicit.

To proceed, given a finite-dimensional vector space, V —which, in the case of interest for us, would be the tangent space at a point p of space-time—we define its dual space, V^* , to be the collection of linear maps from V into \mathbf{R} . It follows that V^* is a vector space of dimension equal to V , but, in the absence of an inner product, there is no natural way of identifying V and V^* . However, given a basis of V , there is a natural corresponding basis of V^* . Since V^* is a vector space, we also can take its dual, thereby producing the “double dual,” V^{**} , of V . It is not difficult to show explicitly that there is a natural way of identifying V^{**} with V .

With this established, a *tensor of type (k, l)* can then be defined as a multilinear map taking k copies of V^* and l copies of V into \mathbf{R} . On account of the isomorphism between

V and V^* , tensors of a given type may be viewed in other equivalent ways. For example, tensors of type $(1,1)$ are isomorphic to the vector space of linear maps from V to V and also are isomorphic to the linear maps from V^* to V^* . There are two basic operations that can be performed on tensors: contraction and taking outer products. All familiar operations can be expressed in terms of these; for example, the composition of two linear maps can be expressed in terms of the outer product of the corresponding tensors followed by a contraction.

All of the assertions of the preceding two paragraphs are entirely straightforward to establish. However, most students are not used to distinguishing between vectors and dual vectors. Indeed, in the familiar context where one has a positive-definite metric, not only can V and V^* be identified, but the components of a vector in an orthonormal basis are equal to the components of the corresponding dual vector in the corresponding dual basis. Students feel that they “know” linear algebra, and they become bored and impatient if one takes the time to carefully explain the above ideas. After all, they took the course to learn about Einstein’s revolutionary ideas about space, time, and gravity, not to learn why a vector space is isomorphic to its double dual. But if one does not carefully explain the above ideas, the students are guaranteed to become quite confused at a later stage. In 30 years of teaching general relativity at the graduate level, I have not found a satisfactory solution to this problem, and I have always found the discussion of tensors to be the “low point” of the course.

Many treatments of general relativity effectively bypass the above treatment of tensors by working only with the components of tensors in bases associated with coordinate systems. Given the “transformation law” for components of tangent vectors under a change of coordinates, the corresponding transformation law for the components of dual vectors can be obtained, and the more general “tensor-transformation law” for a tensor of type (k,l) can be derived. One can then *define* a tensor of type (k,l) at a point p on an n -dimensional manifold to be a collection of n^{k+l} numbers associated with a coordinate system that transform by the tensor-transformation law under a change of coordinates. This approach is taken in many mathematics books written prior to the mid-20th century and in many current treatments of general relativity. It has the advantage that one can then quickly move on to other topics without spending much time talking about tensors. However, it has the obvious disadvantage that although students may still be trained to use tensors correctly in calculations, they usually end up having absolutely no understanding of what they are.

A *metric*, g , on a vector space V can now be defined as a tensor of type $(0,2)$ that is nondegenerate in the sense that the only $v \in V$ satisfying $g(v, w) = 0$ for all $w \in V$ is $v = 0$. A metric is then seen to be equivalent to the specification of an isomorphism between V and V^* . If the metric is positive definite, it is called *Riemannian*, whereas if it is negative definite on a one-dimensional subspace and positive definite on the orthogonal complement of this subspace, it is called *Lorentzian*. (My sign convention on the definition of Lorentzian metrics corresponds to that used by most general relativists; however, most particle physicists use the opposite sign convention, i.e., they take a Lorentzian metric to be positive definite on a one-dimensional subspace and negative definite on the orthogonal complement of this subspace.)

Riemannian metrics describe ordinary curved geometries (like the surface of a potato), whereas curved spacetimes in general relativity are described by Lorentzian metrics.

During the past half-century, a major cultural divide has opened up between mathematicians and physicists with regard to the notation used for tensors. The traditional notation—which is still used by most physicists—is to denote a tensor, T , of type (k,l) by the collection of its components $T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}$, where the “up” indices correspond to vector indices, and the “down” indices correspond to dual vector indices. This notation has the advantage that basic operations on tensors—like taking outer products or performing contractions—are expressed in a clear and explicit way. The isomorphism between vectors and dual vectors that is provided by the presence of a metric can also be nicely incorporated into this notation by using the metric to “raise and lower indices.” However, the notation effectively forces one to think of a tensor as a collection of components rather than an object with legitimate status in its own right that does not require the introduction of a basis. In reaction to this, essentially all modern mathematics books adopt an “index-free” notation for tensors. This notation makes manifest the proper basis/coordinate-independent status of tensors, but it makes it extremely cumbersome to denote even a moderately complicated series of operations. In my view, an excellent compromise is to employ an “abstract-index notation,” which mirrors the component notation, but where a symbol like $T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}$ would now stand for the tensor itself, not its components.

After tensors over an arbitrary vector space have been introduced, one can return to the manifold context and define a *tensor field of type (k,l)* to be an assignment of a tensor of type (k,l) over the tangent space of each point of the manifold. The next key step is to formulate a notion of differentiation of tensor fields. The notation of differentiation of tensor fields is nontrivial because on a manifold M , there is no natural way of identifying the tangent space at a point p with the tangent space at a different point q , so one cannot simply take the difference between the tensors at p and q and then take the limit as q approaches p . In fact, if we had no additional structure present beyond that of a manifold, there would be no unique notion of differentiation; rather there would be a whole class of possible ways of defining the derivative of tensor fields. These can be described directly by providing axioms for a notion of a derivative operator, or, equivalently, it can be done by introducing a notion of “parallel transport” along a curve. In mathematical treatments, the notion of parallel transport is usually introduced in the more general context of a connection on a fiber bundle. The general notions of fiber bundles and connections have many important applications in mathematics and physics (in particular, to the description of gauge theories), but it would normally require far too extensive a mathematical excursion to include a general discussion of these topics in a general-relativity course, even at the graduate level.

Although there is no unique notion of differentiation of tensors in a completely general context, when a metric is present a unique notion of differentiation is picked out by imposing the additional requirement that the derivative of the metric must be zero. In Euclidean geometry (or in special relativity), this notion of differentiation of tensors corresponds to the partial differentiation of the components of the

tensors in Cartesian coordinates (or in global inertial coordinates). However, in nonflat geometries, this notion of differentiation—referred to as the *covariant derivative*—does not correspond to partial differentiation of the components of tensors in any coordinate system.

Once differentiation of tensors has been defined, a *geodesic* can be defined as a curve whose tangent is *parallel transported* along the curve, i.e., the covariant derivative of the tangent in the direction of the tangent vanishes. It is not difficult to show that, in Riemannian geometry, a curve with given endpoints is a geodesic if and only if it is an extremum (though not necessarily a minimum) of length with respect to variations that keep the endpoints fixed. Similarly, in Lorentzian geometry—i.e., in general relativity—a timelike geodesic (i.e., a geodesic whose tangent has everywhere negative “norm” with respect to the space-time metric) can be characterized as an extremum of the proper time, τ , elapsed along the curve. If the curve is described in coordinates x^μ by specifying $x^\mu(t)$, then τ is given by

$$\tau = \int_a^b \sqrt{-\sum_{\mu,\nu} g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} dt. \quad (2)$$

After the above notions have been introduced, curvature may be defined by any of the following three equivalent ways: (1) The failure of successive covariant derivatives on tensor fields to commute; (2) the failure of parallel transport of a vector around an infinitesimal closed curve to return the vector to its original value; (3) the failure of initially parallel, infinitesimally nearby geodesics to remain parallel. Curvature is described by a tensor field of type (1,3), called the *Riemann curvature tensor*. After the Riemann curvature tensor has been defined, all of the essential mathematical material needed for the formulation of general relativity is in place.

IV. TEACHING GENERAL RELATIVITY AT THE UNDERGRADUATE LEVEL

Fortunately, there are not many other courses that are essential prerequisites for an undergraduate general-relativity course. It is, of course, necessary that students have some prior exposure to special relativity, since the conceptual hurdles will be too large for a student with no prior familiarity with special relativity. However, it should suffice to have seen special relativity as normally introduced at the level of first-year introductory-physics courses. It is important that students have taken classical mechanics at the undergraduate level, and thereby have had exposure to “generalized coordinates” and Euler-Lagrange variations. It also is useful (but not essential) for students to have taken an undergraduate electromagnetism course, since one should understand what an electromagnetic wave is before trying to learn what a gravitational wave is.

Teaching general relativity at the undergraduate level poses major challenges, particularly if the course is only one semester (or, worse yet, one quarter) in length. In a one-semester undergraduate course, there is simply not enough time to introduce and properly explain the mathematical material described in the preceding section. Indeed, even in a year-long course, it clearly would be inadvisable to “front load” all of this mathematical material; if one did so, there

would not likely be many students left in the course by the time one got to the interesting physical applications of general relativity.

Clearly, it makes sense to begin an undergraduate relativity course with a discussion or review of special relativity, preferably emphasizing the geometrical point of view described in Sec. II above. It also would make sense to try to explain some of the fundamental ideas and concepts of general relativity at a qualitative level at the beginning of the course, as also described in Sec. II. To proceed further, however, it is necessary to introduce some of the mathematical material discussed in Sec. III. In my view, the minimal amount of mathematical material needed to teach a respectable undergraduate course would include (i) A clear explanation that space-time in general relativity does *not* have the structure of a vector space and that coordinates, x^μ , are merely labels of events in space-time—devoid of any physical significance in their own right. (ii) The introduction of the notion of a tangent vector to a curve, as described in Sec. III above. (iii) The introduction of the notion of a space-time metric as a (Lorentzian) inner product on tangent vectors, and its use for determining the elapsed proper time, τ , along a timelike curve [see Eq. (2) above]. (iv) The introduction of the notion of a timelike geodesic as a curve that extremizes τ . The geodesic equation (for timelike geodesics) can then be derived using Euler-Lagrange variation. (The geodesic equation for null geodesics could then be introduced by a limiting procedure after one has derived the equation for timelike geodesics.) It is worth noting that the same relation between symmetries and conservation laws that one has in Lagrangian mechanics (namely, Noether’s theorem) then automatically applies to geodesics, so in a space-time with a sufficiently high degree of symmetry, one can actually solve the geodesic equation (or, more precisely, “reduce it to quadratures”) using only constants of motion.

The above will give students the necessary tools to interpret what a space-time metric is and what its physical consequences are, since the key things one needs to know are (a) how to calculate elapsed time along arbitrary timelike curves and (b) how to determine the timelike geodesics (which represent the possible paths of freely falling particles) and null geodesics (which represent the possible paths of light rays) in a space-time. However, they will not have the necessary tools to understand Einstein’s equation, so it will be impossible to derive any solutions, i.e., the students will have to accept on faith that the space-times studied do indeed arise as solutions to Einstein’s equation.

After the above mathematical material has been presented, one will be in a good position to discuss the Schwarzschild solution (representing the exterior gravitational field of a spherical body) and the Friedmann-Lemaître-Robertson-Walker (FLRW) solutions (representing spatially homogeneous and isotropic cosmologies). With regard to the Schwarzschild solution, one can solve the timelike and null geodesic equations and thereby derive predictions for the motion of planets and the bending of light. For the FLRW metrics, one can derive the general form of a metric having homogeneous and isotropic symmetry in terms of an unknown “scale factor,” $a(t)$, and explain how a change in a with time corresponds to the expansion or contraction of the universe. Although one cannot, of course, derive the equa-

tions for the scale factor that result from Einstein's equation, one can simply write these equations down and derive their cosmological consequences.

Even in a one-semester undergraduate course, there should still be some time left to discuss some other key topics, such as gravitational radiation and its detection, the black-hole nature of the (extended) Schwarzschild solution, other topics in the theory of black holes, and topics in modern cosmology. In a year-long undergraduate course, one should be able to cover all of these topics and also present the mathematical material related to curvature, so that Einstein's equation may be obtained.

V. TEACHING GENERAL RELATIVITY AT THE GRADUATE LEVEL

In contrast to undergraduates, graduate students will not be satisfied if they are asked to accept a major component of a theory on faith, particularly if they are not even told in a precise and complete way what that component is. Thus, one simply cannot teach a graduate course in general relativity without a full discussion of Einstein's equation. Consequently, it is necessary to introduce the mathematical material needed to define curvature.

When I have taught general relativity at the graduate level, I have spent the first two weeks with a discussion or review of special relativity from the geometrical point of view and a qualitative discussion of the fundamental concepts underlying general relativity. I have then launched into a complete exposition of all of the mathematical material described in Sec. III above, ending with a derivation and discussion of Einstein's equation. This mathematical portion of the course normally occupies approximately 5 weeks. In a one-semester (or, worse yet, a one-quarter) course, this leaves enough time only for a "bare-bones" treatment of the following essential topics: (i) "weak-field" properties of general relativity (Newtonian limit and gravitational radiation), (ii) the FLRW metrics (see above) and their key properties (cosmological redshift, "big-bang" origin, horizons), and (iii) the Schwarzschild solution (planetary motion, the bending of light, and the black-hole nature of the extended Schwarzschild metric). I believe that a course of this nature provides students with a solid introduction to general relativity. By providing the key conceptual ideas and the essential mathematical tools, it leaves students well prepared to continue their study of general relativity. However, a course of this nature has the serious drawback that a high percentage of the effort is spent on mathematical material, and some students are justifiably frustrated with the minimal discussion of physical applications of the theory.

In a one-semester course, the only way one could add significantly more discussion of such physically interesting and relevant topics as gravitational radiation, black holes, relativistic astrophysics, and cosmology would be to significantly cut down on the time spent on the mathematical material. If one introduces coordinates at the outset and works exclusively with the components of tensors in coordinate (or other) bases, then, as described above in Sec. III, one can bypass much of the mathematical discussion of tensors by defining tensors by the tensor-transformation law. One then can define differentiation of tensors by introducing the Christoffel symbol as the "correction term" that needs to be added to the "ordinary derivative" to produce a tensor expression (i.e., to produce a collection of components that

transforms by the tensor-transformation law under coordinate changes). One can then introduce the Riemann curvature tensor as an object constructed out of the Christoffel symbol and its ordinary derivative that—rather magically—can be shown to transform as a tensor. The main price paid by presenting the mathematical material in this way is a sacrifice of clarity in explaining the fundamental conceptual basis of general relativity—particularly its difference from all prior theories with regard to the nonexistence of any nondynamical background structure of space-time—since this conceptual basis is very difficult to understand if one does not formulate the theory in a coordinate-independent way. In addition, students will not have the necessary mathematical tools to advance their study of general relativity to topics involving "global methods"—such as the singularity theorems and the general theory of black holes—where it is essential that the concepts be formulated in a coordinate-independent way. Nevertheless, by proceeding in this manner, one can easily reduce the time spent on mathematical material by a factor of 2 or more, thereby allowing significantly more course time to be spent on physical applications.

VI. RESOURCES

A. Resources for introductory discussions of general relativity

1. **Relativity: The Special and the General Theory, The Masterpiece Science Edition**, A. Einstein (Pi Press, New York, 2005). This reprint of one of Einstein's early, nontechnical expositions of special and general relativity contains an introduction by R. Penrose and commentary by R. Geroch and D. Cassidy. (E)
2. **Flat and Curved Space-Times** (2nd edition), G. F. R. Ellis and R. Williams (Cambridge U. P., Cambridge, 2000). This book provides a discussion of special relativity from a geometrical point of view and an introduction to the basic ideas of general relativity. (E)
3. **General Relativity from A to B**, R. Geroch (University of Chicago Press, Chicago, IL, 1978). This book presents an excellent introduction to the basic ideas of general relativity from a thoroughly geometrical point of view. (E)
4. **Gravity from the Ground Up**, B. Schutz (Cambridge U. P., Cambridge, 2003). This book provides a very readable discussion of the nature of gravitation in general relativity and its implications for astrophysics and cosmology. (E)
5. **Exploring Black Holes: Introduction to General Relativity**, E. F. Taylor and J. A. Wheeler (Addison Wesley Longman, San Francisco, 2000). This book provides a very physically oriented introduction to general relativity and black holes. (E)
6. **Black Holes and Time Warps: Einstein's Outrageous Legacy**, K. S. Thorne (W. W. Norton, New York, 1994). This book provides a very well-written account of some of the most fascinating ideas and speculations to arise from general relativity. (E)
7. **Space, Time, and Gravity: The Theory of the Big Bang and Black Holes** (2nd edition), R. M. Wald (University of Chicago Press, Chicago, 1992). (E)
8. **Was Einstein Right?: Putting General Relativity to the Test** (2nd ed.), C. M. Will (Basic Books, New York, 1993). This book provides an excellent account of the observational and experimental tests of general relativity. (E)

B. Resources for differential geometry

9. **Geometry of Manifolds**, R. L. Bishop and R. J. Crittenden (American Mathematical Society, Providence, RI, 2001). This concise book provides an excellent, high-level account of differential geometry. (A)
10. **Tensor Analysis on Manifolds**, R. L. Bishop and S. Goldberg (Dover, New York, 1987). (I)
11. **Riemannian Geometry**, L. P. Eisenhart (Princeton University Press,

Princeton, NJ, 1997). This is a reprint of the 1925 classic monograph, which gives an excellent presentation of the coordinate-based approach to differential geometry taken by mathematicians prior to the middle of the 20th century and still used by most physicists today. (I,A)

12. **Foundations of Differential Geometry**, volumes 1 and 2, S. Kobayashi and K. Nomizu (Wiley, New York, 1996). This book is an excellent, high-level reference on differential geometry. (A)
13. **Riemannian Manifolds: An Introduction to Curvature**, J. H. Lee (Springer-Verlag, New York, 1997). (I)
14. **Tensors, Differential Forms, and Variational Principles**, D. Lovelock and H. Rund (Dover, New York, 1989). (I)
15. **A Comprehensive Introduction to Differential Geometry**, volumes 1–5, 3rd ed., M. Spivak (Publish or Perish Inc., Houston, 1999). (I)
16. **Tensors and Manifolds: With Applications to Mechanics and Relativity**, R. H. Wasserman (Oxford U. P., Oxford, 1992). This book provides an extremely clear and complete treatment of the basic definitions, constructions, and results associated with tensor fields on manifolds. (I)

C. Undergraduate level texts

17. **Gravity: An Introduction to Einstein's General Relativity**, J. B. Hartle (Addison Wesley, San Francisco, 2003). The philosophy on teaching general relativity to undergraduates expounded in this Resource Letter is adopted directly from the approach taken by Hartle in this text. (I)
18. **General Relativity: A Geometric Approach**, M. Ludvigsen (Cambridge U. P., Cambridge, 1999). (I)
19. **Relativity: Special, General, and Cosmological**, W. Rindler (Oxford U. P., Oxford, 2001). (I)
20. **A First Course in General Relativity**, B. Schutz (Cambridge U. P., Cambridge, 1985). (I)
21. **Relativity: An Introduction to Special and General Relativity**, 3rd edition, H. Stephani (Cambridge U. P., Cambridge, 2004). (I)

D. Graduate level texts and monographs

22. **Spacetime and Geometry: An Introduction to General Relativity**, S. Carroll (Addison Wesley, San Francisco, 2004). This book provides a well-written, pedagogically oriented introduction to general relativity. (I)
23. **The Large Scale Structure of Space-time**, S. W. Hawking and G. F. R. Ellis (Cambridge U. P., Cambridge, 1973). This book is a true masterpiece, containing a complete exposition of the key global results in general relativity, including the singularity theorems and the theory of black holes. It is not light reading, however. (A)
24. **Relativity on Curved Manifolds**, F. de Felice and C. J. S. Clarke (Cambridge U. P., Cambridge, 1990). (I,A)
25. **The Classical Theory of Fields**, L. D. Landau and E. M. Lifshitz (Elsevier, Amsterdam, 1997). This very clear and concise discussion of general relativity from a coordinate-based point of view occupies only about 150 pages of this book. (I, A)
26. **Gravitation**, K. S. Thorne, C. W. Misner, and J. A. Wheeler (W. H. Freeman, San Francisco, 1973). This book, which remains very widely used, was the first text to present general relativity from a modern point of view. It places a strong emphasis on the physical content of the theory. (I,A)
27. **Advanced General Relativity**, J. Stewart (Cambridge Monographs on Mathematical Physics, Cambridge U. P., Cambridge, 1991). (A)
28. **General Relativity**, R. M. Wald (University of Chicago Press, Chicago, 1984). (I,A)
29. **Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity**, S. Weinberg (Wiley, New York, 1972). This book takes an antigeometrical approach and some of the discussion of cosmology is out of date, but it remains one of the best references for providing the details of calculations arising in the applications of general relativity, such as to physical processes occurring in the early universe. (I,A)

SPACE AND TIME

To Isaac Newton, space and time simply were—they formed an inert, universal cosmic stage on which the events of the universe played themselves out. To his contemporary and frequent rival Gottfried Wilhelm von Leibniz, “space” and “time” were merely the vocabulary of relations between where objects were and when events took place. Nothing more. But to Albert Einstein, space and time were the raw material underlying reality. Through his theories of relativity, Einstein jolted our thinking about space and time and revealed the principal part they play in the evolution of physics. They are at once familiar and mystifying; fully understanding space and time has become physics’ most daunting challenge and sought-after prize.

Brian Greene, *The Fabric of the Cosmos: Space, Time, and the Texture of Reality* (Knopf, 2004), p. 6.