Maximum force and minimum distance: physics in limit statements

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Abstract

The newly discovered principle of maximum force makes it possible to summarize special relativity, quantum theory and general relativity in one fundamental limit principle each. The three principles fully contain the three theories and are fully equivalent to their standard formulations. In particular, using a result by Jacobson based on the Raychaudhuri equation, it is shown that a maximum force implies the field equations of general relativity. The maximum force in nature is thus equivalent to the full theory of general relativity. Taken together, the three fundamental principles imply a bound for every physical observable, from acceleration to size. The new, precise limit values differ from the usual Planck values by numerical prefactors of order unity. They are given here for the first time. Among others, a maximum force and thus a minimum length imply that the non-continuity of space-time is an inevitable result of the unification of quantum theory and relativity.

Keywords

Maximum force, force limit, power limits, Planck limits, natural units

Limit values for physical observables are often discussed in the literature. There have been Ref. 1 studies of smallest distance, smallest time intervals and smallest entropy values, as well as largest particle energy and momentum values, largest acceleration values and largest space-time curvature and other extreme values. Usually, these arguments are based on Ref. 2-8 limitations of measurement apparatuses tailored to measure the specific observable under study. In the following we argue that all these limit statements can be deduced in a simpler way, namely by reformulating the main theories of physics themselves as limit statements. The limit value for every physical observable then follows automatically, together with new, corrected numerical prefactors. This aim is achieved by condensing each domain of physics in a straightforward limit principle. Each principle is a limit statement that limits the possibilities of motion in nature. Before we discuss the last principle that was missing in this chain, we summarize special relativity and quantum theory in this way. We then turn to general relativity, where we show that it can be deduced from a new, equally simple principle.

Special relativity in one statement

It is well known that special relativity can be summarised by a single statement on motion: *There is a maximum speed in nature.* For all systems,

$$v \leqslant c \quad . \tag{1}$$

A few well-known remarks set the framework for the later discussions. The speed v is smaller than or equal to the speed of light for *all* physical systems;^{*} in particular, this limit is valid both for composed systems as well as for elementary particles. The statement is valid for all observers. No exception to the statement is known. Only a maximum speed ensures that cause and effect can be distinguished in nature, or that sequences of observations can be defined. The opposite statement, implying the existence of (long-lived) tachyons, has been explored and tested in great detail; it leads to numerous conflicts with observations.

The maximum speed forces us to use the concept of *space-time* to describe nature. The existence of a maximal speed in nature also implies observer-dependent time and space coordinates, to length contraction, time dilation and all other effects that characterise special relativity. Only the existence of a maximum speed leads to the principle of maximum aging that governs special relativity, and thus at low speeds to the principle of least action. In addition, only a finite speed limit allows to define a *unit* of speed. If a speed limit would not exist, no natural measurement standard for speed would exist in nature; in that case, speed would not be a measurable quantity.

Special relativity also limits the size of systems, independently of whether they are composed or elementary. Indeed, the speed limit implies that acceleration a and size l cannot be increased independently without bounds, as the two ends of a system must not interpen-

^{*} A *physical system* is a region of space-time containing mass-energy, whose location can be followed over time and which interacts incoherently with its environment. With this definition, entangled situations are excluded from the definition of system.

etrate. The most important case are massive systems, for which

$$l \leqslant \frac{c^2}{a} \quad . \tag{2}$$

This size limit is also valid for the *displacement* d of a system, if the acceleration measured by an external observer is used. Finally, the limit implies an 'indeterminacy' relation:

$$\Delta l \,\Delta a \leqslant c^2 \quad . \tag{3}$$

This is all textbook knowledge.

Quantum theory in one statement

In the same way, all of quantum theory can be summarised by a single statement on motion: *There is a minimum action in nature.* For all systems,

$$S \geqslant \frac{\hbar}{2}$$
 . (4)

Also this statement is valid both for composite and elementary systems. The action limit is used less frequently than the speed limit. It starts from the usual definition of the action, $S = \int (T - U)dt$, and states that between two observations performed at times t and $t + \Delta t$, even if the evolution of a system is not known, the action is at least $\hbar/2$. The physical action is a quantity that measures the change of state of a physical system. In other words, there is always a minimum change of state taking place between two observations of a system. The quantum of action expresses the well-known fundamental fuzziness of nature at microscopic scale.

It is easily checked that no observation results in a smaller action value, independently of whether photons, electrons or macroscopic systems are observed. No exception to the statement is known.^{**} A minimum action has been observed for fermions, bosons, laser beams, matter systems and for any combination of them. The opposite statement, implying the existence of change that is arbitrary small, has been explored in detail; Einstein's long discussion with Bohr, for example, can be seen as a repeated attempt by Einstein to find experiments which allow to measure arbitrary small changes in nature. In every case, Bohr found that this aim could not be achieved.

The minimum action value implies that in quantum theory, the three concepts of state, measurement operation and measurement result need to be distinguished from each other; a so-called *Hilbert space* needs to be introduced. The minimum action value can be used to deduce the uncertainty relation, the tunnelling effect, entanglement, permutation symmetry, the appearance of probabilities in quantum theory, the information theory aspect of quantum theory and the existence of elementary particle reactions. Details of this discussion can be found in various textbooks.

Ref. 9

Obviously, the existence of a minimal or quantum of action was known right from the beginning of quantum theory. The quantum of action is at the basis of all descriptions

^{**} In fact, virtual particles can be seen as exceptions to this limit.

of quantum theory, including the many-path formulation and the information-theoretic descriptions. The existence of a minimum quantum of action is completely equivalent to all standard developments. In addition, only a finite action limit allows to define a *unit* of action. If an action limit would not exist, no natural measurement standard for action would exist in nature; in that case, action would not be a measurable quantity.

The action bound $S \leq pd \leq mcd$, together with the quantum of action, implies a limit on the displacement d of a system between two observations:

$$d \geqslant \frac{\hbar}{2mc} \quad . \tag{5}$$

In other words, (half) the (reduced) Compton wavelength of quantum theory is recovered as lower limit to the displacement of a system. Since the quantum *displacement* limit applies in particular to an elementary system, the limit is also valid for the size of a composite system. However, the limit is not valid for the size of *elementary* particles.

The action limit of quantum theory also implies Heisenberg's well-known indeterminacy relation for the displacement and momentum of systems:

$$\Delta d \ \Delta p \geqslant \frac{\hbar}{2} \quad . \tag{6}$$

It is valid both for massless and for massive systems. All this is textbook knowledge, of course. One notes that by combining the limits (2) and (5) one obtains

$$a \leqslant \frac{2mc^3}{\hbar} \quad . \tag{7}$$

This maximum acceleration for systems in which gravity plays no role is discussed in many publications. No experiment has ever reached the limit, despite numerous attempts. Ref. 12

General relativity in one statement

Least known of all is the possibility to summarise general relativity in a single statement on motion: There is a maximum force in nature. For all systems,

$$F \leqslant \frac{c^4}{4G} = 3.0 \cdot 10^{43} \,\mathrm{N}$$
 (8)

Let us explore the limit in some detail, as this formulation of general relativity is not common. (In fact, it seems that it has been discovered only 80 years after the general relativity has been around. It might be that the independent derivations of the present author and of Ref. 10 Gary Gibbons have been the first.) The limit statement contains both the speed of light c and the constant of gravitation G; it thus indeed qualifies as a statement from relativistic gravitation. Like the previous limit statements, it is stated to be valid for *all* observers. The Ref. 11 following discussion is given in more detail elsewhere.

The value of the maximum force is the mass-energy of a black hole divided by its diameter. It is also the surface gravity of a black hole times its mass. The force limit thus claims that no physical system of a given mass can concentrated in a region of space-time smaller than a (non-rotating) black hole of that mass. In fact, the mass–energy concentration limit can be easily transformed by algebra into the force limit; both are equivalent.

It is easily checked that the maximum force is valid for all systems *observed* in nature, whether they are microscopic, macroscopic or astrophysical. Neither the 'gravitational force' (as long as it is operationally defined) nor the electromagnetic or the nuclear interactions are found to ever exceed this limit.

The next aspect to check is whether a system can be *imagined* that exceeds the limit. An extensive discussion shows that this is impossible, if the proper size of observers or test masses is taken into account. Even for a moving observer, when the force value is increased by the (cube of the) relativistic dilation factor, or for an accelerating observer, when the observed acceleration is increased by the acceleration of the observer itself, the force limit must still hold. However, no situations allow to exceed the limit, as for high accelerations a, horizons appear at distance a/c^2 ; since a mass m has a minimum diameter given by $l \ge 4Gm/c^2$, we are again limited by the maximum force.

The exploration of the force limit shows that it is achieved only on horizons; the limit is reached in no other situation. The force limit is valid for all observers, all interactions and all imaginable situations.

Alternatively to the maximum force limit, we can use as basic principle the statement: *There is a maximum power in nature.*

$$P \leqslant \frac{c^5}{4G} = 9.1 \cdot 10^{51} W \quad . \tag{9}$$

The value of the force limit is the energy of a Schwarzschild black hole divided by its diameter; here the 'diameter' is defined as the circumference divided by π . The power limit is realized when such a black hole is radiated away in the time that light takes to travel along a length corresponding to the diameter.

In detail, both the force and the power limits state that the flow of momentum or of energy *through any physical surface* – a term defined below – of any size, for any observer, in any coordinate system, never exceeds the limit values. Indeed, due to the lack of nearby black holes or horizons, neither limit value is exceeded in any physical system found so far. This is the case at everyday length scales, in the microscopic world and in astrophysical systems. In addition, even Gedanken experiments do not allow to exceed the limits. However, the limits become evident only when in such Gedanken experiments the size of observers or of test masses is taken into account. Otherwise, apparent exceptions can be constructed; however, they are then unphysical.

Ref. 11

Ref. 11

Deducing general relativity

In order to elevate the force limit to a principle of nature, we have to show that in the same way that special relativity results from the maximum speed, also general relativity results from the maximum force.

The maximum force and the maximum power are only realized on horizons. (In fact, we can define the concept of horizon in this way, and show that it is always a two-dimensional surface, and that it has all properties usually associated with it.) The type of horizon plays no role. Consider a (flat) horizon of area A with surface gravity a through which an energy

Ref. 11

E is flowing. The force limit

$$F \leqslant \frac{c^4}{4G} \tag{10}$$

in the case of a horizon implies the use of the equal sign. For an energy E flowing through a horizon surface A we deduce

$$\frac{EL}{A} = \frac{c^4}{4G} \quad , \tag{11}$$

where L is the proper length of the moving energy pulse or massive test body, taken in the direction perpendicular to the horizon. On a horizon, bodies feel the surface gravity a; now, relativity shows that a pulse or body under acceleration a obeys $aL \le c^2$. Again, on a horizon, the extreme case takes place, so that we have

$$E = \frac{c^2}{4G} aA \quad , \tag{12}$$

where *a* is the surface gravity of the horizon and *A* its area. The relation can be rewritten for the differential case as

$$\delta E = \frac{c^2}{4G} a \,\delta A \quad . \tag{13}$$

In this way, the result can also be used for general horizons, such as horizons that are curved or time-dependent.***

Ref. 14 In a well known paper, Jacobson has given a beautiful proof of a simple connection: if energy flow is proportional to horizon area for all observers and all horizons, then general relativity holds. To see the connection to general relativity, we generalize relation (13) to general coordinate systems and general energy-flow directions. This is achieved by introducing tensor notation.

We start by introducing the local boost Killing field χ that generates the horizon, with suitably defined magnitude and direction; the surface gravity *a* is the acceleration of the Killing orbit with the maximal norm. This specifies what is meant by the term 'perpendicular' used above. We also introduce the general surface element $d\Sigma$. Using the energymomentum tensor T_{ab} the left hand side of relation (13) can then be rewritten as

$$\delta E = \int T_{ab} \chi^a d\Sigma^b \quad . \tag{15}$$

Ref. 14 Jacobson then shows how the right hand side of relation (13) can be rewritten, using the

*** Relation (13) is well-known, though with different names for the observables. Since no communication is possible across a horizon, also the detailed fate of energy flowing through a horizon is unknown. Energy whose detailed fate is unknown is often called *heat*. Relation (13) therefore states that the heat flowing through a horizon is proportional to the horizon area. When quantum theory is introduced into the discussion, the area of a horizon can be called 'entropy' and its surface gravity can be called 'temperature'; relation (13) can then be rewritten as

$$\delta Q = T \,\delta S \quad . \tag{14}$$

However, this translation of the right hand side, which requires the quantum of action, is unnecessary here. We only cite it to show the relation to some of the quantum gravity issues.

Raychaudhuri equation, to give

$$a\delta A = \frac{c^2}{2\pi} \int R_{ab} \chi^a d\Sigma^b \quad , \tag{16}$$

where R_{ab} is the Ricci tensor. The Ricci tensor describes how the shape of the horizon changes over space and time.

In summary, we get

$$\int T_{ab} \chi^a d\Sigma^b = \frac{c^4}{8\pi G} \int R_{ab} \chi^a d\Sigma^b \quad , \tag{17}$$

This equation and the local conservation of energy and momentum can both be satisfied only if

$$T_{ab} = \frac{c^4}{8\pi G} \left(R_{ab} - (\frac{1}{2}R + \Lambda)g_{ab} \right) \quad . \tag{18}$$

These are the full field equations of general relativity, including the cosmological constant Λ , which appears as an unspecified integration constant. By choosing a suitable observer, a horizon can be positioned at any required point in space-time. The equations of general relativity are thus valid generally, for all times and positions.

In other words, *the maximum force principle is a simple way to state that on horizons, energy flow is proportional to area (and surface gravity).* This connection allows to deduce the full theory of general relativity. If no maximum force would exist in nature, it would be possible to send any desired amount of energy through a given surface, including any horizon. In that case, energy flow would not be proportional to area, black holes would not be of the size they are and general relativity would not hold.

The force and power bounds have important consequences. In particular, they imply statements on cosmic censorship, the Penrose inequality, the hoop conjecture, the non-existence of plane gravitational waves, the lack of space-time singularities, new experimental tests of the theory, and on the elimination of competing theories of relativistic gravitation. These consequences are presented elsewhere.

Ref. 11

Deducing universal gravitation

Universal gravitation can be derived from the force limit in case that forces and speeds are much smaller than the maximum values. The first condition implies $\sqrt{4GMa} \ll c^2$, the second $v \ll c$ and $al \ll c^2$. To be concrete, we study a satellite circling a central mass M at distance R with acceleration a. This system, with length l = 2R, has only one characteristic speed. Whenever this speed v is much smaller than c, v^2 must be proportional both to al = 2aR and to $\sqrt{4GMa}$. Together, this implies $a = fGM/R^2$, where the numerical factor f is not yet fixed. A quick check, for example using the observed escape velocity values, shows that f = 1. Low forces and low speeds thus imply that the inverse square law of gravity describes the interaction between systems. In other words, the force limit of nature implies the universal law of gravity, as is expected.

The size of physical systems in general relativity

General relativity provides a limit on the *size* of systems: there is a limit to the amount of matter that can be concentrated into a small volume. The limit appears because a maximum force implies a limit to the depth of free fall, as seen from an observer located far away. Indeed, the speed of free fall cannot reach the speed of light; the ever-increasing red-shift during fall, together with spatial curvature, then gives an effective maximum depth of fall. A maximum depth of fall gives a minimum size or diameter l of massive systems. To see this, we rewrite the force limit in nature as

$$\frac{4Gm}{c^2} \leqslant \frac{c^2}{a} \quad . \tag{19}$$

The right side is the *upper* size limit of systems from special relativity. The left side is the Schwarzschild length of a massive system. The effects of space-time curvature make this length the *lower* size limit of a physical system:

$$l \geqslant \frac{4Gm}{c^2} \quad . \tag{20}$$

The size limit is only achieved for black holes, those well-known systems which swallow everything that is thrown into them. It is fully equivalent to the force limit. All *composite* systems in nature comply with the lower size limit. Whether elementary particles fulfil or even achieve this limit remains one of the open issues of modern physics. At present, neither experiment nor theory allow clear statements on their size. More about this below.

Ref. 13 General relativity also implies an 'indeterminacy relation':

$$\frac{\Delta E}{\Delta l} \leqslant \frac{c^4}{4G} \quad . \tag{21}$$

Since experimental data is available only for composite systems, we cannot say yet whether this inequality also holds for elementary particles. The relation is not as popular as the previous two, but common knowledge in general relativity. In fact, testing the relation, for example with binary pulsars, might lead to new tests to distinguish general relativity from competing theories.

Deducing limit values for all physical observables

The maximum force of nature is equivalent with general relativity and includes universal gravity. As a result, three simple statements on nature can be made:

quantum theory:
$$S \ge \frac{\hbar}{2}$$

special relativity: $v \le c$
general relativity: $F \le \frac{c^4}{4G}$ (22)

The limits are valid for all physical systems, whether composed or elementary, and are valid for all observers. One notes that the limit quantities of special relativity, quantum theory and general relativity can also be seen as the right hand side of the respective indeterminacy relations. Indeed, the set (3, 6, 21) of indeterminacy relations or the set (2, 5, 20) of length limits are fully equivalent to the three limit statements (22). Each set of limits can be seen as a summary of a section of twentieth century physics.

If the three fundamental limits are combined, a limit for a number of observables for physical systems appear. The following limits are valid generally, both for composite and for elementary systems:

time interval:
$$t \ge \sqrt{\frac{2G\hbar}{c^5}} = 7.6 \cdot 10^{-44} \,\mathrm{s}$$
 (23)

time distance product:
$$td \ge \frac{2G\hbar}{c^4} = 1.7 \cdot 10^{-78} \,\mathrm{sm}$$
 (24)

acceleration:
$$a \leqslant \sqrt{\frac{c^7}{2G\hbar}} = 4.0 \cdot 10^{-51} \,\mathrm{m/s^2}$$
 (25)

power, luminosity:
$$P \leqslant \frac{c^3}{4G} = 9.1 \cdot 10^{51} \,\mathrm{W}$$
 (26)

angular frequency:
$$\omega \leq 2\pi \sqrt{\frac{c^5}{2G\hbar}} = 8.2 \cdot 10^{43} / s$$
 (27)

angular momentum:
$$D \ge \frac{h}{2}$$
 = $0.53 \cdot 10^{-34}$ Js (28)

entropy:
$$S \ge k$$
 = 13.8 yJ/K (29)

With the additional knowledge that in nature, space and time can mix, one gets

distance:
$$d \ge \sqrt{\frac{2G\hbar}{c^3}} = 2.3 \cdot 10^{-35} \,\mathrm{m}$$
 (30)

area:
$$A \ge \frac{2G\hbar}{c^3} = 5.2 \cdot 10^{-70} \,\mathrm{m}^2$$
 (31)

volume
$$V \ge \left(\frac{2G\hbar}{c^3}\right)^{3/2} = 1.2 \cdot 10^{-104} \,\mathrm{m}^3$$
 (32)

curvature:
$$K \le \frac{c^3}{2G\hbar} = 1.9 \cdot 10^{69} / \text{m}^2$$
 (33)

mass density:
$$\rho \leqslant \frac{c^3}{8G^2\hbar} = 6.5 \cdot 10^{95} \,\mathrm{kg/m^3}$$
 (34)

Of course, speed, action and force are limited as already stated. Within a small numerical factor, for every physical observable these limits correspond to the Planck value. (The limit values are deduced from the commonly used Planck values simply by substituting *G* with 4*G* and \hbar with $\hbar/2$.) These values are the true *natural units* of nature. In fact, the most aesthetically pleasing solution is to redefine the usual Planck values for every observable to these extremal values by absorbing the numerical factors into the respective definitions. In the following, we call the redefined limits the (*corrected*) *Planck limits* and assume that the factors have been properly included. In other words, *the natural unit or (corrected) Planck unit is at the same time the limit value of the corresponding physical observable*.

Most of these limit statements are found scattered around the literature, though the numerical prefactors are often different. The existence of a smallest measurable distance and time interval of the order of the Planck values are discussed in quantum gravity and string Ref. 2 theory. A largest curvature has been discussed in quantum gravity. The maximal mass Ref. 3 density appears in the discussions on the energy of the vacuum.

With the present deduction of the limits, two results are achieved. First of all, the various arguments found in the literature are reduced to three generally accepted principles. Second, the confusion about the numerical factors is solved. During the history of Planck units, the numerical factors have greatly varied. For example, Planck did not include the factors of 2π . The fathers of quantum theory forgot the 1/2 in the definition of the quantum of action. And the specialists of relativity did not underline the factor 4 too often. With the present framework, the issue of the correct factors in the Planck units can be considered as settled.

The three limits of nature (22) result in a minimum distance and a minimum time interval. These minimum intervals result from the unification of quantum theory and relativity. They do not appear if the theories are kept separate. In short, unification implies that there is a smallest length in nature. The result is important: the formulation of physics as a set of limit statements shows that the continuum description of space and time is not correct. Continuity and manifolds are only approximations valid for large action values, low speed and low force values. However, the way that a minimum distance leads to a homogeneous and isotropic vacuum is still an open issue at this point.

Mass and energy limits

Mass plays a special role in all these arguments. The set of limits (22) does not allow to extract a limit statement on the mass of physical systems. To find one, the aim has to be restricted.

The Planck limits mentioned so far apply for *all* physical systems, whether they are composed or elementary. Additional limits can only be found for elementary systems. In quantum theory, the distance limit is a size limit only for *composed* systems. A particle is elementary if the system size l is smaller than any conceivable dimension:

for elementary particles:
$$l \leq \frac{\hbar}{2mc}$$
 . (35)

By using this new limit, valid only for elementary particles, the well-known mass, energy and momentum limits are found:

for elementary particles:
$$m \leqslant \sqrt{\frac{\hbar c}{8G}} = 7.7 \cdot 10^{-9} \text{ kg} = 0.42 \cdot 10^{19} \text{ GeV/c}^2$$

for elementary particles: $E \leqslant \sqrt{\frac{\hbar c^5}{8G}} = 6.9 \cdot 10^8 \text{ J} = 0.42 \cdot 10^{19} \text{ GeV}$
for elementary particles: $p \leqslant \sqrt{\frac{\hbar c^3}{8G}} = 2.3 \text{ kg m/s} = 0.42 \cdot 10^{19} \text{ GeV/c}$ (36)

Ref. 6

These single particle limits, corresponding to the corrected Planck mass, energy and momentum, were already discussed in 1968 by Andrei Sakharov, though again with different numerical prefactors. They are regularly cited in elementary particle theory. Obviously, all known measurements comply with the limits.

Corrected electromagnetic Planck limits

The discussion of limits can be extended to include electromagnetism. Using the (lowenergy) electromagnetic coupling constant α , one gets the following limits for physical systems interacting electromagnetically:

$$q \geqslant \sqrt{4\pi\varepsilon_{\rm o}\alpha c\hbar} = e = 0.16\,\mathrm{aC}\tag{37}$$

electric field

electric charge:

$$E \leqslant \sqrt{\frac{c^7}{64\pi\varepsilon_0 \alpha\hbar G^2}} = \frac{c^4}{4Ge} = 2.4 \cdot 10^{61} \,\mathrm{V/m} \tag{38}$$

magnetic field (flux density):

$$: B \leqslant \sqrt{\frac{c^5}{64\pi\varepsilon_0\alpha\hbar G^2}} = \frac{c^3}{4Ge} = 7.9 \cdot 10^{52} \,\mathrm{T}$$
(39)

capacitance:

electric resistance:

electric conductivity:

$$U \leqslant \sqrt{\frac{c^4}{32\pi\varepsilon_0 \alpha G}} = \frac{1}{e} \sqrt{\frac{\hbar c^5}{8G}} = 1.5 \cdot 10^{27} \,\mathrm{V} \tag{40}$$

inductance:
$$L \ge \frac{1}{8\pi\varepsilon_0 \alpha} \sqrt{\frac{2\hbar G}{c^7}} = \frac{1}{e^2} \sqrt{\frac{\hbar^3 G}{2c^5}} = 4.4 \cdot 10^{-40} \,\mathrm{H} \tag{41}$$

With the additional assumption that in nature at most one particle can occupy one Planck volume, one gets

charge density:
$$\rho_{\rm e} \leqslant \sqrt{\frac{\pi\varepsilon_0 \alpha}{2G^3}} \frac{c^5}{\hbar} = e\sqrt{\frac{c^9}{8G^3\hbar^3}} = 1.3 \cdot 10^{85} \,{\rm C/m^3} \quad (42)$$

$$C \ge 8\pi\varepsilon_{\rm o}\alpha\sqrt{\frac{2\hbar G}{c^3}} = e^2\sqrt{\frac{8G}{c^5\hbar}} = 1.0 \cdot 10^{-46}\,\mathrm{F} \tag{43}$$

For the case of a single conduction channel, one gets

$$R \ge \frac{1}{8\pi\varepsilon_0 \alpha c} = \frac{\hbar}{2e^2} = 2.1 \,\mathrm{k}\Omega \tag{44}$$

$$G \leqslant 8\pi\varepsilon_{\rm o}\alpha c = \frac{2e^2}{\hbar} = 0.49\,{\rm mS} \tag{45}$$

electric current:
$$I \leqslant \sqrt{\frac{2\pi\varepsilon_0 \alpha c^6}{G}} = e\sqrt{\frac{c^5}{2\hbar G}} = 7.4 \cdot 10^{23} \,\mathrm{A}$$
 (46)

Several electromagnetic limits, such as the magnetic field limit, play a role in the discussion of extreme stars and black holes. The maximal electric field plays a role in the theory of gamma ray bursters. Also the restriction of limit values for current, conductivity and resistance to single channels is well known in the literature. Their values and effects have been studied extensively in the 1980s and 90s.

Ref. 15

Ref. 7

The observation of collective excitations in semiconductors with charge e/3, and of quarks does not invalidate the charge limit for physical systems. In both cases there is no physical system – defined in the sense give above as localized mass-energy interacting incoherently with the environment – with charge e/3.

Thermodynamics

Also thermodynamics can be summarized in a single statement on motion: *There is a smallest entropy in nature*.

$$S \geqslant k$$
 . (47)

Ref. 5 The result is almost 100 years old; it was stated most clearly by Leo Szilard. In the same way as in the other fields of physics, also this result can be phrased as a indeterminacy relation:

$$\Delta \frac{1}{T} \Delta U \geqslant k \quad . \tag{48}$$

This relation has been already given by Bohr and was discussed by Heisenberg and many
Ref. 17 others. It is mentioned here in order to complete the list of indeterminacy relations and
fundamental constants. With the single particle limits, the entropy limit leads to an upper
limit for temperature:

$$T \leqslant \sqrt{\frac{\hbar c^5}{8Gk^2}} = 5.0 \cdot 10^{31} \,\mathrm{K}$$
 (49)

It corresponds to that temperature where the energy of every elementary particle is given by the (corrected) Planck energy.

Paradoxes and curiosities about Planck limits

The (corrected) Planck limits are statements about properties of nature. There is no way to measure values exceeding these limits, whatever experiment is performed. As can be expected, such a claim provokes the search for counter-examples and leads to many paradoxes.

• The minimal angular momentum might surprise at first, especially when one thinks about spin zero particles. However, the angular momentum of the statement is *total* angular momentum, including the orbital part with respect to the observer. The total angular momentum is never smaller than $\hbar/2$.

• If any interaction is stronger than gravity, how can the maximum force be determined by gravity alone, which is the weakest interaction? It turns out that in situations near the maximum force, the other interactions are negligible. This is the reason that gravity must be included in a unified description of nature.

• On first sight, it seems that electric charge can be used in such a way that the acceleration of a charged body towards a charged black hole is increased to a value exceeding the force limit. However, the changes in the horizon for charged black holes prevent this.

• Some limits are of interest if applied to the universe as a whole, such as the luminosity limit (which, together with the age and size of the universe, explains why the sky is dark at night) and the curvature limit (which is of importance near the big bang). The angular rotation limit also provides a limit on the rotation of the observed matter in the sky.

• The limit on power can be challenged when several sources almost as bright as the maximum are combined into one system. Two cases need to be distinguished. If sources are far away, the luminosities cannot added up because the energy from the sources arrives

at different times; for bright sources which are near each other, the combination forms a black hole or at least prevents all radiation to be emitted by swallowing some of it among the sources.

• A precise definition of power, or energy per time, implies the definition of a surface for which the power is measured. This surface must be physical, i.e. must not cross horizons nor have curvatures larger than the maximum possible value. Otherwise, counter-examples to the power limit can be constructed.

• The general connection that to every limit value in nature there is a corresponding indeterminacy relation is also valid for electricity. Indeed, there is an indeterminacy relation for capacitors of the form

$$\Delta C \,\Delta U \geqslant e \tag{50}$$

where e is the positron charge, C capacity and U potential difference, and one between electric current I and time t

$$\Delta I \,\Delta t \geqslant e \tag{51}$$

and both relations are found in the literature.

• The minimum area is twice the uncorrected Planck area. This means that the correct entropy relation for black holes should be $S/S_{\min} = A/2A_{\min}$. The factor 2 replaces the factor 4 that appears when the standard, uncorrected Planck area is used.

Some consequences of the limits

The existence of limit values for the length observable (and all others) has numerous consequences discussed in detail elsewhere. A few are summarized here.

The existence of a smallest length – and a corresponding shortest time interval – implies that no surface is physical if any part of it requires a localization in space-time to dimensions smaller that the minimum length. (In addition, a physical surface must not cross any horizon.) Only through this condition unphysical counter-examples to the force and power limits mentioned above are eliminated. For example, this condition has been overlooked in Bousso's older discussion of the Bekenstein's entropy bound – though not in his newer Ref. 19 ones.

Obviously, a minimum length implies that space, time and space-time are not continuous. The reformulation of general relativity and quantum theory with limit statements makes this result especially clear. The result is thus a direct consequence of the unification of quantum theory and general relativity. No other assumptions are needed.

The corrected value of the Planck length should also be the expression that appears in the so-called theories of 'doubly special relativity'. They try to expand special relativity in such Ref 20 a way that an invariant length appears in the theory.

A force limit in nature implies that no physical system can be smaller than a Schwarzschild black hole of the same mass. The force limit thus implies that *point particles do not exist*. So far, this prediction is not contradicted by observations, as the predicted sizes are so small to be outside experimental reach. If quantum theory is taken into account, this bound is sharpened. Due to the minimum length, elementary particles are now predicted to be larger

Ref. 8

Ref. 2, 18

than the corrected Planck length. Detecting the sizes of elementary particles would allow to check the force limit; this might be possible with future electric dipole measurements.

Ref. 2 In addition, a limit to observables implies that at Planck scales no physical observable can be described by real numbers, that no low-energy symmetry is valid and that matter and

vacuum cannot be distinguished. Ref. 2. 18

Minimum length and measurements

A smallest length, together with the limits for all other observables, has a central consequence. All measurements are limited in precision. A precise discussion shows that Ref. 2 measurement errors increase when the characteristic measurement energy approaches the Planck energy. In that domain, the measurement errors of any observable are comparable to the measurement values.

Limited measurement precision implies that at Planck energy it is impossible to speak about points, instants, events or dimensionality. Limited precision implies that no observable can be described by real numbers. Limited measurement precision also implies that at Planck length it is impossible to distinguish positive and negative time values: particle and anti-particles are thus not clearly distinguished at Planck scales. A smallest length in nature thus implies that there is no way to define exact boundaries of objects or elementary particles. But a boundary is what separates matter from vacuum. In short, a minimum measurement error means that at Planck scales, it is impossible to distinguish objects from vacuum with complete precision.

Ref. 2

So far, the conclusions drawn from the maximum force are essentially the same that are drawn by modern research on unified theories. The limit allows to reach the same conceptual results found by string theory and the various quantum gravity approaches. To show the power of the maximum force limit, we mention a few conclusions which go beyond them.

Measurement precision and sets

The impossibility to completely eliminate measurement errors has an additional consequence. In physics, it is assumed that nature is a *set* of components, all separable from each other. This tacit assumption is introduced in three main situations: it is assumed that matter consists of (separable) particles, that space-time consists of (separable) events or points, and that the set of states consists of (separable) initial conditions. So far, physics has built its complete description of nature on the concept of set.

Ref. 2, 18

A limited measurement precision implies that nature is *not* a set of such separable elements. A limited measurement precision implies that distinction of physical entities is possible only approximately. The approximate distinction is only possible at energies much lower than the Planck energy. As humans we live at such smaller energies; thus we can safely make the approximation. Indeed, the approximation is excellent; we do not notice any error when performing it. But the discussion of the situation at Planck energies shows that a perfect separation is impossible in principle. In particular, at the cosmic horizon, at the big bang, and at Planck scales any precise distinction between two events or between two particles becomes impossible.

Another way to reach this result is the following. Separation of two entities requires

different measurement results, such as different positions, masses, sizes, etc. Whatever observable is chosen, at Planck energy the distinction becomes impossible, due to exploding measurements errors. Only at everyday energies is a distinction approximately possible. Any distinction among two physical systems, such as between a toothpick and a mountain, is thus possible only *approximately*; at Planck scales, a boundary cannot be drawn.

A third argument is the following. In order to *count* any entities whatsoever -a set of particles, a discrete set of points, or any other discrete set of physical observables - the entities have to be separable. The inevitable measurement errors make this impossible. At Planck energy, it is impossible to count physical objects with precision.

In short, at Planck energies a perfect separation is impossible in principle. We cannot distinguish observations at Planck energies. In other words, *at Planck scale it is impossible to split nature into separate entities.* There are no mathematical elements of any kind in nature. Elements of sets cannot be defined. As a result, in nature, neither discrete nor continuous sets can be constructed.

Since sets and elements are only approximations, the concept of 'set', which assumes separable elements, is already too specialized to describe nature. Nature does not contain sets or elements. The result implies that nature cannot be described at Planck scales – i.e., with full precision – if any of the concepts used for its description presupposes sets. However, all concepts used in the past twenty-five centuries to describe nature – space, time, phase space, Hilbert space, Fock space, particle space, loop space or moduli space – are based on sets. They all must be abandoned at Planck energy. In fact, *no approach of theoretical physics so far, not even string theory or the various quantum gravity approaches, satisfies the requirement to abandon sets.* Nature has no parts. *Nature must be described by a mathematical concept which does not contain any set.* This requirement can be used to guide future searches for the unification of relativity and quantum theory.

Hilbert's sixth problem

In the year 1900, David Hilbert gave a well-known lecture in which he listed twenty-three of the great challenges facing mathematics in the twentieth century. Most problems provided challenges to many mathematicians for decades afterwards. Of the still unsolved ones, Hilbert's sixth problem challenges mathematicians and physicists to find an *axiomatic* treatment of physics. The challenge has stayed in the heads of many physicists since that time.

Since nature does not contain sets, we can deduce that such an axiomatic description of nature does not exist. The reasoning is simple; we only have to look at the axiomatic systems found in mathematics. Axiomatic systems define mathematical structures. These structures come into three main types: algebraic systems, order systems, or topological systems. Most mathematical structures – such as symmetry groups, vector spaces, manifolds, or fields – are combinations of the three. But all mathematical structures contain sets. Mathematics does not provide axiomatic systems which do not contain sets. The underlying reason is that every mathematical concept contains at least one set.

Also all physical concepts used in physics so far contain sets. For humans, it is already difficult to simply *think* without first defining a set of possibilities. However, nature is different; nature does not contain sets. Therefore, *an axiomatic formulation of physics is*

Ref. 2

Ref. 2

Ref. 18

Ref. 18

impossible. Of course, this conclusion does not rule out unification in itself; however, it does rule out an axiomatic version of it. The result surprises, as separate axiomatic treatments of quantum theory or general relativity (see above) are possible. Indeed, only their unification does not allow one. This is one of the reasons for the difficulties of unified descriptions of nature.

Outlook

Physics can be summarized in a few limit statements. They imply that in nature every physical observable is limited by a value near the Planck value. The speed limit is equivalent to special relativity, the force limit to general relativity, and the action limit to quantum theory. Even though this summary could have been made (or at least conjectured) by Planck, Einstein or the fathers of quantum theory, it is much younger. The numerical factors for all limit values are new. The limits provoke interesting Gedanken experiments; none of them leads to violations of the limits. On the other hand, the force limit is not yet within direct experimental reach.

The existence of limit values to all observables implies that the description of space-time with a continuous manifold is not correct at Planck scales; it is only an approximation. For the same reason, is predicted that elementary particles are not point-like. Nature's limits also imply the non-distinguishability of matter and vacuum. As a result, the structure of particles and of space-time remains to be clarified. So far, we can conclude that nature can be described by sets only approximately. The limit statements show that Hilbert's sixth problem cannot be solved and that unification requires fresh approaches, taking unbeaten paths into unexplored territory.

The only way to avoid sets in the description of nature seems to be to describe empty space-time, radiation and matter as made of the same underlying entity. The inclusion of space-time dualities and of interaction dualities also seems to be a necessary step. Indeed, both string theory and modern quantum gravity attempt this, though possibly not yet with the full radicalism necessary. The challenge is still open.

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