sis new terms must be added to the Lagrangian function L. If we add terms of type

$$\sum_{s=1}^{n} G_s\left(\frac{\partial \rho_s}{\partial x}\frac{\partial \phi_s}{\partial x} + \frac{\partial \rho_s}{\partial y}\frac{\partial \phi_s}{\partial y} + \frac{\partial \rho_s}{\partial z}\frac{\partial \phi_s}{\partial z}\right)$$

the left-hand side of equation (3) must be replaced by

$$\frac{\partial}{\partial x}\left(G_s\frac{\partial \rho_s}{\partial x}\right) + \frac{\partial}{\partial y}\left(G_s\frac{\partial \rho_s}{\partial y}\right) + \frac{\partial}{\partial z}\left(G_s\frac{\partial \rho_s}{\partial z}\right)$$

and a term

$$-\left[\frac{\partial}{\partial x}(G_s u_s) + \frac{\partial}{\partial y}(G_s v_s) + \frac{\partial}{\partial z}(G_s w_s)\right]$$

must be added to the right-hand side of equation (2). Thus in a non-uniform flow there may be forces on a constituent which arise from diffusion.

[1] An appropriate form of the principle was suggested by the author in a physical seminar at the California Institute of Technology in 1938 and also in a review of H. Ertel's "Methoden und Probleme der Dynamischen Meteorologie," *Zentralblatt für Mathematik und ihre Grenzgebiete*, **18**, 311 (1938). The analysis for the case of a single gas has been given in detail by H. Ertel, *Meteorologische Zeit.*, 105–108 (1939).

[2] Bernard Lewis and Guenther von Elbe, "Combustion, Flames and Explosion of Gases," Cambridge Univ. Press (1938). See also J. B. Zeldovich and D. A. Frank-Kameneckij, "On the Theory of Uniform Flame Propagation," *Compt. rend.* (*Doklady*) *de l'acad. des sciences de l'U. R. S. S.*, **19**, 693–697 (1938).

A QUESTION IN GENERAL RELATIVITY

BY L. I. SCHIFF

DEPARTMENT OF PHYSICS, UNIVERSITY OF CALIFORNIA

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I.—The following question, which provides an illuminating application of general relativity to electrodynamics, has been put to me by Professor Oppenheimer. Consider two concentric spheres with equal and opposite total charges uniformly distributed over their surfaces. When the spheres are at rest, the electric and magnetic fields outside the spheres vanish. When the spheres are in uniform rotation about an axis through their center, the electric field outside vanishes, while the magnetic field does not, since the magnetic moment of each of the spheres is proportional to the square of its radius. Suppose that the spheres are stationary; then an observer traveling in a circular orbit around the spheres should find no field, for since all of the components of the electromagnetic field tensor vanish in one coördinate system, they must vanish in all coördinate systems.

On the other hand, the spheres are rotating with respect to this observer, and so he should experience a magnetic field.[1]

Before discussing the resolution of this apparent paradox, we shall present some general remarks on the relativity of rotational motion.

II.—It is clear in the above arrangement that an observer A at rest with respect to the spheres does not obtain the same results from physical experiments as an observer B who is rotating about the spheres. The very fundamental difference between coördinate systems in which A and in which B are at rest is ascribed, in the general theory of relativity, to the effect of distant masses (extragalactic nebulae), which are at rest with respect to A, but in violent motion with respect to B. That this is a plausible explanation was shown by Thirring's[2] calculation of the motion of a free particle inside a thin spherical shell of matter in uniform rotation about an axis through its center. He found that forces very similar to centrifugal and Coriolis forces appeared, but that they were too small to represent the forces acting on a particle rotating with the same angular velocity as the shell, by a factor of the order of the ratio of the gravitational radius to the actual radius of the shell.

From this relatively simple calculation, one infers that if the gravitational field equations could be solved in a coördinate system in which the actual distribution of matter in the universe is in uniform rotation, one would obtain exactly the usual centrifugal and Coriolis accelerations for a particle. The far simpler procedure that is usually followed is to take the approximately Galilean metric (neglecting local masses) that we know gives the experimentally correct equations of motion of a particle in the coördinate system in which the distant masses are at rest, transform it to the rotating coördinate system, and use this transformed expression for the metric to compute the motion of a particle. Because of the covariance of the gravitational field equations, one feels sure that the two calculations would give identical results. Thirring's direct calculation, however, has the advantage of providing some insight into the effect of rotating distant masses in warping the expression for the metric from its Galilean form.

Similarly, we know experimentally that the fields outside the charged spheres vanish in system A (in which the spheres and the distant masses are at rest), and so the covariance of Maxwell's equations guarantees that the fields will also vanish outside the spheres in system B. It is of interest, however, to see by direct calculation how it is that the spheres, which are rotating with respect to system B, do not give rise to a magnetic field outside. To see this, we must of course know the expression for the metric in system B, and we shall obtain this by transformation from the (approximately) Galilean metric of system A. The warping of the expression for the metric thus obtained for system B can be ascribed to the rotation of the distant masses in this system.

III.—The covariant formulation of Maxwell's equations[3] is:

$$\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} + \frac{\partial F_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial F_{\sigma\mu}}{\partial x^{\nu}} = 0, \tag{1}$$

$$\frac{\partial}{\partial x^{\nu}}(\sqrt{-g}\, F^{\mu\nu}) = \sqrt{-g}\, J^{\mu}. \tag{2}$$

The coördinate transformation from system A (subscript 0) to system B rotating with angular velocity ω can be taken as (velocity of light set equal to unity):

$$\begin{aligned} x &= x_0 \cos \omega t_0 + y_0 \sin \omega t_0, \\ y &= -x_0 \sin \omega t_0 + y_0 \cos \omega t_0, \\ z &= z_0, \\ t &= t_0, \\ r^2 &= x^2 + y^2 + z^2 = x_0^2 + y_0^2 + z_0^2, \end{aligned} \tag{3}$$

so that if the metrical tensor has its Galilean form in system A, it assumes in system B the form:[4]

$$\begin{aligned} &g_{11} = g_{22} = g_{33} = -1, \quad g_{44} = 1 - \omega^2(x^2 + y^2), \\ &g_{14} = g_{41} = \omega y, \qquad\qquad g_{24} = g_{42} = -\omega x, \end{aligned} \tag{4}$$

and all other $g_{\mu\nu}$ vanish. Since the determinant $g = -1$, the electromagnetic field equations (1) and (2) are unaltered by the transformation (3). But the connections between the covariant and contravariant components of the field tensor depend on the metrical tensor:

$$F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}, \tag{5}$$

and so the field equations are different in the two coördinate systems when written entirely in terms of either the covariant or the contravariant components.

The calculations can be made in terms of any one set of components: covariant, contravariant or mixed.[5] We shall use here the covariant form of the tensor since the field equations (1) then have their usual structure, and the extra terms in equations (2) can be put in the form of an additional current. If we define:

$$\begin{aligned} &F_{23} = H_x, \quad F_{31} = H_y, \quad F_{12} = H_z, \\ &F_{14} = E_x, \quad F_{24} = E_y, \quad F_{34} = E_z, \end{aligned} \tag{6}$$

we find from (4) and (5) that:

$$
\begin{aligned}
F^{23} &= (1 - \omega^2x^2)H_x - \omega^2xyH_y - \omega xE_z, \\
F^{31} &= (1 - \omega^2y^2)H_y - \omega^2xyH_x - \omega yE_z, \\
F^{12} &= (1 - \omega^2x^2 - \omega^2y^2)H_z + \omega xE_x + \omega yE_y, \\
F^{41} &= E_x - \omega xH_z, \\
F^{42} &= E_y - \omega yH_z, \\
F^{43} &= E_z + \omega xH_x + \omega yH_y.
\end{aligned} \tag{7}
$$

Then equations (1) have their usual form:[6]

$$\operatorname{div} \mathbf{H} = 0, \quad \mathbf{curl}\ \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = 0, \tag{8}$$

while equations (2) may be written:

$$\operatorname{div} \mathbf{E} - \rho = \sigma, \quad \mathbf{curl}\ \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} - \mathbf{J} = \mathbf{j}, \tag{9}$$

where we define:

$$
\begin{aligned}
\sigma &= 2\omega H_z + \omega x\frac{\partial H_z}{\partial x} + \omega y\frac{\partial H_z}{\partial y} - \omega x\frac{\partial H_x}{\partial z} - \omega y\frac{\partial H_y}{\partial z}, \\
j_x &= 2\omega^2yH_z - \omega E_y - \omega^2xy\frac{\partial H_x}{\partial z} - \omega^2y^2\frac{\partial H_y}{\partial z} - \omega x\frac{\partial H_z}{\partial t} + \\
&\qquad \omega^2(x^2 + y^2)\frac{\partial H_z}{\partial y} - \omega x\frac{\partial E_x}{\partial y} - \omega y\frac{\partial E_y}{\partial y} - \omega y\frac{\partial E_z}{\partial z}, \\
j_y &= -2\omega^2xH_z + \omega E_x + \omega^2x^2\frac{\partial H_x}{\partial z} + \omega^2xy\frac{\partial H_y}{\partial z} - \omega y\frac{\partial H_z}{\partial t} - \\
&\qquad \omega^2(x^2 + y^2)\frac{\partial H_z}{\partial x} + \omega x\frac{\partial E_x}{\partial x} + \omega y\frac{\partial E_y}{\partial x} + \omega x\frac{\partial E_z}{\partial z}, \\
j_z &= \omega^2yH_x - \omega^2xH_y + \omega^2y^2\frac{\partial H_y}{\partial x} - \omega^2x^2\frac{\partial H_x}{\partial y} - \omega y\frac{\partial H_z}{\partial t} + \\
&\qquad \omega^2xy\frac{\partial H_x}{\partial x} - \omega^2xy\frac{\partial H_y}{\partial y} + \omega y\frac{\partial E_z}{\partial x} - \omega x\frac{\partial E_z}{\partial y}.
\end{aligned} \tag{10}
$$

For any spherically symmetrical distribution of charge that is stationary in system A (in particular for our two charged spheres), we have in system B:

$$J_x = \omega y\rho, \quad J_y = -\omega x\rho, \quad J_z = 0. \tag{11}$$

Substituting (10) and (11) into (9), one can either guess a solution or, more satisfyingly, obtain one directly by treating terms in ω as a perturbation, in which case it is possible to carry the perturbation calculation to arbitrarily high orders. The unperturbed solution ($\omega = 0$) that satisfies the boundary conditions (no singularities and vanishing at infinity) is:

$$\mathbf{H} = 0, \quad \mathbf{E} = \frac{4\pi \mathbf{r}}{r^3} \int_0^r \rho(\zeta)\zeta^2 d\zeta. \tag{12}$$

To obtain the fields to first order in ω, we notice that $\mathbf{H}$ is at most of order $\omega\mathbf{E}$, so that to this order we may write equations (10):

$$\begin{aligned}
\sigma &= 0 \\
j_x &= -\omega E_y - \omega x \frac{\partial E_x}{\partial y} + \omega y \frac{\partial E_x}{\partial x} - \omega y \operatorname{div} \mathbf{E}, \\
j_y &= \omega E_x - \omega x \frac{\partial E_y}{\partial y} + \omega y \frac{\partial E_y}{\partial x} + \omega x \operatorname{div} \mathbf{E}, \\
j_z &= \omega y \frac{\partial E_z}{\partial x} - \omega x \frac{\partial E_z}{\partial y}.
\end{aligned} \tag{13}$$

Putting the unperturbed fields (12) into (13), we obtain:

$$\begin{aligned}
j_x &= -\omega y \operatorname{div} \mathbf{E} = -\omega y \rho = -J_x, \\
j_y &= \omega x \operatorname{div} \mathbf{E} = \omega x \rho = -J_y, \\
j_z &= 0,
\end{aligned} \tag{14}$$

and equations (9) are satisfied by the fields (12) to the first order. This result is readily extended to arbitrarily high orders in ω, and it is seen that the fields (12) are an exact solution of the field equations (9) and (10) with the current (11). Since for our two oppositely charged spheres, the integral in (12) vanishes for values of r greater than the radius of the larger sphere, both the electric and magnetic fields vanish outside the spheres in system B, in agreement with the result obtained by transformation from system A.

It is of interest to note that the vanishing of the fields in this calculation is due to the cancellation of the actual current $\mathbf{J}$ with other terms (right side of (9)) that behave in this respect like a current. The appearance of this extra current (10) is due to the action of the rotating distant masses, via the metric, on the electromagnetic fields.

[1] From these remarks it is evident, for example, that one cannot calculate the magnetic field about a single charged rotating sphere by transforming the electrostatic field of such a sphere at rest.

[2] Thirring, *Phys. Zeits.*, **19**, 33 (1918); **22**, 29 (1921).

[3] Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford, 1934), p. 259.

[4] The indices 1, 2, 3, 4 refer to x, y, z, t, respectively.

[5] It is the mixed tensor that gives the acceleration of a charged particle (cf. Tolman, reference 2, p. 260); however, any one form is readily obtained from any other.

[6] Bold-face symbols indicate ordinary three-dimensional vectors.