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A Spinor Approach to General Relativity

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A calculus for general relativity is developed in which the basic role of tensors is taken over by spinors. The Riemann–Christoffel tensor is written in a spinor form according to a scheme of Witten. It is shown that the curvature of empty space can be uniquely characterized by a totally symmetric four-index spinor which satisfies a first order equation formally identical with one for a zero rest-mass particle of spin two. However, the derivatives used here are covariant, so that on iteration, instead of the usual wave equation, a nonlinear "source" term appears. The case when a source-free electromagnetic field is present is also considered. (No quantization is attempted here.)

The "gravitational density" tensor of Robinson and Bel is obtained in a natural way as a striking analogy with the spinor expression for the Maxwell stress tensor in the electromagnetic case. It is shown that the curvature tensor determines four gravitational principal null directions associated with flow of "gravitational density", which supplement the two electromagnetic null directions of Synge. The invariants and Petrov type of the curvature tensor are analyzed in terms of these, and a natural classification of curvature tensors is given.

An essentially coordinate-free method is outlined, by which any analytic solution of Einstein's field equations may, in principle, be found. As an elementary example the gravitational and gravitational-electromagnetic plane wave solutions are obtained.

1. INTRODUCTION

An essentially coordinate-free attitude to general relativity will be adopted here. The tensors and spinors occurring are best thought of not as sets of components, but as geometric objects subject to certain formal rules of manipulation. A spinor formalism will be used instead of the usual tensor one, spinors appearing to fit in with general relativity in a remarkably natural way. This adds to a belief that spinors are basically simpler and perhaps more deep-rooted than tensors.

The usual correspondence between tensors and spinors (*1*, *2*) is obtained by the use of a mixed quantity† $\sigma_{\mu}{}^{AB'}$ satisfying the equation

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† For each of the four values of μ, $\sigma_{\mu}^{AB'}$ is a (2 x 2) Hermitian matrix.

$$\sigma_\mu{}^{A}{}_{C'}\sigma_\nu{}^{BC'} + \sigma_\nu{}^{A}{}_{C'}\sigma_\mu{}^{BC'} = g_{\mu\nu}\epsilon^{AB}, \tag{1.1}$$

where ϵ^{AB}, together with ϵ_{AB}, $\epsilon^{A'B'}$, and $\epsilon_{A'B'}$ is a skew-symmetric "metric" spinor for the 2-dimensional complex spin space. The components of the ϵ's may be taken as $\pm 1, 0$. (To raise or lower a spinor index, one of the ϵ's must be used, e.g., $\xi^A = \epsilon^{AB}\xi_B$, $\xi_B = \xi^A\epsilon_{AB}$.) Primed indices* refer to the complex conjugate spin space. Roman capitals are used here for spinor indices and Greek letters for tensor indices. The spinor equivalent of any tensor is a quantity which has an unprimed and a primed spinor index replacing each tensor index. For example, for a tensor $X^{\lambda\mu}{}_\nu$, we have

$$X^{\lambda\mu}{}_\nu \leftrightarrow X^{AB'CD'}{}_{EF'},$$

where

$$X^{AB'CD'}{}_{EF'} = \sigma_\lambda{}^{AB'}\sigma_\mu{}^{CD'}X^{\lambda\mu}{}_\nu\sigma^\nu{}_{EF'}$$

and

$$X^{\lambda\mu}{}_\nu = \sigma^\lambda{}_{AB'}\sigma^\mu{}_{CD'}X^{AB'CD'}{}_{EF'}\sigma_\nu{}^{EF'},$$

(with $\sigma^\mu{}_{AB'} = g^{\mu\nu}\sigma_\nu{}^{CD'}\epsilon_{CA}\epsilon_{D'B'}$). We have

$$g_{AB'CD'} = \epsilon_{AC}\epsilon_{B'D'},\ \delta^{AB'}_{CD'} = \delta^A_C\delta^{B'}_{D'},\ g^{AB'CD'} = \epsilon^{AC}\epsilon^{B'D'}. \tag{1.2}$$

The algebraic tensor operations can now all be interpreted as spinor operations. Also the notions of reality of tensors, and of complex conjugate, are interpreted in spinor form with

$$\bar{X}^{\lambda\mu}{}_\nu \leftrightarrow \bar{X}^{A'BC'D}{}_{E'F}$$

so that the roles of primed and unprimed indices are interchanged.[1] Thus reality of tensors is expressed as a Hermitian property of the corresponding spinors.

In addition to the usual correspondence between tensors and spinors given above, there is also a well-known correspondence between real skew-symmetric second rank tensors and symmetric second rank spinors (*2*). Thus if $F_{\mu\nu}$ is real and skew-symmetric, we have

$$F_{AB'CD'} = \tfrac{1}{2}\{\phi_{AC}\epsilon_{B'D'} + \epsilon_{AC}\bar{\phi}_{B'D'}\}, \tag{1.3}$$

where ϕ_{AB} is a uniquely defined symmetric spinor. The right-hand side of (1.3) expresses $F_{AB'CD'}$ as the sum of the part symmetric in A, C (and therefore skew in B′, D′) and the part skew in A, C (and symmetric in B′, D′). (Any skew

* Primed indices are used here rather than the more usual dotted indices, for typographical reasons.

[1] Many authors would omit the bar on the right-hand side. The choice here here is made for reasons of clarity.

pair of spinor indices may be split off as an ϵ-spinor.) A corresponding procedure can be applied to *any* skew-symmetric pair of tensor indices. A tensor with r skew-symmetric pairs of indices thus gives rise to 2^r spinors each with r symmetric pairs of indices in a decomposition similar to (1.3). If the tensor is real, these spinors are paired off as complex conjugates. For an example, see (2.2).

If the tensor $H_{\mu\nu}$ "dual" to $F_{\mu\nu}$ is defined by

$$H_{\mu\nu} = \tfrac{1}{2}\sqrt{-g}F^{\rho\sigma}\epsilon_{\mu\nu\rho\sigma}, \tag{1.4}$$

we have

$$H_{AB'CD'} = \tfrac{1}{2}\{-i\phi_{AC}\epsilon_{B'D'} + i\epsilon_{AC}\bar{\phi}_{B'C'}\}, \tag{1.5}$$

since if

$$\mathcal{E}^{\alpha\beta}_{\mu\nu} = \sqrt{-g}\epsilon_{\rho\sigma\mu\nu}g^{\alpha\rho}g^{\beta\sigma},$$

then

$$\mathcal{E}^{AB'CD'}_{EF'GH'} = i\delta^A_E\delta^C_G\delta^{B'}_{H'}\delta^{D'}_{F'} - i\delta^A_G\delta^C_E\delta^{B'}_{F'}\delta^{D'}_{H'}. \tag{1.6}$$

(Actually, formulas (1.5) and (1.6) are only correct for one class of choices of $\sigma_\mu{}^{AB}$ satisfying (1.1). If $\sigma_\mu{}^{AB}$ had been chosen from the other class of solutions, the signs of the right-hand sides of (1.5) and (1.6) would be reversed. It will be supposed that the $\sigma_\mu{}^{AB}$ have, in fact, been selected from the appropriate class.) In a similar way any tensor possessing a pair of skew-symmetric indices may be "dualized" with respect to that pair of indices. The spinor decomposition of the "dualized" tensor then differs from that of the original tensor in that the relevant ϵ_{AC} and $\epsilon_{B'D'}$ are, respectively, multiplied by i and by $-i$. This again follows from (1.6). For an example, see (2.6).

General relativity requires, in addition to algebraic properties of tensors, the notion of covariant derivative. The symbol ∂_μ, or correspondingly $\partial_{AB'}$, will be used here to denote *covariant* differentiation. The covariant derivatives of $g_{\mu\nu}$ and of $\sigma_\mu{}^{AB'}$ are both required to be zero.[2] This implies that

$$\partial_\mu\{\epsilon_{AB}\epsilon_{C'D'}\} = 0$$

(see *2*). The stronger conditions

$$\partial_\mu\epsilon_{AB} = 0 \quad \text{and} \quad \partial_\mu\epsilon_{A'B'} = 0 \tag{1.7}$$

will be adopted here (*3*). This enables one to raise and lower spinor indices under the derivative symbol, but it precludes the use of phase transformations of the spinors to generate the electromagnetic field. However, the electromagnetic field

[2] "Spin affinities" $\Gamma^A{}_{B\mu}$, $\bar{\Gamma}^{A'}{}_{B'\mu}$ are introduced to deal with the spinor indices. The conditions (1.7) imply that these spin affinities can be expressed explicitly in terms of $\sigma_\mu{}^{AB'}$ and its coordinate derivatives (see Ruse (*3*)).

will appear here as being associated with spinor transformations in a different way (see 3.13). These two procedures do not appear to combine in an altogether natural way. The simplest formalism, when charges are not present, seems to be obtained when such phase transformations are not permitted.

The point of view adopted here is nearer to that of Rainich (*4*) and of Misner and Wheeler (*5*) in which the electromagnetic field is obtained from the curvature of space-time alone. These phase transformations would not be related in any way to the geometry of the space-time.

2. THE CURVATURE SPINORS

Since the symbol ∂_μ here stands for covariant differentiation, we have

$$\partial_\mu \partial_\nu \neq \partial_\nu \partial_\mu ,$$

the commutation of two ∂'s giving rise to the Riemann-Christoffel tensor $R_{\mu\nu\rho\sigma}$. In fact, we have

$$\{\partial_\mu \partial_\nu - \partial_\nu \partial_\mu\} X_\rho = R_{\nu\mu\rho\sigma} X^\sigma. \tag{2.1}$$

The tensor $R_{\mu\nu\rho\sigma}$ is skew-symmetric in μ, ν and in ρ, σ. Thus, following Witten (*6*), we can apply the procedure outlined in Section 1 and obtain

$$\begin{aligned} R_{AE'BF'CG'DH'} = \tfrac{1}{2}\{\chi_{ABCD}\epsilon_{E'F'}\epsilon_{G'H'} + \epsilon_{CD}\phi_{ABG'H'}\epsilon_{E'F'} \\ + \epsilon_{AB}\bar{\phi}_{E'F'CD}\epsilon_{G'H'} + \epsilon_{AB}\epsilon_{CD}\bar{\chi}_{E'F'G'H'}\}. \end{aligned} \tag{2.2}$$

The spinors χ_{ABCD} and ϕ_{ABGH} are the uniquely defined *curvature spinors*. However, this differs from Witten's form by a factor $\frac{1}{2}$ which is included here for reasons of convenience. From the symmetries of $R_{\mu\nu\rho\sigma}$, it follows that

$$\chi_{ABCD} = \chi_{BACD} = \chi_{ABDC} = \chi_{CDAB} \tag{2.3}$$

and

$$\phi_{ABC'D'} = \phi_{BAC'D'} = \phi_{ABD'C'} = \bar{\phi}_{C'D'AB} . \tag{2.4}$$

Let the right dual $S_{\mu\nu\rho\sigma}$ of $R_{\mu\nu\rho\sigma}$ be defined by

$$S_{\mu\nu\rho\sigma} = \tfrac{1}{2} \sqrt{-g} R_{\mu\nu}{}^{\alpha\beta} \epsilon_{\alpha\beta\rho\sigma} . \tag{2.5}$$

Then from (1.6), we have

$$\begin{aligned} S_{AE'BF'CG'DH'} = \frac{i}{2} \{ -\chi_{ABCD}\epsilon_{E'F'}\epsilon_{G'H'} + \epsilon_{CD}\phi_{ABG'H'}\epsilon_{E'F'} - \epsilon_{AB}\bar{\phi}_{E'F'CD}\epsilon_{G'H'} \\ + \epsilon_{AB}\epsilon_{CD}\chi_{E'F'G'H'} \}. \end{aligned} \tag{2.6}$$

Now, the symmetry relation $R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0$ is equivalent to

$$S_{\mu\nu\rho}{}^{\nu} = 0,$$

so that multiplying (2.6) by $\epsilon^{BD}\epsilon^{F'H'}$ should give zero (see 1.2). Hence,

$$-\chi_{ABC}{}^{B}\epsilon_{E'G'} - \phi_{ACG'E'} + \bar{\phi}_{E'G'CA} + \epsilon_{AC}\bar{\chi}_{E'F'G'}{}^{F'} = 0$$

The ϕ terms cancel by (2.4), so we have

$$\chi_{ABC}{}^{B} = \lambda\epsilon_{AC}\,, \tag{2.7}$$

where λ is *real* and given by

$$\lambda = \tfrac{1}{2}\chi_{AB}{}^{AB} = \tfrac{1}{2}\bar{\chi}_{E'F'}{}^{E'F'}. \tag{2.8}$$

The reality of λ is, in fact, the only thing new we get out of this identity since (2.7) is implied by (2.3) in any case.

The relations (2.3), (2.4), and (2.8) are the only algebraic relations necessarily satisfied by χ_{ABCD} and $\phi_{ABC'D'}$ for a general Riemannian space, since they imply that an $R_{\mu\nu\rho\sigma}$ given by (2.2) has the required symmetry properties. These relations are all to be found in Witten's paper. However, χ_{ABCD} and $\phi_{ABC'D'}$ also satisfy a differential relation obtained from the Bianchi identity

$$\partial_\tau R_{\mu\nu\rho\sigma} + \partial_\rho R_{\mu\nu\sigma\tau} + \partial_\sigma R_{\mu\nu\tau\rho} = 0.$$

This is equivalent to

$$\partial^\sigma S_{\mu\nu\rho\sigma} = 0,$$

i.e., (by 2.6)

$$\partial^{D}{}_{G'}\chi_{ABCD}\epsilon_{E'F'} - \partial_C{}^{H'}\phi_{ABG'H'}\epsilon_{E'F'} + \epsilon_{AB}\partial^{D}{}_{G'}\bar{\phi}_{E'F'CD} - \epsilon_{AB}\partial_C{}^{H'}\bar{\chi}_{E'F'G'H'} = 0.$$

Separating this into the two equations obtained by, respectively, symmetrizing and skew-symmetrizing with respect to A, B, we get

$$\partial^{D}{}_{G'}\chi_{ABCD} = \partial_C{}^{H'}\phi_{ABG'H'} \tag{2.9}$$

and its complex conjugate. The Bianchi identity is therefore equivalent to (2.9).

There are also relations connecting χ_{ABCD} and $\phi_{ABG'H'}$ with covariant second derivatives of spinors, corresponding to the vector relation (2.1). Let ξ_A be an arbitrary spinor field and define

$$X_{PR'QS'} = \xi_P\xi_Q\epsilon_{R'S'}\,. \tag{2.10}$$

Now (2.1) generalizes to (and in fact implies)

$$\{\partial_\mu\partial_\nu - \partial_\nu\partial_\mu\}X_{\rho\sigma} = R_{\nu\mu\rho\alpha}X^{\alpha}{}_{\sigma} + R_{\nu\mu\sigma\alpha}X_{\rho}{}^{\alpha}. \tag{2.11}$$

But $\partial_\mu\partial_\nu - \partial_\nu\partial_\mu$ is skew-symmetric in μ, ν so that the decomposition (1.3) can be applied:

$$\begin{aligned}\partial_{AC'}\partial_{BD'} - \partial_{BD'}\partial_{AC'} \equiv{}& \tfrac{1}{2}\epsilon_{C'D'}\{\partial_{AF'}\partial_B{}^{F'} + \partial_{BF'}\partial_A{}^{F'}\}\\ &+ \tfrac{1}{2}\epsilon_{AB}\{\partial_{EC'}\partial^{E}{}_{D'} + \partial_{ED'}\partial^{E}{}_{C'}\}.\end{aligned} \tag{2.12}$$

Thus, (2.11) can be split into two equations each of which must hold separately, one symmetric in A, B (and skew in C′, D′) and the other skew in A, B (and symmetric in C′, D′). Also, any skew pair of indices can be split off as an ϵ-spinor and these may be cancelled throughout the equation. Hence by (2.2), the equation symmetric in A, B is

$$\begin{aligned}\{\partial_{AF'}\partial_{B}{}^{F'} + \partial_{BF'}\partial_{A}{}^{F'}\}\xi_P\xi_Q\epsilon_{R'S'} &= \chi_{ABPC}\xi^C\xi_Q\epsilon_{R'S'} \\ &+ \chi_{ABQC}\xi_P\xi^C\epsilon_{R'S'} - \phi_{ABR'C'}\xi_P\xi_Q\delta_{S'}^{C'} + \phi_{ABS'C'}\xi_P\xi_Q\delta_{R'}^{C'}.\end{aligned} \tag{2.13}$$

The ϕ terms cancel and, because of (1.7), the $\epsilon_{R'S'}$ term may be divided out. Also,

$$\partial_\mu\partial_\nu(\xi_P\xi_Q) = \xi_P\partial_\mu\partial_\nu\xi_Q + \xi_Q\partial_\mu\partial_\nu\xi_P + (\partial_\nu\xi_P)(\partial_\mu\xi_Q) + (\partial_\mu\xi_Q)(\partial_\nu\xi_P),$$

whence

$$\{\partial_\mu\partial_\nu - \partial_\nu\partial_\mu\}(\xi_P\xi_Q) = \xi_P\{\partial_\mu\partial_\nu - \partial_\nu\partial_\mu\}\xi_Q + \xi_Q\{\partial_\mu\partial_\nu - \partial_\nu\partial_\mu\}\xi_P .$$

It follows that

$$\begin{aligned}\{\partial_{AF'}\partial_{B}{}^{F'} + \partial_{BF'}\partial_{A}{}^{F'}\}(\xi_P\xi_Q) &= \xi_P\{\partial_{AF'}\partial_{B}{}^{F'} + \partial_{BF'}\partial_{A}{}^{F'}\}\xi_Q \\ &\quad + \xi_Q\{\partial_{AF'}\partial_{B}{}^{F'} + \partial_{BF'}\partial_{A}{}^{F'}\}\xi_P \\ &= \xi_P\chi_{ABQC}\xi^C + \xi_Q\chi_{ABPC}\xi^C\end{aligned}$$

by (2.13). Multiplying this equation by $\eta^P\eta^Q$ where η^A is chosen arbitrarily, we get

$$2(\eta^P\xi_P)(\eta^Q\{\partial_{AF'}\partial_{B}{}^{F'} + \partial_{BF'}\partial_{A}{}^{F'}\}\xi_Q) = 2(\eta^P\xi_P)(\eta^Q\chi_{ABQC}\xi^C).$$

Since η^A is arbitrary, we may divide by $2(\eta^P\xi_P)$ and obtain

$$\{\partial_{AF'}\partial_{B}{}^{F'} + \partial_{BF'}\partial_{A}{}^{F'}\}\xi_Q = \chi_{ABQC}\xi^C. \tag{2.14}$$

Also the equation obtained from (2.11) which is skew in A, B and symmetric in C′, D′ gives rise to

$$\{\partial_{EC'}\partial^{E}{}_{D'} + \partial_{ED'}\partial^{E}{}_{C'}\}\xi_Q = \phi_{QAC'D'}\xi^A \tag{2.15}$$

in an exactly similar way. The corresponding results for a primed spinor $\zeta_{A'}$ are obtained by taking the complex conjugates of (2.14) and (2.15). Thus,

$$\{\partial_{EC'}\partial^{E}{}_{D'} + \partial_{ED'}\partial^{E}{}_{C'}\}\zeta_{A'} = \bar{\chi}_{C'D'A'B'}\zeta^{B'} \tag{2.16}$$

and

$$\{\partial_{AF'}\partial_{B}{}^{F'} + \partial_{BF'}\partial_{A}{}^{F'}\}\zeta_{C'} = \phi_{ABC'D'}\zeta^{D'}. \tag{2.17}$$

The corresponding relations for spinors with more than one index can be obtained from (2.14), . . ., (2.17) since any spinor can be expressed as a linear combination of products of one-index spinors. Spinors with upper indices present no extra problem because the derivative of an ϵ-spinor is zero. As an example, we have

$$\{\partial_{AF'}\partial_B{}^{F'} + \partial_{BF'}\partial_A{}^{F'}\}\beta_C{}^{DE'} = \chi_{ABCP}\beta^{PDE'} + \chi_{AB}{}^D{}_P\beta_C{}^{PE'} + \phi_{AB}{}^{E'}{}_{Q'}\beta_C{}^{DQ'}.$$

In particular, by applying this to a "vector" $X^{DE'}$, and using (2.12) and (2.2), we can get back to (2.1). (It is not so easy to obtain (2.14), . . ., (2.17) directly from (2.1) rather than from (2.11), since the fact that the ϵ-spinors are constant must be used somewhere in the argument.)

The geometry of a Riemannian space (with signature $+---$) can thus be described entirely in spinor terms, with the role of the curvature tensor being taken over by spinors χ_{ABCD}, $\phi_{ABE'F'}$ satisfying (2.3), (2.4), (2.8), (2.9), (2.14), (2.15), (2.16), and (2.17).

3. THE EINSTEIN CONDITIONS

The theory of Section 2 will now be specialized to two cases of particular note, namely empty space-time and source-free electromagnetic field.

The Ricci tensor $R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}$ has the spinor form

$$\begin{aligned} R_{AC'BD'} &= \tfrac{1}{2}\{\chi_{EA}{}^E{}_B\epsilon_{C'D'} - 2\phi_{ABC'D'} + \epsilon_{AB}\bar{\chi}_{F'C'}{}^{F'}{}_{D'}\} \\ &= \lambda\epsilon_{AB}\epsilon_{CD} - \phi_{ABC'D'} \end{aligned}$$

by (2.2), (2.7), (2.8). The scalar curvature $R = R^\sigma{}_\sigma$ is given by

$$R = 4\lambda \tag{3.1}$$

because of the symmetry of $\phi_{ABC'D'}$. The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R$ takes the form

$$G_{AC'BD'} = -\lambda\epsilon_{AB}\epsilon_{C'D'} - \phi_{ABC'D'}\,. \tag{3.2}$$

Einstein's equations $G_{\mu\nu} = 0$ for empty space clearly give

$$\phi_{ABC'D'} = 0 \tag{3.3}$$

and

$$\lambda = 0.$$

On the other hand, if it is required to include a cosmological term in Einstein's equations, we have only $\phi_{ABC'D'} = 0$, the cosmological constant being equal to λ by (3.1).

Supposing for the moment that the cosmological constant is zero, (2.7) gives

$$\chi_{ABC}{}^B = 0,$$

that is, χ_{ABCD} is symmetric in B and D. But by (2.3), it is also symmetric A, B and in C, D. It is therefore *completely symmetric* in all its indices.

It is a remarkable and perhaps significant fact, that only for a manifold with the apparently arbitrary $+---$ signature of our space-time, and which satisfies the Einstein equations for empty space, can its curvature be characterized by so natural an object as a totally symmetric four-index spinor. The geometry of this spinor will be dealt with in Section 3.

If a cosmological term (or matter) is present, we can write

$$\chi_{ABCD} = \psi_{ABCD} + \frac{\lambda}{3}\{\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC}\} \tag{3.4}$$

and then ψ_{ABCD} will be totally symmetric even if $\lambda \neq 0$. The spinor ψ_{ABCD} defined by (3.4) will be called here the *gravitational* spinor (even in cases where $\phi_{ABC'D'} \neq 0$). It corresponds uniquely to Weyl's conformal tensor $C_{\mu\nu\rho\sigma}$.

Relation (2.9) gives (with $\phi_{ABC'D'} = 0$)

$$\partial^{DE'}\psi_{ABCD} = 0 \tag{3.5}$$

and of course $\partial^{DE'}\lambda = 0$ also. Equation (3.5) has the suggestive appearance of being formally identical with a spinor equation for a zero rest-mass particle of spin two. (See Dirac (7) and compare (3.10).) However, the differentiation used here is covariant, so that derivatives do not commute. Hence, new features arise with second and higher derivatives. In particular, it is not true that Eq. (3.5) leads to the covariant wave equation upon iteration with $\partial_{FE'}$. We have

$$\partial_{FE'}\partial_D{}^{E'} \equiv \tfrac{1}{2}\{\partial_{FE'}\partial_D{}^{E'} + \partial_{DE'}\partial_F{}^{E'}\} + \tfrac{1}{2}\epsilon_{FD}\Box, \tag{3.6}$$

where

$$\Box \equiv \partial_\mu\partial^\mu \equiv \partial_{FE'}\partial^{FE'}.$$

Also,

$$\{\partial_{FE'}\partial_D{}^{E'} + \partial_{DE'}\partial_F{}^{E'}\}\xi_A = \psi_{FDAB}\xi^B - \frac{\lambda}{3}\{\xi_D\epsilon_{FA} + \xi_F\epsilon_{DA}\} \tag{3.7}$$

by (2.14) and (3.4). Now (3.6) gives

$$0 = \partial_{FE'}\partial_D{}^{E'}\psi_{ABC}{}^D = \tfrac{1}{2}\{\partial_{FE'}\partial_D{}^{E'} + \partial_{DE'}\partial_F{}^{E'}\}\psi_{ABC}{}^D - \tfrac{1}{2}\Box\psi_{ABCF}.$$

By (3.7), this leads to

$$\begin{aligned}\Box\psi_{ABCD} &= \psi_{ABEF}\psi_{CD}{}^{EF} + \psi_{ACEF}\psi_{DB}{}^{EF} + \psi_{ADEF}\psi_{BC}{}^{EF} - 2\lambda\psi_{ABCD} \\ &= 3\psi_{(AB}{}^{EF}\psi_{CD)EF} - 2\lambda\psi_{ABCD}\end{aligned} \tag{3.8}$$

where the indices between the brackets are to be symmetrized.[3] Thus, even when $\lambda = 0$ there is the nonlinear term on the right. This shows that the ψ-field can perhaps be thought of as acting as its own source to a certain extent. If ψ_{ABCD} is small we have

$$\Box\psi_{ABCD} \doteq 0$$

since λ is small in any case. Equation (3.8) indicates that we can only expect to have *exact* solutions for plane gravitational waves moving with the velocity of light when $\lambda = 0$ and $\psi_{(AB}{}^{EF}\psi_{CD)EF} = 0$. This question will be returned to in Section 4 where this condition will be interpreted geometrically and in Section 5 where such an exact solution will be given.

The tensor $T_{\mu\nu\rho\sigma}$ whose spinor equivalent is given by

$$T_{AE'BF'CG'DH'} = \psi_{ABCD}\bar{\psi}_{E'F'G'H'} \tag{3.9}$$

is of considerable interest. It has the properties of complete symmetry in its tensor indices, vanishing traces (as easily follows from (3.9)) and vanishing covariant divergence (with or without λ-term in Einstein's equations), since by (3.5)

$$\partial^{DH'}\{\psi_{ABCD}\bar{\psi}_{E'F'G'H'}\} = 0.$$

It would therefore appear that $T_{\mu\nu\rho\sigma}$ is a multiple of the "gravitational density" (or "super-energy") tensor due independently to I. Robinson (unpublished seminars) and to Bel (*8*, *9*).[4] As is easily verified, $T_{\mu\nu\rho\sigma}$ is, in fact, proportional to the Robinson-Bel tensor. Equation (3.9) bears a striking resemblance to the corresponding Eq. (3.11) for the electromagnetic case.

Let us now suppose that there is a source-free electromagnetic field present. The field tensor $F_{\mu\nu}$ can be expressed according to (1.3) in terms of a symmetric spinor ϕ_{AB}. This spinor can be used instead of $F_{\mu\nu}$ to represent the electromagnetic field (*2*), and the Maxwell field equations (in covariant form) become

$$\partial^{AC'}\phi_{AB} = 0. \tag{3.10}$$

[3] The tensor form of this relation is

$$\Box R_{\mu\nu\rho\sigma} = R_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta\rho\sigma} + 4R^{\alpha}{}_{\mu\beta[\rho}R^{\beta}{}_{\sigma]\alpha\nu} - 2\lambda R_{\mu\nu\rho\sigma}\,.$$

[4] This tensor was also found by R. Sachs (*10*) working with the group at Hamburg, and by A. Komar. However, only Robinson noticed the *total* symmetry of the tensor expression. It is not hard to see in the spinor formalism that, if $R_{\mu\nu} = 0$, *any* four-index tensor, quadratic and homogeneous in $R^{\mu}{}_{\nu\rho\sigma}$ and with vanishing divergence, must be totally symmetric. Robinson's tensor expression (with ($S_{\mu\nu\rho\sigma}$ as in (2.5)) is

$$T_{\mu\nu\rho\sigma} = R_{\mu\alpha\nu\beta}R_{\rho}{}^{\alpha}{}_{\sigma}{}^{\beta} + S_{\mu\alpha\nu\beta}S_{\rho}{}^{\alpha}{}_{\sigma}{}^{\beta}.$$

The energy-momentum tensor $T_{\mu\nu}$ for the electromagnetic field is given by

$$T_{AC'BD'} = \tfrac{1}{2}\phi_{AB}\bar{\phi}_{C'D'} \tag{3.11}$$

(see *10*). Now Einstein's equations with cosmological term are

$$G_{\mu\nu} + \lambda g_{\mu\nu} = -\kappa T_{\mu\nu} .$$

The λ defined by (2.8) is still the cosmological constant, because $T^{\nu}{}_{\nu} = 0$ and by (3.1), $4\lambda = R = -G^{\nu}{}_{\nu}$. Choosing units suitably so that $\kappa = 2$ (or absorbing the constant into the definition of ϕ_{AB}) we have, from (3.2), (3.11)

$$\phi_{ABC'D'} = \phi_{AB}\bar{\phi}_{C'D'} .$$

Equation (2.9) now gives

$$\partial^{D}{}_{G'}\psi_{ABCD} = \bar{\phi}_{G'H'}\partial_{C}{}^{H'}\phi_{AB} \tag{3.12}$$

by (3.4) and (3.10) since λ is necessarily constant. Thus the ϕ-field appears as a kind of source term to the ψ-field (here in the first-order equation).

From (2.17) we have

$$\{\partial_{FE'}\partial_{D}{}^{E'} + \partial_{DE'}\partial_{F}{}^{E'}\}\zeta_{G'} = \phi_{DF}\bar{\phi}_{G'H'}\zeta^{H'} \tag{3.13}$$

and (3.7) still holds. The second order equation arising from (3.10) turns out to be

$$\Box\phi_{AB} = \psi_{ABCD}\phi^{CD} - \tfrac{4}{3}\lambda\phi_{AB}$$

so that even the Maxwell field does not exactly satisfy the covariant wave equation (compare Eddington, *11*, Section 74). Also, (3.5) leads to

$$\Box\psi_{ABCD} = 3\psi_{(AB}{}^{EF}\psi_{CD)EF} - 2\lambda\psi_{ABCD} - 2\bar{\phi}_{G'H'}\partial_{A}{}^{G'}\partial_{B}{}^{H'}\phi_{CD}$$

4. THE GEOMETRY AND INVARIANTS OF ψ_{ABCD}

It is known that a general electromagnetic field determines two real principal null directions at each point (*12*). These are given in the general case by the real eigenvectors of the field tensor $F^{\mu}{}_{\nu}$ considered as a matrix. (There are also two complex null directions given by the complex eigenvectors, but these add nothing to the geometry as they are determined by their orthogonality with the real ones.) An alternative method of obtaining these principal null directions is to use a spinor approach. Any null vector x^{μ} corresponds to the product of a dotted with an undotted spinor

$$x^{AB'} = \eta^{A}\theta^{B'}.$$

If x^{μ} is real, $\theta^{B'}$ is a multiple of $\bar{\eta}^{B'}$, positive if x^{μ} points to the future. Any direction along the light cone therefore corresponds uniquely to a one-index spinor ray (set of spinors proportional to a given spinor). Now $F_{\mu\nu}$ corresponds uniquely to ϕ_{AB} (by 1.3) and we have

$$F^{AC'}{}_{BD'} = -\tfrac{1}{2}\{\phi^{A}{}_{B}\delta^{C'}_{D'} + \delta^{A}_{B}\bar{\phi}^{C'}{}_{D'}\}.$$

It is easily verified from this that the eigenvectors of $F^{AC'}{}_{BD'}$ are $\eta^{A}\bar{\eta}^{B'}$, $\zeta^{A}\bar{\zeta}^{B'}$ (corresponding to the real null vectors) and $\eta^{A}\bar{\zeta}^{B'}$, $\zeta^{A}\bar{\eta}^{B'}$ (corresponding to the complex null vectors) where

$$\phi_{AB} = \tfrac{1}{2}\{\eta_{A}\zeta_{B} + \eta_{B}\zeta_{A}\} = \eta_{(A}\zeta_{B)}\,. \tag{4.1}$$

See also Witten (*13*). A decomposition exactly analogous to (4.1) exists for the gravitational spinor. We have

$$\psi_{ABCD} = \alpha_{(A}\beta_{B}\gamma_{C}\delta_{D)}\,, \tag{4.2}$$

which expresses the gravitational spinor uniquely (except for scale factors) as a symmetrized product of one-index spinors. The bracket here denotes symmetrization as before, so that written out in full, there would be 24 terms on the right-hand side. The existence and uniqueness of (4.2) follows from the fundamental theorem of algebra:

$$\psi_{ABCD}\xi^{A}\xi^{B}\xi^{C}\xi^{D} = (\alpha_{A}\xi^{A})(\beta_{B}\xi^{B})(\gamma_{C}\xi^{C})(\delta_{D}\xi^{D}) \tag{4.3}$$

expresses the general binary quartic form as a product of linear factors. These factors are essentially unique, and equating coefficients gives (4.2).

Now the spinors α_A, β_B, γ_C, δ_D determine four directions along the light cone. These are uniquely determined by ψ_{ABCD} and will be called the *gravitational principal null directions*.[5] They supplement the two electromagnetic principal null directions corresponding to η_A and ζ_A. The gravitational principal null directions are only undefined if $\psi_{ABCD} = 0$ but they may coincide in special cases. In particular, for the case of the Schwarzschild solution, it follows from the symmetry that they must coincide in pairs at every point, one pair pointing towards the origin along the light cone and the other pair pointing away from it. (Time reversal symmetry shows that they cannot all four coincide or coincide three and one.) The coincidence of the two electromagnetic null directions is the condition for the electromagnetic field to be null. (The electromagnetic directions are, of course, only undetermined if $\phi_{AB} = 0$.) Thus, for an electromagnetic plane wave, the principal null directions coincide and, naturally enough, point in the direction of motion of the wave. Similarly, it turns out that for a gravitational plane wave, all the gravitational null directions coincide. This

[5] These four null directions are implicit in the work of Ruse (*14*). They correspond to the self-conjugate lines of the Riemannian complex. *Note added in proof:* They have been further exploited by Debever (*15*, *16*).

question will be returned to later. Gravitational radiation is sometimes analysed in terms of the invariants of the Riemann tensor (*17*) and it will be useful first to relate these invariants to the null directions defined above.

The number of independent invariants of the Riemann tensor in empty space is well known to be four. These may be interpreted as the real and imaginary parts of two independent complex invariants of ψ_{ABCD}, e.g.,

$$I = \psi_{ABCD}\psi^{ABCD}, \; J = \psi^{AB}{}_{CD}\psi^{CD}{}_{EF}\psi^{EF}{}_{AB} \tag{4.4}$$

(see Witten, *6*, p. 359). These may be thought of as invariants of the binary quartic form (4.3). According to the theory of invariants of binary forms, I and J are independent and any invariant of the quartic form (4.3) is a function of them (see Grace and Young, *18*). Thus the real and imaginary parts of I and J are a complete set of curvature invariants for empty space. The invariants I and J take the following tensor form if $R_{\mu\nu} = \lambda g_{\mu\nu}$:

$$I = \frac{1}{2} R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{i}{4}\sqrt{-g}R_{\mu\nu}{}^{\alpha\beta}\epsilon_{\alpha\beta\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{4}{3}\lambda^2,$$

$$J = \frac{1}{2}\left\{R^{\mu\nu}{}_{\rho\sigma} + \frac{i}{2}\sqrt{-g}R^{\mu\nu\alpha\beta}\epsilon_{\alpha\beta\rho\sigma}\right\} R^{\rho\sigma}{}_{\gamma\delta}R^{\gamma\delta}{}_{\mu\nu} - 2\lambda I - \frac{8}{9}\lambda^3$$

with $\lambda = \frac{1}{4}R$. These relations are obtained from (2.2), (2.5), (2.6), and (3.4). For a general curvature tensor,[6] the tensor $R_{\mu\nu\rho\sigma}$ in the above expressions must be replaced by

$$\tfrac{1}{2}R_{\mu\nu\rho\sigma} + \tfrac{1}{8}gR^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\mu\nu}\epsilon_{\gamma\delta\rho\sigma}\,.$$

Binary forms have a geometrical interpretation as sets of points on a complex projective line. The equation

$$\psi_{ABCD}\xi^A\xi^B\xi^C\xi^D = 0$$

is satisfied if and only if at least one of the factors $\alpha_A\xi^A$, $\beta_B\xi^B$, $\gamma_C\xi^C$, $\delta_D\xi^D$ vanishes, each of the conditions $\alpha_A\xi^A = 0, \cdots, \delta_D\xi^D = 0$ representing a point on the line. Thus ψ_{ABCD} corresponds to four points A, B, C, D on a complex projective line the coordinates of these points being the components of α_A, β_A, γ_A, δ_A, respec-

[6] It is perhaps worth remarking that a general method of converting expressions involving ψ_{ABCD} into the corresponding expressions for $R_{\mu\nu\rho\sigma}$ would be to use the formula

$$\psi_{ABCD} = \tfrac{1}{2}R_{AE'BF'CG'DH'}\epsilon^{E'F'}\epsilon^{G'H'} - \tfrac{1}{12}R\{\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC}\}$$

but the conversion of spinor contractions to an equivalent tensor form is sometimes complicated.

tively. Now any four collinear points have a projective invariant, namely, their cross-ratio

$$\mu = \frac{(\alpha_A \beta^B)(\gamma_B \delta^B)}{(\alpha_C \delta^C)(\gamma_D \beta^D)}.$$

This cross-ratio is the only independent invariant of the four points and is therefore the only independent invariant of ψ_{ABCD} which is unchanged if ψ_{ABCD} is multiplied by a non-zero complex number. Thus, the four real invariants of the curvature of empty space can be interpreted as a complex cross-ratio,[7] and a phase and a magniture[8] for ψ_{ABCD}.

This phase is associated with duality rotations of the curvature tensor (suggested to me first by I. Robinson) which are exactly analogous to electromagnetic duality rotations (*5*). In each case the duality rotation invariance of the first-order equation (3.5), (3.10) is broken only when sources are present. Letting

$$\psi_{ABCD} \rightarrow e^{i\theta} \psi_{ABCD},$$

where θ is a real constant, we have, assuming for simplicity that $\phi_{ABC'D'}$ and λ both vanish,

$$R_{\mu\nu\rho\sigma} \rightarrow \cos\theta\, R_{\mu\nu\rho\sigma} - \sin\theta\, S_{\mu\nu\rho\sigma}$$

by (2.2) and (2.6), $S_{\mu\nu\rho\sigma}$ being the right (or equivalently the left) dual of $R_{\mu\nu\rho\sigma}$ defined by (2.5). This is exactly analogous to

$$\phi_{AB} \rightarrow e^{i\theta} \phi_{AB}$$

giving

$$F_{\mu\nu} \rightarrow \cos\theta\, F_{\mu\nu} - \sin\theta\, H_{\mu\nu}$$

where the dual $H_{\mu\nu}$ of $F_{\mu\nu}$ is given by (1.4). Unlike the electromagnetic case, however, duality rotations of the ψ-field of an empty space solution do not in general give rise to new exact solutions of the field equations. (See, for example, Eq. (3.8).)

It will be observed that the Robinson–Bel tensor $\psi_{ABCD}\bar{\psi}_{E'F'G'H'}$ determines ψ_{ABCD} up to a duality rotation in the same way that $\phi_{AB}\bar{\phi}_{C'D'}$ determines ϕ_{AB}

[7] The idea of using a complex cross-ratio as an invariant defined by four null rays has also been independently suggested by I. Robinson (unpublished).

[8] This phase and magnitude of ψ_{ABCD} can be interpreted in an invariant way as the argument and modulus of, say, $\sqrt{I}$. This is not really satisfactory, however, since I may vanish. It might be better to use the argument and modulus of the κ which is defined by the relations (4.5). This only need vanish if $I = J = 0$, the condition for three of the null directions to coincide. Its definition depends on an arbitrary ordering of the null directions, however, as does the definition of μ.

up to a duality rotation. The principal null directions are therefore associated even more closely with these "energy" expressions than with the field quantities themselves. These expressions are completely characterized by the principal null directions, apart from their actual magnitude. It might be expected that the gravitational null directions are in some way associated with flow of "gravitational density." There does appear to be such a connection, as may be seen from the following argument.

Let x_μ be any null vector pointing into the future, so that

$$x_{\mathrm{AB'}} = \xi_{\mathrm{A}}\bar{\xi}_{\mathrm{B'}}\,.$$

Then by (3.9) and (4.3)

$$\begin{aligned} \mathrm{T}_{\mu\nu\rho\sigma}x^\mu x^\nu x^\rho x^\sigma &= (\psi_{\mathrm{ABCD}}\xi^{\mathrm{A}}\xi^{\mathrm{B}}\xi^{\mathrm{C}}\xi^{\mathrm{D}})(\bar{\psi}_{\mathrm{E'F'G'H'}}\bar{\xi}^{\mathrm{E'}}\bar{\xi}^{\mathrm{F'}}\bar{\xi}^{\mathrm{G'}}\bar{\xi}^{\mathrm{H'}}) \\ &= (a_\mu x^\mu)(b_\nu x^\nu)(c_\rho x^\rho)(d_\sigma x^\sigma), \end{aligned}$$

where

$$a_{\mathrm{AB'}} = \alpha_{\mathrm{A}}\bar{\alpha}_{\mathrm{B'}}\,,\ b_{\mathrm{AB'}} = \beta_{\mathrm{A}}\bar{\beta}_{\mathrm{B'}}\,,\ c_{\mathrm{AB'}} = \gamma_{\mathrm{A}}\bar{\gamma}_{\mathrm{B'}}\,,\ d_{\mathrm{AB'}} = \delta_{\mathrm{A}}\bar{\delta}_{\mathrm{B'}}\,.$$

The vectors a_μ , b_μ , c_μ , d_μ are null vectors, pointing into the future, corresponding to the gravitational principal null directions. Thus $\mathrm{T}_{\mu\nu\rho\sigma}x^\mu x^\nu x^\rho x^\sigma$ only vanishes for null vectors x^μ which point in one of the gravitational principal null directions. Otherwise it is positive. But for any time-like vector t^μ, the expression

$$\frac{\mathrm{T}_{\mu\nu\rho\sigma}t^\mu t^\nu t^\rho t^\sigma}{(t_\alpha t^\alpha)^2} \tag{4.5}$$

measures the gravitational density for an observer whose time axis is t^μ (see Bel (*8*, *9*)). It is positive (for empty space) unless $\mathrm{R}_{\mu\nu\rho\sigma} = 0$. Thus the gravitational principal null directions are characterized by the fact that for observers travelling with a given velocity infinitessimally less than c, the gravitational density will be a minimum for those observers who travel approximately along a principal null direction.

It is convenient, from a geometrical point of view, to represent null directions as points on a sphere. This sphere may be thought of as the field of vision of some observer. It may also be interpreted as a realization of the complex projective line mentioned above. (A complex projective line is, topologically, a real 2-sphere.) This sphere is the Argand sphere of the ratio of the two components of a one-index spinor (see Penrose, *19*, p. 138). Any Lorentz transformation corresponds to a bilinear transformation of this ratio and therefore to a projective (conformal) transformation of the sphere, which sends circles into circles.

Four points on the sphere are concyclic if and only if their cross-ratio is real. A particular case of this is harmonic points for which the cross-ratio is -1, 2, or

$\frac{1}{2}$ according to the order in which the points are taken. The symmetry of a harmonic set is best exhibited when the points are equally spaced around a great circle. The symmetries are then just the symmetries of a square. Any harmonic set can be brought into this form by a suitable projective (Lorentz) transformation, since any three points on the sphere can be transformed into any three others. Harmonic sets are of interest here because they have a greater symmetry than a general set of four points. They correspond to the vanishing of the invariant J (see Grace and Young, *18*, p. 206). Also of interest is the equianharmonic set which has an even greater symmetry. The cross-ratio here is $-\omega$ or $-\omega^2$ where $\omega = e^{i2\pi/3}$. By a suitable projective transformation these four points can be made the vertices of a regular tetrahedron. Equianharmonic points correspond to the vanishing of the invariant I (*18*, p. 206).

In the case of a general cross-ratio μ the symmetry is given by the Klein 4-group (except that there are also some reflectional symmetries if μ is real or has modulus unity). There is a unique projective transformation (involution) which interchanges any pair of the points with the remaining pair. These and the identity constitute the complete projective symmetry group provided that μ is different from $-1, 2, \frac{1}{2}, -\omega, -\omega^2, 0, 1$, or ∞, the cases 0, 1, and ∞ occurring when a pair of points coincide. The value of μ can be obtained from I^3/J^2 since it can be shown that

$$I = 6\kappa^2(\mu + \omega)(\mu + \omega^2), \quad J = 6\kappa^3(\mu + 1)(\mu - \tfrac{1}{2})(\mu - 2) \qquad (4.6)$$

for some κ (see (*16*) p. 205). There are in general six values of μ for a given value of I^3/J^2. They correspond to different orders in which the four points can be taken. The values are μ, $1 - \mu$, $1/\mu$, $1 - (1/\mu)$, $1/(1 - \mu)$, $\mu/(\mu - 1)$. The symmetries in the general case can also be realized as rotational symmetries of the sphere similarly to the two cases considered above. By a suitable projective transformation the four points, A, B, C, D can be transformed into the vertices of a tetrahedron which has opposite edges equal in pairs (a disphenoid). Such a tetrahedron has three orthogonal dyad axes of symmetry. These axes are the joins of the midpoints of opposite edges. If the cross-ratio is real, the tetrahedron is flattened into a rectangle but the three symmetry axes remain.

To see that such a transformation exists consider the three pairs (E,F), (G,H), (K,L) of united points for the three involutions which send (A,B,C,D) into (B,A,D,C), (C,D,A,B) and (D,C,B,A), respectively. Now the involution which sends (A,B,C,D) into (B,A,D,C) transforms the other involutions into themselves. It therefore sends G into H and K into L. Hence, (E,F) is harmonic with respect to (G,H) and also with respect to (K,L). Similarly (G,H) is harmonic with respect to (K,L). Now, E, G, F, H can be transformed (as above) into four points equally spaced, in that order, around the equator. K and L will then be the north and south poles, so that the six points form the vertices of a regular octahedron. The three involutions are then represented as rotations through π about the three axes EF, GH, KL. The point A is rotated into B, C, D by means of these involutions giving the symmetrical tetrahedron described above.

This symmetrical representation of the points A, B, C, D is of interest because it is related to Petrov's canonical representation of the Riemann tensor with $R_{\mu\nu} = 0$ (*17*, *20*). The rest frame in which the gravitational principal null directions appear to have this symmetrical form determines the canonical time axis, the three canonical space axes arising from the three axes of symmetry. These four axes are orthogonal to each other and are called the Riemann principal directions. They are uniquely defined provided that A, B, C, D are all distinct. If A, B, C, D coincide in pairs they can still be considered to exist but they are not uniquely defined.

The rotational symmetries of the tetrahedron A B C D in the general case give rise to the corresponding symmetries for $R_{\mu\nu\rho\sigma}$, since being dyad axes the only other possibility would be $R_{\mu\nu\rho\sigma} \to -R_{\mu\nu\rho\sigma}$ (a duality rotation of π).[9] Such an alternative is easily ruled out as impossible. It follows that, for the canonical choice of axes,

$$R_{ijkl} = 0 \quad \text{whenever} \quad i = k \quad \text{and} \quad j \neq l$$

as is required in Petrov's canonical form. Conversely, the above condition is sufficient for the Riemann principal directions to be the axes.

The usual definitions of the Riemann principal directions is in terms of the intersections of certain planes which are determined by the "eigenbivectors" of $R_{\mu\nu\rho\sigma}$, i.e., from the nonzero (complex) skew tensors $x^{\mu\nu}$ which satisfy a relation

$$R^{\mu\nu}{}_{\rho\sigma}x^{\rho\sigma} = \alpha x^{\mu\nu}. \tag{4.7}$$

Writing this in a spinor form with

$$x^{AC'BD'} = \tfrac{1}{2}\{\eta^{AB}\epsilon^{C'D'} + \epsilon^{AB}\bar{\zeta}^{C'D'}\}$$

(see 1.3) η^{AB} and ζ^{AB} being symmetric, (4.7) becomes

$$\psi^{AB}{}_{EF}\eta^{EF}\epsilon^{C'D'} + \epsilon^{AB}\bar{\psi}^{C'D'}{}_{E'F'}\bar{\zeta}^{E'F'} = \alpha\{\eta^{AB}\epsilon^{C'D'} + \epsilon^{AB}\bar{\zeta}^{C'D'}\}$$

(since $\phi_{ABC'D'} = 0$, $\lambda = 0$) so that

$$\psi^{AB}{}_{EF}\eta^{EF} = \alpha\eta^{AB}, \quad \psi^{AB}{}_{EF}\zeta^{EF} = \bar{\alpha}\zeta^{AB}.$$

One or the other of η^{AB}, ζ^{AB} may be zero. The eigenbivectors of $R^{\mu\nu}{}_{\rho\sigma}$ are thus

[9] In the special cases where the set of points A, B, C, D has an additional rotational symmetry, this does not always lead to a corresponding symmetry of $R_{\mu\nu\rho\sigma}$, although it does for the case when A, B, C, D coincide in pairs. In particular, in the equianharmonic case, the triad axes of symmetry give rise to duality rotations through angles $2\pi/3$, $4\pi/3$.

expressible in terms of "eigenspinors" of $\psi^{AB}{}_{CD}$, the eigenvalues of $R^{\mu\nu}{}_{\rho\sigma}$ being being those of $\psi^{AB}{}_{CD}$ and their complex conjugates. Witten (*6*) also considers these eigenspinors.

Now if the eigenvalues of $\psi^{AB}{}_{CD}$ are α_1, α_2, α_3 (the space of symmetric ξ^{AB} being three-dimensional) we have

$$\alpha_1 + \alpha_2 + \alpha_3 = \psi^{AB}{}_{AB} = 0,$$

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \psi^{AB}{}_{CD}\psi^{CD}{}_{AB} = I,$$

$$\alpha_1^3 + \alpha_2^3 + \alpha_3^3 = \psi^{AB}{}_{CD}\psi^{CD}{}_{EF}\psi^{EF}{}_{AB} = J.$$

With the expressions for I and J given in (4.6), it is easily verified that these relations are satisfied by

$$\alpha_1 = \kappa(2\mu - 1), \quad \alpha_2 = \kappa(2 - \mu), \quad \alpha_3 = \kappa(-1 - \mu). \tag{4.8}$$

The six eigenvalues of $R^{\mu\nu}{}_{\rho\sigma}$ are therefore these three numbers and their complex conjugates. It will be seen that the vanishing of just one of the eigenvalues (4.8) is the condition for the principal null directions to form a harmonic set. If two of them vanish they must all vanish and $I = J = 0$. This is the condition for at least three of the principal null directions to coincide (since they form both a harmonic and an equianharmonic set). If two of the eigenvalues (4.8) coincide, this is the condition $\mu = 0, 1$, or ∞ for a pair of principal null directions to coincide. This is the case $I^3 = 6J^2$ (*18*, p. 198).

The three eigenspinors η^{AB}, ζ^{AB}, θ^{AB} of $\psi^{AB}{}_{CD}$ will next be considered. They are symmetric and therefore each is expressible as a symmetrized product of a pair of one-index spinors (see 4.1). Each of η^{AB}, ζ^{AB}, θ^{AB} corresponds to a pair of points on the projective line considered earlier, so in the general case we have six points on this line determined by A, B, C, D. These can only be E, F, G, H, K, L since a general quartic form has only one sextic covariant (*18*, pp. 92, 94). This sextic covariant is

$$\psi_{PQRA}\psi^{PQ}{}_{BC}\psi^{R}{}_{DEF}\xi^A\xi^B\xi^C\xi^D\xi^E\xi^F,$$

whence

$$\psi_{PQR(A}\psi^{PQ}{}_{BC}\psi^{R}{}_{DEF)} = \eta_{(AB}\zeta_{CD}\theta_{EF)}$$

choosing the scale factor suitably. The vanishing of this expression is the condition for A, B, C, D to coincide in pairs, since E, F, G, H, K, L are not then defined uniquely. It does not vanish if just two of A, B, C, D coincide, or if they coincide three and one.

The planes determined by the eigenbivectors of $R^{\mu\nu}{}_{\rho\sigma}$ are those determined by η^{AB}, ζ^{AB}, θ^{AB}. They are therefore the three planes of the pairs of null directions corresponding to EF, GH, KL and the three orthogonal complements of these planes. Their intersections give the Riemann principal directions defined here, as is required. This is easily seen from the symmetrical representation of A, B, C, D given above.

These considerations have so far been essentially only concerned with Petrov's tensors $R_{\mu\nu\rho\sigma}$ of Type I. This is the case when the eigenbivectors of $R^{\mu\nu}{}_{\rho\sigma}$ span the six-dimensional space of bivectors. In special cases these eigenbivectors span only a four-dimensional space (Type II) and in very special cases, a two-dimensional space (Type III). In spinor terms, this means that Type I occurs when the eigenspinors of $\psi^{AB}{}_{CD}$ span a three-dimensional space, Type II when they span a two-dimensional space and Type III when they span only a one-dimensional space. Thus, Type II can only occur when at least two of the eigenvalues (4.8) are equal and Type III when they are all equal (and therefore all zero). We have seen that equality of eigenvalues implies coincidences among A, B, C, D so the cases where such coincidences occur must now be considered.

There are six different cases to be distinguished including the general case [1111] where the null directions are all distinct. There is the case [211] where exactly two of them coincide, [22] where they coincide in pairs, [31] where they coincide three and one, and [4] where all four directions are the same. Finally, there is the case [—] when $\psi_{ABCD} = 0$ and the null directions are undefined. This gives us a natural classification of Riemann tensors in empty space into six types (see also Géhéniau (*21*) for a closely related procedure[10]). In each case, the eigenspinors can be obtained by observing what happens to E, F, G, H, K, L when A, B, C, D are specialized. However, this must be done with care so that possible limiting positions of E, F, G, H, K, L are not omitted. Figure 1 shows how the different special cases arise from one another. The vertical specializations can be carried out keeping the positions of E, F, G, H, K, L fixed, but in the diagonal specializations, further pairs of them are forced to coincide. (For example, in the case [1111] $\rightarrow$ [211] if B $\rightarrow$ X and A $\rightarrow$ X, we have (G, H $\rightarrow$ (X, X), (K, L) $\rightarrow$ (X, X) and (E, F) $\rightarrow$ (X, Y) where Y is the harmonic conjugate of X with respect to the limiting positions of C and D.) The Petrov type for each case may be obtained in this way and the results are shown in Fig. 1. Each column corresponds to a particular type. Thus, [1111], [22], and [—] are Type I, [211] and [4] are Type II, while [31] is Type III. The different rows can be distinguished by the invariants I and J (or by the eigenvalues). Hence the invariants and Petrov type together serve to characterize ψ_{ABCD}.

It is of interest to see how this classification is in accord with that given by the classical canonical form of $\psi^{AB}{}_{CD}$ considered as a (3×3) matrix. These corresponding canonical forms are given in Fig. 2.

The various algebraic conditions for each case (or one of its specializations)

[10] *Note added in proof:* See also, more explicitly, Debever (*15, 16*).

$$\begin{tikzcd} & & {[1111]} \arrow[dl] \arrow[d] & I^3 \neq 6J^2 \\ & {[211]} \arrow[r] \arrow[dl] \arrow[d] & {[22]} \arrow[dl] \arrow[d] & I^3 = 6J^2 \neq 0 \\ {[31]} \arrow[r] & {[4]} \arrow[r] & {[-]} & I = J = 0 \\ \text{III} & \text{II} & \text{I} & \end{tikzcd}$$

Petrov type: III II I

FIG. 1. Classification scheme for ψ_{ABCD} in terms of coincidences between principal null directions.

$$\begin{tikzcd}[column sep=small] & & \begin{pmatrix} \kappa(2\mu-1) & & \\ & \kappa(2-\mu) & \\ & & \kappa(-1-\mu) \end{pmatrix} \arrow[dl] \arrow[d] \\ & \begin{pmatrix} \kappa & 1 & \\ & \kappa & \\ & & -2\kappa \end{pmatrix} \arrow[r] \arrow[dl] \arrow[d] & \begin{pmatrix} \kappa & & \\ & \kappa & \\ & & -2\kappa \end{pmatrix} \arrow[dl] \arrow[d] \\ \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix} \arrow[r] & \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix} \arrow[r] & \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix} \end{tikzcd}$$

FIG. 2. Classification in terms of matrix canonical form of $\psi^{AB}{}_{CD}$.

to occur may be collected together as follows:

$$[211]\colon I^3 = 6J^2, \quad [22]\colon \psi_{PQR(A}\psi^{PQ}{}_{BC}\psi^{R}{}_{DEF)} = 0, \quad [31]\colon I = J = 0,$$
$$[4]\colon \psi_{(AB}{}^{EF}\psi_{CD)EF} = 0, \quad [-]\colon \psi_{ABCD} = 0.$$

The only case that has not already been dealt with is the condition for [4] to occur. The quartic form $\psi_{AB}{}^{EF}\psi_{CDEF}\xi^A\xi^B\xi^C\xi^D$ is the Hessian of the form

$$\psi_{ABCD}\xi^A\xi^B\xi^C\xi^D$$

and its vanishing is known to be the condition for the latter form to be a perfect fourth power (*18*, p. 235). The interest of this condition lies in the fact that $\psi_{(AB}{}^{EF}\psi_{CD)EF}$ is precisely the term (in the case $\lambda = 0$) which prevents Eq. (3.8) from being a covariant wave equation[11] for ψ_{ABCD}. Thus, plane wave solutions can only reasonably be expected in case [4].[12] This is Petrov's Type II with vanishing invariants and is apparently the case characteristic of a "*pure*" gravitational radiation field (*8*, *22*, and *23*). The other cases which might conceivably also be considered as "pure gravitational radiation" are [211] and [31] (see *17*).

[11] Case [4] therefore appears to be the only case (apart from [—]) in which the gravitational field has no "gravitational mass". See also Bondi *et al.* (*23*), p. 532.

[12] However, a point perhaps worth mentioning is that in case [22], ψ_{ABCD} and $\psi_{(AB}{}^{EF}\psi_{CD)EF}$ are proportional.

Case [211] would seem to be wrong since [22], which is a special case of it, would also have to be considered as pure gravitational radiation. But we have seen that the Schwarzschild solution is [22].

Case [31] is, however, worthy of consideration in this respect since it shares with Case [4] the property that the gravitational density (4.5) can be made as small as we please by a suitable choice of time axis ("following the wave"). If

$$\psi_{ABCD} = \alpha_{(A}\alpha_B\alpha_C\beta_{D)}$$

and

$$t_\mu = a_\mu + \epsilon x_\mu, \quad a_{AB'} = \alpha_A\bar{\alpha}_{B'}, \quad b_{AB'} = \beta_A\bar{\beta}_{B'},$$

where $\epsilon > 0$ is small and x_μ is time-like pointing to the future, we have

$$\frac{T_{\mu\nu\rho\sigma}t^\mu t^\nu t^\rho t^\sigma}{(t_\tau t^\tau)^2} \doteq \frac{\frac{1}{4}\epsilon^3 \mid \beta_A\alpha^A \mid^2 (\alpha_B\bar{\alpha}_{C'}x^{BC'})^3}{4\epsilon^2(a_\tau x^\tau)^2},$$

$$= \epsilon\frac{(b_\mu a^\mu)(a_\nu x^\nu)^3}{16(a_\tau x^\tau)^2}.$$

If $\beta_A = \alpha_A$, the right-hand side would be of order ϵ^2 instead of ϵ. Thus, the gravitational density tends to zero for observers, whose velocity approaches the multiple principal null direction, both in Case [31] and in Case [4], but it tends to zero more rapidly in Case [4]. It would appear to be correct to call Case [4] "pure" radiation field[13] but not Case [31]. Case [4] is like a null electromagnetic field ("pure" electromagnetic radiation field) in that it determines only one null direction, and in that it is the general limiting case obtained as a result of a high-velocity Lorentz transformation (see also *24*). (However, it is worth remarking that for a null electromagnetic field, the energy $T_{\mu\nu}t^\mu t^\nu/(t_\tau t^\tau)$ can only be made to tend to zero to order ϵ by "following the wave", like Case [31] above.)

The invariants of ψ_{ABCD} have been treated in considerable detail above. It now remains to give a brief discussion of the combined system ψ_{ABCD}, ϕ_{AB} for the case when electromagnetic field is present. We expect to find just three more complex invariants, since ϕ_{AB} is determined by its phase and magnitude, and by the positions relative to A, B, C, D of the two complex points Y and Z on the argand sphere, corresponding to the electromagnetic principal null directions. There is the obvious invariant

$$K = \phi_{AB}\phi^{AB}$$

of ϕ_{AB} alone. This is the discriminant of the binary form $\phi_{AB}\xi^A\xi^B$, the condition

[13] It is probably preferable, however, to call case [4] simply a *null* gravitational field (as suggested by Robinson) analogously to the electromagnetic case.

$K = 0$ being necessary and sufficient for the points X and Y to coincide, that is, for the field to be null.[13a] The list is completed by the two independent invariants

$$L = \phi_{AB}\psi^{AB}{}_{CD}\phi^{CD}, \quad M = \phi_{AB}\psi^{AB}{}_{CD}\psi^{CD}{}_{EF}\phi^{EF}.$$

The fact that I, J, K, L, M are in general independent is most easily seen if $\psi^{AB}{}_{CD}$ is thought of as a matrix and ϕ^{AB} as a "vector" which may then be expanded in terms of the eigenspinors of $\psi^{AB}{}_{CD}$ with arbitrary coefficients. K, L, and M then become independent linear functions of the squares of these coefficients.

However, I, J, K, L, and M do not form a complete system of invariants in the sense of invariant theory (*18*). That is, not every algebraic invariant of ψ_{ABCD} and ϕ_{AB} can be expressed as a polynomial in I, ... , M. The invariant

$$N = \phi_{AB}\psi^{AB}{}_{CD}\psi^{CD}{}_{EF}\phi^{E}{}_{G}\psi^{FG}{}_{PQ}\phi^{PQ}$$

clearly is not even a rational function of I, ... , M since every such functions is of even order in ϕ_{AB}. Also N does not vanish identically. On the other hand, N is *algebraically* dependent on I, ... , M, there being the *syzygy*

$$N^2 = \tfrac{1}{2}JKLM - \tfrac{1}{6}JL^3 - \tfrac{1}{2}M^3 - \tfrac{1}{8}I^2KL^2 - \tfrac{1}{6}IJK^2L - \tfrac{1}{18}J^2K^3 + \tfrac{1}{4}IKM^2 + \tfrac{1}{4}IL^2M.$$

The system I, J, K, L. M, N does, in fact, form a complete system of invariants for ψ_{ABCD} and ϕ_{AB}.

The condition for an electromagnetic principal null direction to coincide with a gravitational principal null direction is that the resultant of the quartic and quadratic forms should vanish. Expressed in terms of invariants this condition turns out to be

$$2K^2I - 4KM + L^2 = 0.$$

The condition for both electromagnetic null directions to lie along a gravitational null direction is therefore

$$K = 0, \quad L = 0.$$

The electromagnetic and gravitational fields together have ten independent real invariants, namely the real and imaginary parts of I, J, K, L, M. However, only nine of these are determined by the curvature $R_{\mu\nu\rho\sigma}$ since it is unaffected by duality rotations of the electromagnetic field. These are the nine independent

[13a] The real and imaginary parts of K are the usual invariants $F_{\mu\nu}F^{\mu\nu}$ and $1/2\,\sqrt{-g}\,F^{\mu\nu}F^{\rho\sigma}\epsilon_{\mu\nu\rho\sigma}$ respectively, of $F_{\mu\nu}$.

real invariants of ψ_{ABCD} and $\phi_{ABC'D'} = \phi_{AB}\bar{\phi}_{C'D'}$. The phase of ϕ_{AB} is undetermined by $\phi_{ABC'D'}$, so we can take for these invariants[14]

$$\mathrm{I}, \mathrm{J}, |\mathrm{K}|, |\mathrm{L}|, |\mathrm{M}|$$

and the arguments of the two ratios

$$\mathrm{K}:\mathrm{L}:\mathrm{M}.$$

(The invariants $|\mathrm{K}|^2$, $|\mathrm{L}|^2$, $|\mathrm{M}|^2$, $\mathrm{K}\bar{\mathrm{L}}$, $\mathrm{L}\bar{\mathrm{M}}$, $\mathrm{M}\bar{\mathrm{K}}$ are easily expressible in terms of ψ_{ABCD} and $\phi_{ABC'D'}$.)

5. ANALYTIC SOLUTIONS OF EINSTEIN'S EQUATIONS

Let $\mathfrak{M}$ be an analytic (connected) Riemannian manifold. Then starting from any point O on $\mathfrak{M}$ at which the curvature tensor $\mathrm{R}_{\mu\nu\rho\sigma}$ and all its covariant derivatives are known, it is possible to calculate the curvature tensor (and its derivatives) at any other point by means of a power series:

$$(\mathrm{R}_{\mu\nu\rho\sigma})_x = (\mathrm{R}_{\mu\nu\rho\sigma})_0 + x^\alpha(\partial_\alpha \mathrm{R}_{\mu\nu\rho\sigma})_0 + \frac{1}{2!}x^\alpha x^\beta(\partial_\alpha\partial_\beta \mathrm{R}_{\mu\nu\rho\sigma})_0 + \cdots. \quad (5.1)$$

The point x is that point on $\mathfrak{M}$ whose geodesic distance from O is $\sqrt{(x_\alpha x^\alpha)}$ and which lies on the geodesic through O which starts off in the direction of x^α(Riemannian coordinates). (If x^α is null this has to be interpreted suitably.) The $\mathrm{R}_{\mu\nu\rho\sigma}$ at the point x is referred to axes which are those at O transferred in parallel along this geodesic. If the power series does not converge, the point x may be reached in several steps, using intermediate points, in the manner of analytic continuation. This power series expression and its convergence is considered by Thomas (*25*, p. 234).

Equation (5.1) is a special case of the more general situation, whereby any analytic tensor field may be calculated from a knowledge of the tensor and all its covariant derivatives at the point 0 alone:

$$\begin{aligned}(f\ldots)_x &= (f\ldots)_0 + x^\alpha(\partial_\alpha f\ldots)_0 + \frac{1}{2!}x^\alpha x^\beta(\partial_\alpha\partial_\beta f\ldots)_0 + \cdots \\ &= [\exp(x^\alpha\partial_\alpha)f\ldots]_0 = \left[\lim_{n\to\infty}\left(1+\frac{1}{n}x^\alpha\partial_\alpha\right)^n f\ldots\right]_0.\end{aligned} \quad (5.2)$$

[14] When the electromagnetic field is null there still remain the seven real invariants given by I, J, $|\mathrm{L}|$, $|\mathrm{M}|$ and the argument of L/M. Thus Witten (*6*) is mistaken when he claims that there remain only the four real invariants of ψ_{ABCD} in this case. For example, the invariant $\phi_{ABE'F'}\psi^{AB}{}_{CD}\bar{\psi}^{E'F'}{}_{G'H'}\phi^{CDG'H'} = |\mathrm{L}|^2$ need not vanish when $\mathrm{K} = 0$. Such an invariant could appear as a quotient of invariants built up from Witten's list.

The ∂_α's are to be taken as acting only on $f \cdots$ and not on x^α. (This last expression can be used to obtain the power series expression since

$$(f\ldots)_{\epsilon x} = [(1 + \epsilon x^\alpha \partial_\alpha) f \ldots]_0 + O(\epsilon^2),$$

which may be applied n times with $\epsilon = 1/n$, giving $(f\ldots)_x$ correct to order $1/n$).

These power series can be used as the basis for a coordinate-free approach to Riemannian geometry. Instead of specifying a space by giving the metric tensor $g_{\mu\nu}$ as a function of some coordinates, the space may be determined (except possibly for some of its topological properties in the large) by specifying $R_{\mu\nu\rho\sigma}$, $\partial_\alpha R_{\mu\nu\rho\sigma}$, $\partial_\alpha \partial_\beta R_{\mu\nu\rho\sigma}$, $\cdots$ at a point O. To specify a set of tensors at a point does not require coordinates since their algebraic tensorial properties need only be given. The metric tensors $g_{\mu\nu}$, $g^{\mu\nu}$ and the alternating tensor $\sqrt{\pm g}\epsilon_{\mu\ldots\sigma}$ are also supposed to be specified at the point O. They are an essential part of the tensor algebra at O.

A difficulty about specifying a space in this way is that $R_{\mu\nu\rho\sigma}$, $\partial_\alpha R_{\mu\nu\rho\sigma}$, $\partial_\alpha \partial_\beta R_{\mu\nu\rho\sigma}$, $\cdots$ are not algebraically (tensorially) independent of one another. Relation (2.1) implies identities (Ricci) connecting second derivatives with the curvature tensor, and also there is the Bianchi identity which is the consistency condition for (2.1). The Bianchi identity is in fact the only consistency condition required (*25*, pp. 131, 132). Applying these two types of identity to the higher derivatives of $R_{\mu\nu\rho\sigma}$ a host of relations is obtained. It is therefore of importance to be able to single out a set of tensors which are algebraically independent (in the general case) and from which $R_{\mu\nu\rho\sigma}$ and all its derivatives are obtainable by algebraic operations. It is possible to show that the following set of tensors, in fact, has all these properties:

$$Q^{\mu\nu}{}_{\rho\sigma} = R^{\mu}{}_{(\rho}{}^{\nu}{}_{\sigma)}, \quad Q^{\mu\nu}{}_{\rho\sigma\alpha} = \partial_{(\alpha} R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma)}, \quad Q^{\mu\nu}{}_{\rho\sigma\alpha\beta} = \partial_{(\alpha}\partial_\beta R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma)}, \quad \text{etc.}$$

Each $Q\ldots$ has the symmetry given by a Young tableau operator corresponding to a partition $(r - 2, 2)$. That is to say, we have

$$Q_{\mu\nu\rho\sigma\cdots\beta} = Q_{(\mu\nu)(\rho\sigma\cdots\beta)} \quad \text{and} \quad Q_{\mu(\nu\rho\sigma\cdots\beta)} = 0.$$

Apart from these symmetries and from certain considerations of convergence, the Q's may be chosen arbitrarily.[15] Unfortunately, however, if it is required to impose a condition such as Einstein's $R^{\mu}{}_{\nu\mu\sigma} = 0$ (or $= \lambda g_{\nu\sigma}$) on the space, this implies a condition not only on $Q^{\mu\nu}{}_{\rho\sigma}$, but also on $Q^{\mu\nu}{}_{\rho\sigma\alpha}$, $Q^{\mu\nu}{}_{\rho\sigma\alpha\beta}$, etc. These conditions are all linear, but they appear to be somewhat complicated. It seems for this reason that an approach based explicitly on these Q's would not be usually very convenient for general relativity. (However, in a later paper it is proposed

[15] These Q's are somewhat analogous to (but different from) the "normal tensors" (see Thomas *25*, p. 102).

to give a class of special solutions using this method.) On the other hand, if a spinor approach is used, these linear conditions take on a particularly simple form. This approach will now be described in more detail.

Suppose that $\mathfrak{M}$ has four dimensions and signature $(+---)$, and that $R^{\mu}{}_{\nu\mu\sigma} = \lambda g_{\nu\sigma}$. Then we have seen that $R_{\mu\nu\rho\sigma}$ can be represented uniquely by a totally symmetric spinor ψ_{ABCD} (λ being known). We wish to find a set of algebraically independent spinors from which

$$\psi_{ABCD}\,, \quad \partial_E{}^{P'}\psi_{ABCD}\,, \quad \partial_E{}^{P'}\partial_F{}^{Q'}\psi_{ABCD}\,, \cdots \tag{5.3}$$

(at the point O) can be constructed by means of algebraic spinor operations. The identities relating the spinors (5.3) arise from the equivalent of the Bianchi identity, namely (3.5):

$$\partial^{AP'}\psi_{ABCD} = 0 \quad \text{or} \quad \epsilon^{EA}\partial_E{}^{P'}\psi_{ABCD} = 0 \tag{5.4}$$

and the equivalent of (2.1), namely, (2.14), (2.15), (2.16), and (2.17):

$$\left.\begin{aligned} &\epsilon_{R'S'}\{\partial_G{}^{R'}\partial_H{}^{S'} + \partial_H{}^{R'}\partial_G{}^{S'}\}\xi_A = \psi_{GHAB}\xi^B - \frac{\lambda}{3}\{\xi_G\epsilon_{HA} + \xi_H\epsilon_{GA}\}, \\ &\epsilon_{R'S'}\{\partial_G{}^{R'}\partial_H{}^{S'} + \partial_H{}^{R'}\partial_G{}^{S'}\}\eta^{P'} = 0 \end{aligned}\right\} \tag{5.5}$$

$$\left.\begin{aligned} &\epsilon^{GH}\{\partial_G{}^{R'}\partial_H{}^{S'} + \partial_G{}^{S'}\partial_H{}^{R'}\}\xi_A = 0, \\ &\epsilon^{GH}\{\partial_G{}^{R'}\partial_H{}^{S'} + \partial_G{}^{S'}\partial_H{}^{R'}\}\eta^{P'} = \bar{\psi}^{R'S'P'}{}_{Q'}\eta^{Q'} - \frac{\lambda}{3}\{\eta^{R'}\epsilon^{S'P'} + \eta^{S'}\epsilon^{R'P'}\} \end{aligned}\right\} \tag{5.6}$$

(see 3.7 and 3.3) applied to ψ_{ABCD} and its derivatives.

The various derivatives of (5.4) must all hold identically also. Hence the algebraic relations on the spinors (5.3) arising from (5.4) are

$$\epsilon^{HA}(\partial_E{}^{P'} \cdots \partial_G{}^{R'}\partial_H{}^{S'}\psi_{ABCD}) = 0. \tag{5.7}$$

This expresses a condition on (namely, the vanishing of) the part of $\partial_E{}^{P'} \cdots \partial_H{}^{S'}\psi_{ABCD}$ which is skew in H, A and says nothing about the part symmetric in H, A. Moreover the relations (5.5) connect

$$\epsilon_{R'S'}(\partial_E{}^{P'} \cdots \partial_G{}^{R'}\partial_H{}^{S'} \cdots \partial_K{}^{V'}\psi_{ABCD}) + \epsilon_{R'S'}(\partial_E{}^{P'} \cdots \partial_H{}^{R'}\partial_G{}^{S'} \cdots \partial_K{}^{V'}\psi_{ABCD})$$

with lower derivatives of ψ_{ABCD}, while (5.6) connect

$$\epsilon^{GH}(\partial_E{}^{P'} \cdots \partial_G{}^{R'}\partial_H{}^{S'} \cdots \partial_K{}^{V'}\psi_{ABCD}) + \epsilon^{GH}(\partial_E{}^{P'} \cdots \partial_G{}^{S'}\partial_H{}^{R'} \cdots \partial_K{}^{V'}\psi_{ABCD'})$$

with lower derivatives of ψ_{ABCD}. These express conditions only on parts of $\partial_E{}^{P'} \cdots \partial_K{}^{V'}\psi_{ABCD}$ which are skew in a pair of primed indices or in a pair of unprimed indices. Thus the algebraic relations arising from (5.4), (5.5), and (5.6) connecting the spinors (5.3) are all concerned with parts of $\partial_E{}^{P'} \cdots \partial_K{}^{V'}\psi_{ABCD}$

which are skew in at least one pair of indices. They imply no conditions on the parts totally symmetric in all primed indices and in all unprimed indices. (It might, perhaps, be thought that other relations could be obtained by expanding skew parts of $\partial_E{}^{P'} \cdots \partial_K{}^{V'} \psi_{ABCD}$ in two different ways. However, these all lead back to (5.7) which is the only consistency condition implied.) Hence the spinors

$$\psi_{ABCD}\,, \quad \psi_{ABCDE}{}^{P'} = \partial_{(E}{}^{P'}\psi_{ABCD)}\,, \quad \psi_{ABCDEF}{}^{P'Q'} = \partial_{(E}{}^{(P'}\partial_F{}^{Q')}\psi_{ABCD)}\,, \cdots \tag{5.8}$$

are all algebraically independent and can therefore be specified arbitrarily (apart from convergence considerations) at the point 0.

The problem is now to show, conversely, that all the spinors (5.3) can be obtained algebraically from the spinors (5.8). For then ψ_{ABCD}, ψ_{ABCDEP}, $\psi_{ABCDEFP'Q'}$, $\cdots$ will be a complete set of algebraically independent spinors at O, which can be used to generate the space $\mathfrak{M}$. In order to show that they form such a complete set, an argument by induction will be used. We wish to express $\partial_E{}^{P'} \cdots \partial_K{}^{V'} \psi_{ABCD}$ in terms of $\psi_{ABCDE\cdots K}{}^{P'\cdots V'}$ and lower order derivatives of ψ_{ABCD} since it may be supposed as the inductive hypothesis that all these lower derivatives have already been expressed algebraically in terms of symmetrized derivatives $\psi_{AB\cdots G}{}^{P'\cdots R'}$. Now, if we add together all the spinors obtained from $\partial_E{}^{P'} \cdots \partial_K{}^{V'} \psi_{ABCD}$ by permuting P′, $\cdots$, V′ in all possible ways and A, B, C, D, E, $\cdots$, K in all possible ways, we get a multiple of $\psi_{AB\cdots K}{}^{P'\cdots V'}$. Thus, if it can be shown that each of the spinors obtained by such permutations differs from $\partial_E{}^{P'} \cdots \partial_K{}^{V'} \psi_{ABCD}$ by expressions involving only lower derivatives of ψ_{ABCD} the result will be proved. The spinor $\partial_E{}^{P'} \cdots \partial_K{}^{V'} \psi_{ABCD}$ will then be seen to differ from $\psi_{AB\cdots K}{}^{P'\cdots V'}$ by a spinor built up from lower derivatives of ψ_{ABCD}.

Any two spinors obtained by such a permutation of indices from

$$\partial_E{}^{P'} \cdots \partial_K{}^{V'} \psi_{ABCD}$$

will be called *equivalent* (denoted by $\sim$) if they differ from each other by expressions built up from lower order derivatives of ψ_{ABCD}. This is clearly an equivalence relation. It is required to show that all such spinors are, in fact, equivalent to one another. Now since

$$\partial_W{}^{X'}\partial_Y{}^{Z'} - \partial_Y{}^{Z'}\partial_W{}^{X'} \equiv \tfrac{1}{2}\epsilon^{X'Z'}\epsilon_{M'N'}\{\partial_W{}^{M'}\partial_Y{}^{N'} + \partial_Y{}^{M'}\partial_W{}^{N'}\} + \tfrac{1}{2}\epsilon_{WY}\epsilon^{ST}\{\partial_S{}^{X'}\partial_T{}^{Z'} + \partial_S{}^{Z'}\partial_T{}^{X'}\}$$

(see 2.12), we have, applying (5.5) and (5.6)

$$\cdots \partial_W{}^{X'}\partial_Y{}^{Z'} \cdots \psi_{ABCD} \sim \cdots \partial_Y{}^{Z'}\partial_W{}^{X'} \cdots \psi_{ABCD}\,.$$

Hence any permutation of the $\partial_M{}^{N'}$ symbols gives rise to an equivalent spinor. (Any permutation can be expressed as a product of transpositions of adjacent elements.) That is, any permutation of P′, $\cdots$, V′ can be applied to

$\partial_E{}^{P'} \cdots \partial_K{}^{V'} \psi_{ABCD}$ provided that the same permutation is applied to E, $\cdots$, K and an equivalent spinor is obtained. It remains to show that E, $\cdots$, K, A, B, C, D can be permuted independently and an equivalent spinor is still obtained. The symmetry of ψ_{ABCD} implies that A, B, C, D can be permuted without change. Furthermore, from 5.7, K and A can be interchanged in $\partial_E{}^{P'} \cdots \partial_K{}^{V'} \psi_{ABCD}$. Also,

$$\cdots \partial_Y{}^{Z'} \cdots \partial_K{}^{V'} \psi_{ABCD} \sim \cdots \partial_K{}^{V'} \cdots \partial_Y{}^{Z'} \psi_{ABCD},$$
$$\sim \cdots \partial_K{}^{V'} \cdots \partial_A{}^{Z'} \psi_{YBCD} \sim \cdots \partial_A{}^{Z'} \cdots \partial_K{}^{V'} \psi_{YBCD}$$

so that A can be interchanged with any other unprimed index and an equivalent spinor is obtained. It follows that any pair of unprimed indices can be interchanged since

$$\cdots \partial_W{}^{X'} \cdots \partial_Y{}^{Z'} \cdots \psi_{ABCD} \sim \cdots \partial_W{}^{X'} \cdots \partial_A{}^{Z'} \cdots \psi_{YBCD},$$
$$\sim \cdots \partial_Y{}^{X'} \cdots \partial_A{}^{Z'} \cdots \psi_{WBCD} \sim \cdots \partial_Y{}^{X'} \cdots \partial_W{}^{Z'} \cdots \psi_{ABCD}.$$

Hence all the spinors are equivalent and the result is proved.

As examples of the above, we have

$$\partial_E{}^{P'} \psi_{ABCD} = \psi_{ABCDE}{}^{P'},$$
$$\partial_E{}^{P'} \partial_F{}^{Q'} \psi_{ABCD} = \psi_{ABCDEF}{}^{P'Q'} + \epsilon_{EF}\epsilon^{P'Q'}\{\tfrac{3}{4}\psi_{(AB}{}^{GH}\psi_{CD)GH} - \tfrac{1}{2}\lambda\psi_{ABCD}\}$$
$$+ \epsilon^{P'Q'}\{\psi_{(ABC}{}^{G}\psi_{D)EFG} + \tfrac{1}{3}\lambda\psi_{E(ABC}\epsilon_{D)F} + \tfrac{1}{3}\lambda\psi_{F(ABC}\epsilon_{D)E}\}.$$

Higher derivatives involve $\bar{\psi}_{A'B'C'D'}$, $\bar{\psi}_{A'B'C'D'E'P}$, $\cdots$ also. We have from (5.1), with $\psi_{ABCD} = (\psi_{ABCD})_0$, etc.,

$$(\psi_{ABCD})_x = \psi_{ABCD} + x^{EP'}\partial_{EP'}\psi_{ABCD} + \frac{1}{2!}x^{EP'}x^{FQ'}\partial_{EP'}\partial_{FQ'}\psi_{ABCD} + \cdots.$$

Hence

$$(\psi_{ABCD})_x = \psi_{ABCD} + x^{EP'}\psi_{ABCDEP'} + \tfrac{1}{2}x^{EP'}x^{FQ'}\psi_{ABCDEFP'Q'}$$
$$+ \tfrac{1}{2}(x_{EP'}x^{EP'})\{\tfrac{3}{4}\psi_{(AB}{}^{GH}\psi_{CD)GH} - \tfrac{1}{2}\lambda\psi_{ABCD}\} + O(x^3).$$

It is possible to obtain a class of exact solutions for gravitational plane waves using this method. Such solutions, obtained using more conventional methods, have been known for some time (for references, see Bondi *et al.*, *23*). Let

$$\psi_{ABCD} = \alpha_0\pi_A\pi_B\pi_C\pi_D, \quad \psi_{ABCDEP'} = \alpha_1\pi_A\pi_B\pi_C\pi_D\pi_E\bar{\pi}_{P'},$$
$$\psi_{ABCDEFP'Q'} = \alpha_2\pi_A \cdots \pi_F\bar{\pi}_{P'}\bar{\pi}_{Q'}, \cdots \tag{5.9}$$

at the point O, where π_A is a spinor corresponding to the null direction giving the direction of motion of the wave and α_0, α_1, $\cdots$ are complex numbers. Suppose

$\lambda = 0$. It will now be shown that the unsymmetrized derivatives of ψ_{ABCD} are all equal to the symmetrized derivatives, so the situation is much simplified in this case. As an inductive hypothesis we assume that all the derivatives of ψ_{ABCD} of lower order than $\partial_E{}^{P'} \cdots \partial_K{}^{V'} \psi_{ABCD}$ are already symmetric and therefore equal to the corresponding expressions 5.9. The argument given above shows that $\partial_E{}^{P'} \cdots \partial_K{}^{V'} \psi_{ABCD}$ differs from $\psi_{ABCDE\cdots K}{}^{P'\cdots V'}$ by expressions obtained by applying rule (5.5) and (5.6) to ψ_{ABCD} and derivatives of ψ_{ABCD}, and perhaps differentiating further. Since $\lambda = 0$, this leads to terms of the form

$$\psi_{XAG\cdots K}{}^{T'\cdots V'} \psi^X{}_{B\cdots F}{}^{P'\cdots S'} \quad \text{or} \quad \bar{\psi}_{Y'}{}^{P'T'\cdots V'}{}_{G\cdots K} \psi_{A\cdots F}{}^{Y'Q'\cdots S'}$$

only. (By the inductive hypothesis all the derivatives of ψ_{ABCD} which occur are equal to the $\psi_{\cdots}^{\cdots}$'s.) These terms all involve contractions between the $\psi_{\cdots}^{\cdots}$'s. But with $\psi_{\cdots}^{\cdots}$'s given by (5.9), any contraction must clearly vanish (since $\pi_X \pi^X = 0$). Hence

$$(\partial_E{}^{P'} \cdots \partial_K{}^{V'} \psi_{ABCD})_0 = \alpha_r \pi_A \pi_B \cdots \pi_K \bar{\pi}^{P'} \cdots \bar{\pi}^{V'}$$

as required.

The curvature at points other than O can now be calculated:

$$\begin{aligned}(\psi_{ABCD})_x &= \alpha_0 \pi_A \pi_B \pi_C \pi_D + \alpha_1 x^{EP'} \pi_A \\ &\qquad \cdots \pi_E \bar{\pi}_{P'} + \frac{1}{2!} \alpha_2 x^{EP'} x^{FQ'} \pi_A \cdots \pi_F \bar{\pi}_{P'} \bar{\pi}_{Q'} + \cdots \\ &= f(x^{EP'} \pi_E \bar{\pi}_{P'}) \pi_A \pi_B \pi_C \pi_D = f(x^\mu p_\mu) \pi_A \pi_B \pi_C \pi_D,\end{aligned}$$

where

$$f(s) = \alpha_0 + \alpha_1 s + \frac{1}{2!} \alpha_2 s^2 + \frac{1}{3!} \alpha_3 s^3 + \cdots \tag{5.10}$$

and $p_{AB'} = \pi_A \bar{\pi}_{B'}$. Thus the curvature is a function of the one parameter $x^\mu p_\mu$ only. It is constant along the (null) 3-spaces $x^\mu p_\mu =$ constant. Furthermore, by (5.2),

$$\begin{aligned}(\partial_{EP'} \psi_{ABCD})_x &= (\partial_{EP'} \psi_{ABCD})_0 + x^{FQ'} (\partial_{FQ'} \{\partial_{EP'} \psi_{ABCD}\})_0 \\ &\qquad + \frac{1}{2!} x^{FQ'} x^{GR'} (\partial_{FQ'} \partial_{GR'} \{\partial_{EP'} \psi_{ABCD}\})_0 + \cdots \\ &= f'(x^\mu p_\mu) \pi_A \pi_B \pi_C \pi_D \pi_E \bar{\pi}_{P'}, \\ (\partial_{EP'} \partial_{FQ'} \psi_{ABCD})_x &= f''(x^\mu p_\mu) \pi_A \cdots \pi_F \bar{\pi}_{P'} \bar{\pi}_{Q'},\end{aligned}$$

etc. Hence ψ_{ABCD}, $\psi_{ABCDEP'}$, $\psi_{ABCDEFP'Q'}$, $\cdots$ are all constant along the 3-space $x^\mu p_\mu = 0$. It follows that the whole space $\mathfrak{M}$ admits the three-parameter group of translations[16] in the directions lying in this 3-space. The space $\mathfrak{M}$ thus represents a plane wave which moves uniformly with the velocity of light in the direction represented by p_μ. The intensity and polarization of the wave are determined by the modulus and argument of the function $f(s)$.

Particular cases of interest are:

(i) the constant gravitational field with ψ_{ABCD} constant everywhere. Here $f(s)$ = constant, i.e., $\alpha_1 = \alpha_2 = \cdots = 0$, and $\mathfrak{M}$ admits additional translational motions.

(ii) Sinusoidal waves;

$$f(s) \equiv ae^{ins} + be^{ins}, \quad \text{i.e.,} \quad \alpha_r = a(in)^r + b(-in)^r.$$

In this case $\mathfrak{M}$ admits an additional discrete group of translations.

(iii) Gravitational pulse; for example,

$$f(s) = \begin{cases} b \exp\left(\dfrac{c}{s-a} - \dfrac{c}{s+a}\right) & \text{if } -a < s < a \\ 0 & \text{if } s \leqq -a \text{ or } s \geqq a. \end{cases}$$

Case (iii) is not strictly an analytic manifold. $\mathfrak{M}$ has to be constructed from three analytic pieces (two of which are flat). The middle piece fits on smoothly to the other two pieces, the join being C^∞. The space is exactly flat before the pulse arrives and is again exactly flat after the pulse has departed (*23*, p. 523).

An advantage of a method such as this for obtaining spaces satisfying Einstein's equations is that the usual problem of deciding whether an effect is real or merely due to a bad choice of coordinates simply does not arise. The curvature at any point is found directly. However, it will naturally be convenient to be able to introduce coordinates into a space defined in this way, if desired. A coordinate system on $\mathfrak{M}$ may be thought of as a set of four scalar fields $u_{(i)}$ $(i = 0, \cdots 3)$. The symmetric derivatives $\partial_{(\alpha} \cdots \partial_{\gamma)} u_{(i)}$ of each $u_{(i)}$ may be specified arbitrarily at the point 0. The values of the coordinates $u_{(i)}$ and their derivatives at any other point may then be calculated using (5.2), after some of the unsymmetrized derivatives have been obtained using (2.1). The expression for the metric at each point can be obtained from the first derivatives of the $u_{(i)}$ at that point. This method will be described in detail in a later paper.

The case when an electromagnetic field is present in the space can be treated by an extension of the coordinate-free method for empty space described above. The spinors

$$\psi_{ABCDE\cdots G}{}^{P'\cdots R'} = \partial_{(E}{}^{(P'} \cdots {}_{G}{}^{R')} \psi_{ABCD)}$$

[16] $\mathfrak{M}$ also admits a two parameter group of rotational (Lorentz) symmetries given by the unimodular matrices $t^A{}_B$ satisfying $t^A{}_B\pi^B = \pm\pi^A$, and disconnected from these, the rotations for which $t^A{}_B\pi^B = \pm i\pi^A$. There may also be some reflectional symmetries in special cases. This five parameter group of motions serves to characterize the plane wave solutions (see Bondi *et al.*, (*23*)).

are defined as before and spinors ϕ_{AB}, $\phi_{ABC}{}^{P'}$, $\phi_{ABCD}{}^{P'Q'}$, $\cdots$ are introduced, defined similarly by

$$\phi_{ABC\cdots E}{}^{P'\cdots R'} = \partial_{(C}{}^{(P'} \cdots \partial_{E}{}^{R')}\phi_{AB)} .$$

By the same kind of argument as before, it follows that ϕ_{AB}, $\phi_{ABC}{}^{P'}$, $\cdots$, ψ_{ABCD}, $\psi_{ABCDE}{}^{P'}$, $\cdots$ are all algebraically independent. Instead of (5.4) we have

$$\epsilon^{CA}\partial_{C}{}^{P'}\phi_{AB} = 0 \quad \text{and} \quad -\epsilon^{EA}\partial_{E}{}^{P'}\psi_{ABCD} = \bar{\phi}^{P'}{}_{Q'}\partial_{D}{}^{Q'}\phi_{BC}$$

from (3.10) and (3.12). The first of these states the symmetry of

$$\partial_{C}{}^{P'} \cdots \partial_{E}{}^{R'}\phi_{AB}$$

in E, A, while the second expresses the part of

$$\partial_{E}{}^{P'} \cdots \partial_{G}{}^{R'}\psi_{ABCD}$$

skew in G, A in terms of derivatives of ϕ_{AB} of at most the same order. They imply no condition on the symmetrized derivatives of ϕ_{AB} or ψ_{ABCD}. Nor do the equivalents of (5.5) and (5.6), which differ from them only in that the second relation (5.5) is replaced by

$$\epsilon_{R'S'}\{\partial_{G}{}^{R'}\partial_{H}{}^{S'} + \partial_{H}{}^{R'}\partial_{G}{}^{S'}\}\eta^{P'} = \phi_{GH}\bar{\phi}^{P'}{}_{Q'}\eta^{Q'}$$

(see 3.13) and the first relation (5.6) by

$$\epsilon^{GH}\{\partial_{G}{}^{R'}\partial_{H}{}^{S'} + \partial_{G}{}^{S'}\partial_{H}{}^{R'}\}\xi_{A} = \bar{\phi}^{R'S'}\phi_{AB}\xi^{B}$$

The argument to show that the unsymmetrized derivatives can be expressed algebraically in terms of the symmetrized derivatives is exactly analogous to that for pure gravitational case. The derivative $\partial_{C}{}^{P'} \cdots \partial_{E}{}^{R'}\phi_{AB}$ differs from $\phi_{ABC\cdots E}{}^{P'\cdots R'}$ by expressions constructed from lower order derivatives of ϕ_{AB} and ψ_{ABCD}, while $\partial_{E}{}^{P'} \cdots \partial_{G}{}^{R'}\psi_{ABCD}$ differs from $\psi_{ABCDE\cdots G}{}^{P'\cdots R'}$ by expressions constructed from derivatives of ϕ_{AB} of the same order or lower and from lower order derivatives of ψ_{ABCD}. Thus, we can construct $\partial_{C}{}^{P'}\phi_{AB}$, $\partial_{E}{}^{P'}\psi_{ABCD}$, $\partial_{C}{}^{P'}\partial_{D}{}^{Q'}\phi_{AB}$, $\partial_{E}{}^{P'}\partial_{F}{}^{Q'}\psi_{ABCD}$, $\cdots$, in that order, from the symmetrized derivatives. The symmetric spinors ϕ_{AB}, $\phi_{ABC}{}^{P'}$, $\phi_{ABCD}{}^{P'Q'}$, $\cdots$, ψ_{ABCD}, $\psi_{ABCDE}{}^{P'}$, $\cdots$ can therefore be specified arbitrarily at a point 0 (apart from convergence considerations) and ϕ_{AB}, ψ_{ABCD} at any other point can be determined from them by (5.2).

A simple example is the case of a combined gravitational-electromagnetic wave (see also *22*). Here ψ_{ABCD}, $\psi_{ABCDEP'}$, $\cdots$ are given by (5.9) and

$$\phi_{AB} = \beta_0\pi_A\pi_B , \quad \phi_{ABCP'} = \beta_1\pi_A\pi_B\pi_C\bar{\pi}_{P'} ,$$

$$\phi_{ABCDP'Q'} = \beta_2\pi_A\pi_B\pi_C\pi_D\bar{\pi}_{P'}\bar{\pi}_{Q'} , \cdots$$

at the point O. As was the case, considered earlier, with the pure gravitational

TABLE I

SUMMARY OF SOME OF THE RESULTS OF THIS PAPER ON THE COMPARISONS BETWEEN ELECTROMAGNETIC AND GRAVITATIONAL FIELDS IN SPINOR FORM

	Maxwell field	Curvature tensor with $R^{\mu}{}_{\gamma\mu\sigma} = 0$
Tensor-spinor correspondence	$F_{\mu\gamma} \leftrightarrow \frac{1}{2}\{\phi_{AB}\epsilon_{C'D'} + \epsilon_{AB}\bar{\phi}_{C'D'}\}$	$R_{\mu\gamma\rho\sigma} \leftrightarrow \frac{1}{2}\{\psi_{ABCD}\epsilon_{E'F'}\epsilon_{G'H'} + \epsilon_{AB}\epsilon_{CD}\bar{\psi}_{E'F'G'H'}\}$
First order equation	Maxwell equations: $\partial^{AC'}\phi_{AB} = 0$	Bianchi identities: $\partial^{AE'}\psi_{ABCD} = 0$
(Super-)energy tensor	Maxwell stress tensor $\leftrightarrow$ $\frac{1}{2}\phi_{AB}\bar{\phi}_{C'D'}$	Robinson-Bel tensor $\leftrightarrow$ $\psi_{ABCD}\bar{\psi}_{E'F'G'H'}$
Duality rotations	$\phi_{AB} \to e^{i\theta}\phi_{AB}$	$\psi_{ABCD} \to e^{i\theta}\psi_{ABCD}$
Canonical representation	$\phi_{AB} = \eta_{(A}\zeta_{B)}$	$\psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$
Classification scheme	[11] $K \neq 0$ $\swarrow \ \downarrow$ [2] $\to$ [—] $K = 0$	[1111] $I^3 \neq 6J^2$ $\swarrow \ \downarrow$ [211] $\to$ [22] $I^3 = 6J^2 \neq 0$ $\swarrow \ \downarrow \ \swarrow \ \downarrow$ [31] $\to$ [4] $\to$ [—] $I = J = 0$
Plane wave	$\phi_{AB}(x^{\mu}) = g(x^{\mu}p_{\mu})\pi_A\pi_B$	$\psi_{ABCD}(x^{\mu}) = f(x^{\mu}p_{\mu})\pi_A\pi_B\pi_C\pi_D$

wave, the unsymmetrized derivatives of ϕ_{AB} and ψ_{ABCD} turn out to be equal to the symmetrized derivatives provided that $\lambda = 0$. Hence

$$(\phi_{AB})_x = g(x^{\mu}p_{\mu})\pi_A\pi_B, \quad (\psi_{ABCD})_x = f(x^{\mu}p_{\mu})\pi_A\pi_B\pi_C\pi_D,$$

where

$$g(s) \equiv \beta_0 + \beta_1 s + \frac{1}{2!}\beta_2 s^2 + \cdots$$

and $f(s)$ is given by (5.10) as before. The discussion given in the pure gravitational case applies here also. The function $g(s)$ determines the intensity and polarization of the electromagnetic part of the wave and $f(s)$ the "purely gravitational" part. The electromagnetic field is null everywhere and the gravitational field is [4]. All six principal null directions coincide and point in the direction p_{μ} giving the motion of the wave.

Table I summarizes some of the many analogies between the electromagnetic and gravitational fields, that are brought out by the spinor formalism.

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