The fourth test of general relativity

1982 Rep. Prog. Phys. 45 631

(http://iopscience.iop.org/0034-4885/45/6/002)

Rep. Prog. Phys., Vol. 45, 1982. Printed in Great Britain

The fourth test of general relativity

Kenneth Nordtvedt Jr

Montana State University, Bozeman, Montana 59715, USA

Abstract

In the 'fourth test of general relativity' the gravitational acceleration of celestial bodies—the Earth and the Moon—were experimentally compared in the gravitational field of the Sun. Because such bodies obtain an appreciable fraction of their total mass-energy from their internal gravitational self-energy (5×10^{-10} for the Earth), this comparison of free-fall rates measures, among other things, how gravity pulls on gravitational energy and how gravitational energy contributes to the inertial mass of celestial bodies.

By using high-precision laser ranging between Earth and reflectors on the Moon's surface, it was found that the Earth and Moon's acceleration in the Sun's gravitational field are the same to one part in 10^{11}. Hypothesising that the gravitational to inertial mass ratio of a celestial body may differ from one by the order of the gravitational self-energy content of the body divided by the total mass-energy:

$$M_G/M_I = 1 + \eta(U_G/Mc^2)$$

η being a dimensionless constant determined by gravitational theory, the lunar laser ranging experiment limits $|\eta|$ to less than 1.4×10^{-2}.

This experiment is consistent with general relativity which predicts $\eta = 0$. However, scalar-metric tensor theories such as the Brans–Dicke theory, vector-metric tensor theories and two-tensor theories of gravity are, in most cases, inconsistent with this experiment unless sufficient adjustable parameters are used in such theories.

This review was received in November 1981

0034–4885/82/060631 + 21\$08.00

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1. Phenomenology of the 'fourth test' of general relativity

The fourth test of general relativity is a precise comparison of the free-fall rate of the Earth and the Moon in the gravitational field of the Sun. Laser ranging between the surfaces of the Earth and Moon permits the detection of anomalous time-dependent range variations which would result if the two bodies' free-fall rates differ by as little as one part in 10^{11}. Analysis of relativistic theories of gravity indicates that Einstein's general theory of relativity is almost unique among theories in predicting a null result for this experiment.

In a historical sense this new test of gravity theory can be considered as a continuation of experiments confirming the universality of free-fall rates for various bodies in gravitational fields performed by Galileo, Eötvös (1922), Dicke (Roll 1964) and Braginsky (1971). In a more theoretical sense, this experiment tests the full post-Newtonian structure of the general relativity theory of gravity.

Eötvös (1922) employed a torsion balance on which platinum and copper, as well as other material masses, were placed and the balance was turned in the Earth's gravitational field. No detectable torques were developed, which was interpreted to mean that these substances fell at the same rate in the Earth's gravitational field to an accuracy of $\delta g/g < 3\times 10^{-9}$.

Dicke and co-workers (Roll 1964) altered the experiment to some extent, using a three-mass torsion balance to make it insensitive to nearby Newtonian quadrupole perturbations. The balance orientation was kept fixed relative to the Earth's gravitational field, and instead the Earth's daily rotation turned the apparatus in the Sun's gravitational field. Finding no anomalous torques, they concluded that gold and aluminium fall in the Sun's gravitational field at the same rate to an accuracy of $\delta g/g < 3\times 10^{-11}$. Braginsky (1971) repeated a similar experiment and quoted somewhat more accurate confirmation of the universality of free-fall rates: $\delta g/g < 10^{-12}$.

A theoretical interpretation of these historic Eötvös-type experiments can be stated as follows: all forms of energy in laboratory bodies—mass, nuclear, electromagnetic, kinetic, thermal, etc—contribute identically to both a body's inertial mass M_{I} and (passive) gravitational mass M_{GP} to a precision of one part in 10^{11} or so:

$$M_{\mathrm{I}}\boldsymbol{a} = M_{\mathrm{GP}}\boldsymbol{g} \qquad (1.1(a))$$

or

$$\boldsymbol{a} = (M_{\mathrm{GP}}/M_{\mathrm{I}})\boldsymbol{g}. \qquad (1.1(b))$$

Universality of $\boldsymbol{a}$ in a given gravitational field $\boldsymbol{g}$ implies universality of the $M_{\mathrm{G}}/M_{\mathrm{I}}$ ratio for laboratory bodies of varying material composition.

However, all these experiments compare the acceleration rates of laboratory-sized objects which possess a negligible fraction of gravitational self-energy (internal gravitational potential energy). The ratio of the internal gravitational self-energy of a laboratory object to its total mass energy is of the order of

$$|(-GM^2/D)/Mc^2| \simeq GM/c^2D \leqslant 10^{-25} \qquad (1.2)$$

with G being Newton's gravitational constant, M being the body's mass and D being its size.

Celestial bodies possess more significant fractions of gravitational self-energy. GM/c^2D is of the order of 10^{-6} for the Sun, 10^{-8} for Jupiter, and 5×10^{-10} for the Earth. It is therefore worthwhile to hypothesise that celestial bodies may have anomalous gravitational to inertial mass ratios proportional to their gravitational self-energy content (Dicke 1962, Nordtvedt 1968a):

$$M_{\mathrm{GP}}/M_{\mathrm{I}} = 1 + \eta U_{\mathrm{G}}/Mc^2 \tag{1.3}$$

with

$$U_{\mathrm{G}} = -\frac{G}{2}\int \rho(\boldsymbol{x})\rho(\boldsymbol{x}')\,\mathrm{d}^3x\,\mathrm{d}^3x'/|\boldsymbol{x}-\boldsymbol{x}'|.$$

η is a dimensionless constant which experiment would seek to detect or limit and which any particular theory of gravity would determine by appropriate calculation.

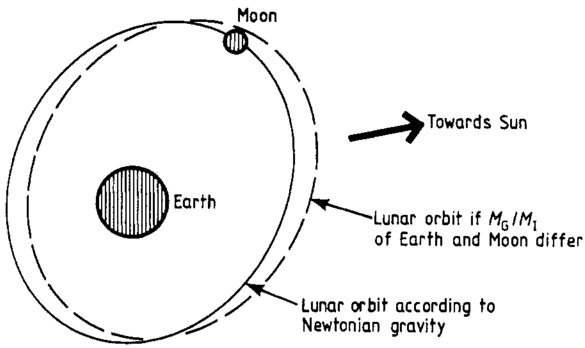

Figure 1. If the gravitational to inertial mass ratio of the Earth and Moon differ from each other, then their acceleration toward the Sun will differ, producing a polarisation of the lunar orbit towards (or away from) the Sun. This manifests itself in the Earth–Moon distance as an anomalous range oscillation with a mean period of 29.53 d—the synodic month. If the $M_{\mathrm{G}}/M_{\mathrm{I}}$ ratio of the Earth differed from 1 by an amount of the order of the Earth's internal gravitational potential energy divided by the Earth's total mass-energy, then this polarisation amplitude would be about 11 m.

Standard celestial dynamics does not readily detect a non-zero η because Kepler's laws are unaltered except for minute rescalings of Kepler's third law (Nordtvedt 1968a). The celestial three-body system offers a more hopeful method to test for a non-zero η. Consider the Earth and Moon in the Sun's gravitational field as shown in figure 1. In a coordinate frame free-falling with the Earth–Moon centre of mass (1.3) produces an anomalous relative acceleration of the Earth and Moon (Nordtvedt 1968c):

$$\boldsymbol{a} = \eta[(U_{\mathrm{G}}/Mc^2)_{\mathrm{E}} - (U_{\mathrm{G}}/Mc^2)_{\mathrm{M}}]\boldsymbol{g}_{\mathrm{S}} \tag{1.4}$$

with $\boldsymbol{g}_{\mathrm{S}}$ being the Sun's gravitational field. For $\boldsymbol{g}_{\mathrm{S}}$ lying approximately in the Earth–Moon orbital plane and a near-circular lunar orbit (1.4) produces an Earth–Moon range oscillation (Nordtvedt 1968c):

$$\delta r(t) = 3\eta(U_{\mathrm{G}}/Mc^2)_{\mathrm{E}}g_{\mathrm{S}}[(\omega_0^2-(\omega-\Omega)^2)]^{-1}\cos(\omega-\Omega)t \tag{1.5(a)}$$

or

$$\delta r(t) = 1100\eta\cos(\omega-\Omega)t \qquad \mathrm{cm} \tag{1.5(b)}$$

using $|U_G/Mc^2| = 5 \times 10^{-10}$ for the Earth. Ω is the mean angular frequency of the Earth–Moon system around the Sun, ω is the mean angular frequency of the Moon around the Earth, and ω_0 is the natural angular frequency for radial perturbations of the lunar orbit, i.e. frequency of periastron occurrence.

Measuring the range oscillation amplitude of frequency $\omega - \Omega$ to high accuracy is accomplished by laser ranging between the Earth and reflecting stations on the Moon's surface (Faller 1970, Bender 1973). Ranging data to several reflectors placed on the Moon by the Apollo astronauts has been collected over the past decade. Analysis of several years' data by two independent groups (Williams *et al* 1976, Shapiro *et al* 1976) measures the appropriate range Fourier amplitude to 15 cm accuracy, and on finding no anomaly from Newtonian predictions one concludes that

$$|\eta| \leqslant 1.4 \times 10^{-2}. \tag{1.6}$$

To almost 1% accuracy then, the gravitational self-energy of celestial bodies contributes properly to altering both a body's gravitational and inertial mass.

In § 6 we show that this result offers a comprehensive test of the full non-linear, non-static, non-spherically symmetric post-Newtonian structure of Einstein's general relativity theory of gravity.

2. Previous tests of general relativity

When Einstein presented his full field equations for gravity in 1915–16, he found that there were corrections to Newtonian gravity which produced an additional 43″ per century precession of Mercury's perihelion, in very good agreement with a known discrepancy in the astronomical observations. In fact, there is evidence that Einstein felt constrained during the decade 1906–16 to explain, among other things, that precession anomaly by a proper relativistic theory of gravity.

Einstein's theory yields perihelion precession through two additions to Newtonian gravity. First, there is a non-linear potential in general relativity proportional to $(GM/cr)^2$, and secondly there are corrections to the coupling of a moving object in a gravitational field proportional to $(v/c)^2\boldsymbol{g}$ and $\boldsymbol{v} \cdot \boldsymbol{g}\boldsymbol{v}/c^2$.

Prior to the development of a specific relativistic theory of gravity, Einstein (1911), using only arguments from the equivalence of uniform gravitational fields to accelerated coordinate frames (equivalence principle), predicted that light trajectories should be deflected by gravitational fields at the rate

$$\mathrm{d}\theta/\mathrm{d}x = g_{\perp}/c^2 \tag{2.1}$$

where $g_{\perp}$ is the gravitational field perpendicular to the light propagation direction. For a light ray passing the edge of the Sun (radius D) from a distant star (2.1) gives a total deflection angle of the apparent position of the star of

$$\theta = 2GM/c^2D. \tag{2.2}$$

Einstein's full theory of gravity (1916), however, predicts a deflection which is twice that given by (2.2). The difference in these two predictions can be understood as follows. Figure 2 illustrates the geometry of rigid rods and light-ray trajectories in the vicinity of a massive body like the Sun. The light-ray trajectories do, in fact, deflect relative to the rigid 'straight' rods by an amount given by Einstein's equivalence principle argument. But Einstein's general theory of relativity also predicts that the

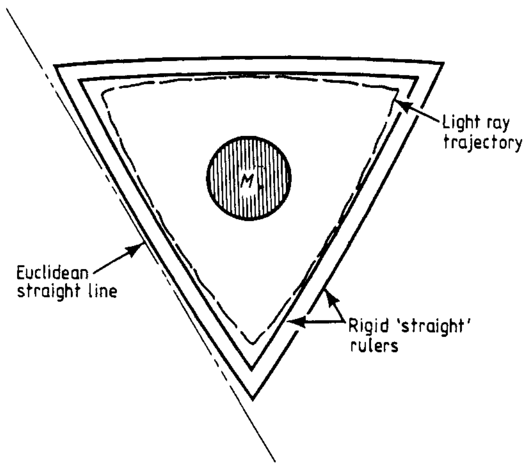

Figure 2. In metric theories of gravity the geometry of 'straight' rulers deviates from Euclidean geometry. The rigid-ruler triangle illustrated would have a sum of angles differing from 180° when a massive body M were in the proximity. This spatial geometry alteration differs from one metric theory to another and is parametrised by the PPN coefficient γ. Light rays deflect relative to these rigid 'straight' rulers. This deflection is universal in all metric theories and is calculable by Einstein's equivalence principle hypothesis. Deflection of a light ray with respect to the distant fixed stars will have contributions from the sum of these effects.

spatial structure of 'straight' rods becomes non-Euclidean in the vicinity of matter; the 'straight' rods are curved with respect to distant star orientations. This spatial curvature produces an additional light deflection with respect to those distant stars.

Measurements of the apparent location of stars during a solar eclipse in 1919 and in subsequent experiments have been found to be consistent with the predictions of Einstein's full theory of gravity. Modern observations using interferometric techniques on radio signals from quasi-stellar sources passing behind the Sun (Fomalont 1976) have improved the quantitative verification of general relativity's deflection prediction to about 1%.

Shapiro (1964) pointed out that one could measure a time delay of radio signals that necessarily accompanies these deflections in a gravitational field. In general relativity the speed of light as measured by a distant observer is reduced near a massive body:

$$c(r)=c_\infty(1-2GM/c^2r). \tag{2.3}$$

A radar pulse sent from a body at distance D_1 from a massive body and returned by another body at distance D_2 from the body, with the pulse passing the massive body at closest distance D_0 is anomalously delayed by

$$\Delta t=4GM/c^3\ln(4D_1D_2/D_0^2) \tag{2.4}$$

which amounts to about 250 μs for D_1 and D_2 being Earth's and Mars' orbital radii and D_0 being the solar radius. Radar time delay experiments to Viking satellites sent to Mars have confirmed the time delay (2.4) to 0.2% accuracy (Reasenberg 1979).

It has been traditional to view experiments which measure the effect of gravitational potential on clock rates as tests of Einstein's general relativity theory. In actuality, such experiments test Einstein's equivalence principle (Einstein 1911) which he employed to predict that clocks would go slower when near massive bodies. For small changes in gravitational potential clock rates are expected to vary as

$$\Delta t'/\Delta t = \{1 - [U(r') - U(r)]/c^2\} \qquad \text{for } U(r) = GM/r. \tag{2.5}$$

This was confirmed to 1% precision (Pound and Snider 1965) by comparing γ-ray nuclear emitters and absorbers separated vertically in the Earth's gravitational field and employing the Mössbauer effect to obtain recoiless nuclei for the emitters and absorbers. More recent 1% confirmation of the gravitational clock rate shift has become routine with high-precision atomic clocks flown on high-altitude aircraft (Alley 1979).

0.01% confirmation of (2.5) was accomplished using one-way and two-way Doppler tracking of the trajectory of a sub-orbital rocket flight, an atomic clock on board the rocket being compared with another precision clock on the Earth's surface (Vessot 1981).

The clock rate variation caused by gravity (2.5) is present in all metric theories of gravity. This is a very broad class of possible theories, of which Einstein's general relativity is one specific case. In fact, it has been shown (Nordtvedt 1975, Unruh 1979, Haugan 1979) that the relation (2.5) is implied simply by conservation of energy and universality of free-fall rates in gravitational fields.

3. Metric theories of gravity

The very high accuracy to which the universality of free-fall for laboratory bodies has been confirmed, along with other high-precision experiments such as the Hughs–Drever experiment (Hughs 1960), implies that gravity must be represented by a metric field $g_{\mu\nu}(\boldsymbol{r}, t)$. This second-rank, symmetric tensor field (ten components at every space–time point) must in turn couple in a universal and prescribed way to all matter and fields of laboratory physics. A 'space–time' geometry then results given by the invariant interval between neighbouring space–time points;

$$\mathrm{d}\tau^2 = g_{\mu\nu}(\boldsymbol{r}, t)\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu \tag{3.1}$$

the indices μ and ν are summed over 0, 1, 2, 3; $\mathrm{d}x^0 = \mathrm{d}t$, $\mathrm{d}x^{1,2,3} = \mathrm{d}x/c, \mathrm{d}y/c, \mathrm{d}z/c$. In metric theories of gravity the equation of motion for small mechanical particles is obtained from the variational principle

$$\delta \int (g_{\mu\nu}(\boldsymbol{r}, t)\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu)^{1/2} = 0 \tag{3.2}$$

which defines the geodesics (extremum trajectories) of the space–time. Light rays travel on 'null-geodesics':

$$\mathrm{d}\tau = 0. \tag{3.3}$$

Any physical clock at location $\boldsymbol{r}$ at time t ticks with an intrinsic elapsed interval given by (3.1), while the coordinate span of all physical rulers is also determined at all space–time locations by the metric field $g_{\mu\nu}(\boldsymbol{r}, t)$. This control of $g_{\mu\nu}(\boldsymbol{r}, t)$ over the chrono-metrical properties of clocks and rulers is not a postulational one in metric theories: it emerges from the actual coupling of $g_{\mu\nu}(\boldsymbol{r}, t)$ to all of the dynamical equations of motion of matter from which physical clocks and rulers are built.

The only additional specification needed for a unique metric theory of gravity and matter is the field equations for the metric gravity field $g_{\mu\nu}$.

4. The historic parametrised metric field expansions

Eddington (1923) and Robertson (1962) suggested that in the vicinity of, but external to, a spherically symmetric mass M, the general form of the components of the metric field $g_{\mu\nu}$ for most metric theories of gravity must be a power series in m/r, with $m = GM/c^2$ being a gravitational 'length' unit of measure for the central mass M, and r is a radial coordinate:

$$g_{00} = 1 - 2(m/r) + 2\beta(m/r)^2 + \ldots \tag{4.1(a)}$$

$$g_{kl} = -[1 + 2\gamma(m/r)]\delta_{kl} + \ldots \tag{4.1(b)}$$

and

$$g_{0k} = 0 \tag{4.1(c)}$$

for k and l equal to 1, 2, 3. The asymptotic form for $g_{\mu\nu}$ far from the mass must necessarily be the Minkowski metric of special relativity. The leading term $-2(m/r)$ in g_{00} is necessary in order to obtain Newtonian gravity in lowest-order approximation. The dimensionless parameters γ, β, etc, are dependent on the specific theory of gravity. For example, in general relativity both γ and β are 1. In the Brans–Dicke theory $\beta = 1$ but $\gamma \neq 1$.

The γ parameter indicates the degree of curvature of three-space produced by matter (see figure 2), while the β parameter indicates the post-Newtonian non-linearity of the central gravitational field.

Figure 3 outlines the logic of how a gravitational phenomenon can be evaluated within the entire class of metric theories of gravity, without the knowledge or assumption of a specific theory of gravity. A general, parametrised expansion for the metric field in terms of the matter distribution permits an evaluation of the matter's dynamical response to gravity in the general case. Experimental effects are finally expressed in terms of the parameters of the metric field expansion which multiplied the components of $g_{\mu\nu}$ which participated in that particular effect.

In terms of the Eddington–Robertson parametrised metric expansion for a static, spherically symmetric mass source it is straightforward to confirm that the clock rate shift (2.5) is universally valid for any metric theory of gravity. The light deflection (2.2) and retardation (2.4) effects become generally in any metric theory

$$\theta = (1+\gamma)2GM/c^2D \tag{4.2}$$

and

$$\Delta t = 2(1+\gamma)GM/c^3 \ln(4D_1D_2/D_0^2). \tag{4.3}$$

The relativistic perihelion precession of a planet or other body orbiting a central mass is

$$\omega = (2 + 2\gamma - \beta)8\pi^2a^2/c^2T^3(1-e^2) \tag{4.4}$$

a, T and e are the semi-major axis, period and eccentricity of the orbit.

The historic tests of general relativity are analysable in terms of the Eddington–Robertson metric field for the static, spherically symmetric gravitational environment of the Sun. These experiments, then, can be said to test the post-Newtonian structure of gravity only in this restricted situation.

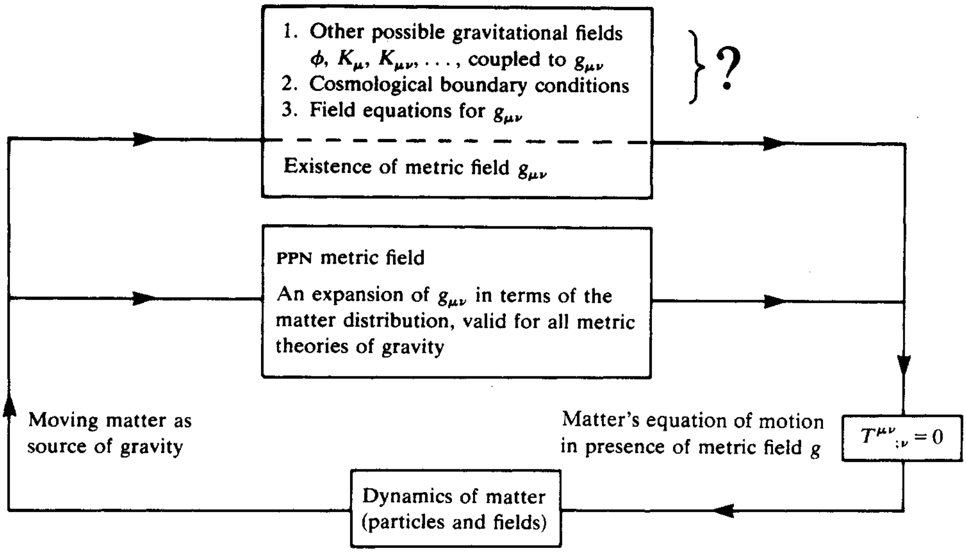

Figure 3. The logic of the PPN metric theory is shown in the closed operational loop for solving dynamical problems of matter plus gravity. Assuming the existence of a metric gravitational field $g_{\mu\nu}$ which interacts with matter's stress-energy tensor according to the equation $T^{\mu\nu}_{;\nu}=0$, the matter dynamics can be determined if $g_{\mu\nu}$ is known (i.e. we can close the outer loop). But if the full theory of metric gravity is not known (possible unknowns are listed outside the loop as 1, 2 and 3), making it impossible to calculate $g_{\mu\nu}$ from the matter distribution, $g_{\mu\nu}$ can instead be given as a general expansion (PPN expansion) in terms of the matter distribution, and we close via the inner calculational loop without the assumption of a specific theory of gravity.

L Schiff (1961 private communication) enlarged the parametrised metric field expansion in order to include a stationary gravitational vector potential produced by a rotating mass. Defining $\boldsymbol{h}=(g_{0x}, g_{0y}, g_{0z})$

$$\boldsymbol{h}=2\Delta G(\boldsymbol{J}\times\boldsymbol{r})/c^3r^3 \tag{4.5}$$

where $\boldsymbol{J}$ is the rotating body's angular momentum, and the new parameter Δ introduced by Schiff is 1 in general relativity. This 'Lense–Thirring' potential has the effect of dragging the inertial coordinate frames slowly around in the vicinity of a rotating body. Schiff introduced this potential in order to derive the general effect of such a gravitational vector potential on the precession of a free-falling gyroscope. He obtained

$$\boldsymbol{\Omega}=\boldsymbol{\nabla}\times\boldsymbol{h}c. \tag{4.6}$$

The gravitational potential (4.5) also perturbs the orbits of bodies and trajectories of light rays passing near a rotating celestial body in, for example, an experiment where a satellite was put into an orbit which closely approached the Sun and was tracked by radar (Nordtvedt 1977b).

5. The complete parametrised, post-Newtonian (PPN) metric field

The limited metric expansions of § 4 are not sufficiently general and complete to calculate the gravitational to inertial mass ratio of celestial bodies. A more complete collection of post-Newtonian potentials play a role in determining M_G/M_I for such bodies. Figure 4 illustrates the configuration of masses involved in a theoretical derivation of M_G/M_I. M is a distant mass toward which a celestial body accelerates. Mass elements m_i, m_j, m_k, etc, make up the celestial body. These mass elements must, for purposes of this calculation, be considered to be in general motion at velocities $\boldsymbol{v}_i$ and accelerations $\boldsymbol{a}_i$. Each mass element m_i moves in the gravitational metric field of M plus the $m_{j\neq i}$, as well as in response to any internal non-gravitational forces in the celestial body such as solid-state electrical forces. Our metric field expansion must be sufficiently general to have such a dynamical matter distribution as its source.

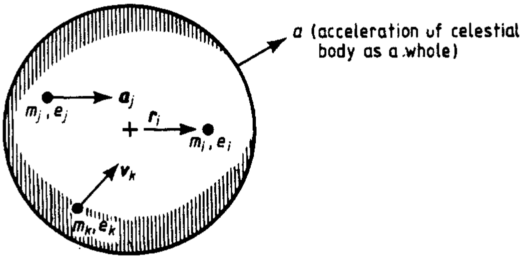

Figure 4. The material configuration used in a calculation of the gravitational to inertial mass ratio for a celestial body is shown. The mass elements m_i (m_j, m_k, etc) making up the celestial body are located at positions $\boldsymbol{r}_i(t)$ from the body's centre; each mass element generally has a velocity $v_k(t) = \mathrm{d}r_k(t)/\mathrm{d}t$ due to thermal, kinetic or rotational motion in the body, and has an acceleration $\boldsymbol{a}_j(t) = \mathrm{d}\boldsymbol{v}_j(t)/\mathrm{d}t$ due to acceleration of the body as a whole toward an external mass M ($\boldsymbol{a}$) and internal accelerations due to internal gravitational and electrical forces for mass elements having non-zero charge e_i. A gravitational metric field expansion must be sufficiently general to represent the gravitational fields for such a dynamical matter source.

Because of the motion of the mass sources, and because of the superposition of sources, the phenomenological metric field expansions of Eddington, Robertson and Schiff must be enlarged and generalised to include all the post-Newtonian potential terms. Baierlein (1967) did some work in this direction in order to study general post-Newtonian perturbations of the Moon's orbit, but his metric expansion was not sufficiently complete for our purposes.

Following an approach used by Einstein, Infeld and Hoffman who obtained a post-Newtonian expansion for general relativity theory, we consider gravitational environments in which the gravitational potentials are relatively weak and the sources are slow moving. Two small dimensionless quantities then exist in which a post-Newtonian metric field can be expanded:

$$(GM/c^2r) \ll 1 \qquad \text{and} \qquad (v/c)^2 \ll 1.$$

M, v and r are any of the masses, velocities or spatial intervals in figure 4. Nordtvedt's first generalised post-Newtonian metric field (Nordtvedt 1968b) took the form:

$$g_{00} = 1 - 2U + 2\beta U^2 + 2\alpha' \sum_i \sum_j m_i m_j / |\boldsymbol{r}_i - \boldsymbol{r}_j||\boldsymbol{r} - \boldsymbol{r}_i| + \chi \sum_i m_i \boldsymbol{a}_i \cdot (\boldsymbol{r} - \boldsymbol{r}_i)/|\boldsymbol{r} - \boldsymbol{r}_i|$$

$$- \alpha'' \sum_i m_i v_i^2 / |\boldsymbol{r} - \boldsymbol{r}_i| + \alpha''' \sum_i m_i [\boldsymbol{v}_i \cdot (\boldsymbol{r} - \boldsymbol{r}_i)]^2 / |\boldsymbol{r} - \boldsymbol{r}_i|^3 + \ldots \qquad (5.1(a))$$

(the speed of light c is set equal to 1 in the metric expansions)

$$g_{0k} = 4\Delta \sum_i m_i (\boldsymbol{v}_i)^k / |\boldsymbol{r} - \boldsymbol{r}_i| + 4\Delta' \sum_i m_i (\boldsymbol{r} - \boldsymbol{r}_i) \cdot \boldsymbol{v}_i (\boldsymbol{r} - \boldsymbol{r}_i)^k / |\boldsymbol{r} - \boldsymbol{r}_i|^3 + \ldots \qquad (5.1(b))$$

and

$$g_{kl} = -(1+2\gamma U)\delta_{kl} + \ldots \tag{5.1(c)}$$

for k and l equal to 1, 2, 3. U is the Newtonian potential function:

$$U = \sum_i m_i/|\boldsymbol{r} - \boldsymbol{r}_i|. \tag{5.2}$$

This form of the metric for a distribution of moving particles was obtained by imposing several conditions on the metric field.

(i) g_{00} and g_{kl} were required to be even under time reversal, while g_{0k} was required to be odd. This guaranteed time-reversal invariance for the equations of motion of the interacting masses.

(ii) g_{00} was required to transform as a scalar under spatial rotations, g_{0k} as a spatial vector and g_{kl} as a spatial second-rank tensor.

(iii) When there were several mass sources, the metric field was required to be properly symmetric under interchange of mass sources and was required to become equal to the Eddington–Robertson metric (4.1) in the appropriate limits of mass sources coalescing into single sources.

If the linear part of $g_{\mu\nu}$ is form-invariant under Lorentz transformations of the space–time coordinates, the various PPN coefficients in (5.1) are not all independent of each other. Nordtvedt (1969, 1970a) found that Lorentz invariance implied that $\Delta = \alpha'' = (1+\gamma)/2$, and that proper retardation of the static Newtonian potential implied $\chi = 1$ and $\Delta' = 0$. Will (1971b) derived similar conditions within his hydrodynamic PPN formalism.

Using this metric field expansion Nordtvedt (1968b) calculated the gravitational to inertial mass ratio for a celestial body held in equilibrium solely by internal kinetic motion and pressure and internal gravitational forces, thereby avoiding in his initial investigation the necessity of any assumptions about the equations of interacting matter in gravitational fields. Nordtvedt's result was

$$\begin{aligned} M_G/M_I &= 1 + [(4\beta + 3\gamma - 8\Delta + \chi) + (2 + \alpha' - 2\beta - \chi - 8\Delta')/3]U_G/Mc^2 \\ &= 1 + \eta U_G/Mc^2 \end{aligned} \tag{5.3}$$

where U_G is the internal gravitational self-energy of the celestial body and M is the total mass-energy of the body.

More generally, for a rotating celestial body Nordtvedt (1969) found that M_G/M_I became a spatially anisotropic tensor:

$$\{M_G/M_I\}_{kl} = (1 + \eta U_G/Mc^2)\delta_{kl} + \eta'(J^2/2I\delta_{kl} - 3J_kJ_l/2I)/Mc^2 \tag{5.4}$$

with $\eta' = (8\Delta' + 2\beta + \chi - \alpha' - 2)$. $\boldsymbol{J}$ is the body's angular momentum and I is its moment of inertia. However, rotational kinetic energy is much less than the gravitational self-energy for almost all celestial bodies ($T_{\text{rot}} \simeq 10^{-3}U_G$ for Earth). Will (1971c) found the same anisotropic expression in the fluid PPN formalism described below.

In general relativity the PPN coefficients $\gamma = \beta = \chi = \Delta = \alpha' = 1$ and $\Delta' = 0$, so there is no anomalous M_G/M_I in Einstein's theory. But in the Brans–Dicke (1961) theory of gravity $\gamma = (1+\omega)/(2+\omega)$ and $\Delta = (3+2\omega)/(4+2\omega)$; ω is a dimensionless coupling constant in their theory which controls the amount of scalar gravitational field affecting matter. Therefore M_G/M_I differs from 1 in the Brans–Dicke theory:

$$(M_G/M_I)_{\text{B-D}} = 1 + (U_G/Mc^2)/(2+\omega). \tag{5.5}$$

Using the experimental limit (1.6) on $|\eta|$ gives a lower limit on the Brans–Dicke coupling constant

$$\omega \geqslant 70.$$

Will (1971a) considered celestial bodies as continuous fluids described by a stress-energy tensor

$$T^{\mu\nu} = \rho v^{\mu} v^{\nu} + p(g^{\mu\nu} - v^{\mu} v^{\nu}) \tag{5.6}$$

where ρ is the fluid's scalar mass density and p is the fluid pressure. An equation of state $p(\rho)$ would also be assumed for any specific body. The fluid's response to gravity was given by the covariant divergence condition

$$T^{\mu\nu}_{;\nu} = 0. \tag{5.7}$$

Will developed a version of a PPN metric with the components of the fluid stress-energy tensor acting as sources of gravity. He calculated the M_G/M_I ratio for celestial bodies and found the ratio to be independent of the body's fluid equation of state and in agreement with Nordtvedt's result (5.3). Will's work also first made clear the gauge freedom available in metric gravity. Under gauge (coordinate) transformations, some of the PPN coefficients change; however, linear combinations of PPN coefficients expressing physical effects must in all cases be unaltered by choice of coordinates (gauge).

Nordtvedt (1971a) later added electromagnetic interactions between the particles forming the celestial bodies (solid-state model of matter), with the electromagnetic fields properly coupled to the PPN metric field. The M_G/M_I ratio for a body was found to be identical to (5.4) for arbitrary equilibrium configuration of internal and gravitational forces and kinetic motion.

These generalisations of the M_G/M_I calculation by Will and Nordtvedt show that (5.4) and (5.5) are universally valid for celestial bodies which are crystalline, fluid, gaseous, etc.

Both Nordtvedt's and Will's metric field expansions were found to need modification in order to permit consistent use of the PPN metric by all asymptotic inertial observers. Suppose there were several distant observers in different inertial frames, and each of them employed a PPN metric to calculate an observable effect. The several observers must agree (up to appropriate special relativity adjustments) in their calculations of a physical effect for a theory to be consistent—indeed, to be meaningful.

Will and Nordtvedt (1972) found that some of the PPN coefficients were related to the emergence of preferred inertial frame (lack of Lorentz invariance in post-Newtonian gravitational phenomena) in the metric field expansion. Will and Nordtvedt's extended PPN metric formalism contained new gravitational potentials which are dependent on the velocity of the observer's inertial frame relative to a preferred inertial frame. These potentials produce observable gravitational effects on systems moving relative to the preferred frame. These modifications do not alter the calculation of M_G/M_I for massive bodies, but some of the PPN coefficients used to express the ratio are changed. The new PPN language facilitates tracing origins of anomalies in the M_G/M_I ratio to separate anomalous behaviour of M_G or M_I.

In examining two-tensor theories of gravity Nordtvedt (1976) found that such gravitational theories will generally have intrinsic anisotropies in the various post-Newtonian as well as Newtonian potentials, and the PPN metric field expansion was generalised accordingly. Earth-based gravimeter experiments have provided the best experimental evidence against such theories.

The original PPN formalism includes metric theories of gravity which lack post-Newtonian energy or momentum conservation laws. However, relationships among the PPN coefficients exist guaranteeing post-Newtonian conservation of energy, momentum and angular momentum, as well as uniform centre-of-energy motion (Nordtvedt 1970a, b, Will 1971b, Haugan 1979).

The most general extended PPN metric field with energy, momentum and angular momentum conservation laws has five parameters—γ, β, α_1, α_2 and ζ. Its form in the preferred inertial frame is

$$g_{00}=1-2U+2\beta U^2+(4\beta-2)\sum_{j,k} m_j m_k/|\boldsymbol{r}-\boldsymbol{r}_j||\boldsymbol{r}_j-\boldsymbol{r}_k|$$

$$-(2\gamma+1+\zeta)\sum_j m_j v_j^2/|\boldsymbol{r}-\boldsymbol{r}_j|+\zeta\sum_j m_j[\boldsymbol{v}_j\cdot(\boldsymbol{r}-\boldsymbol{r}_j)]^2/|\boldsymbol{r}-\boldsymbol{r}_j|^3 \qquad (5.8(a))$$

$$\boldsymbol{h}=(g_{0x}, g_{0y}, g_{0z})=\left(\frac{4\gamma+3+\alpha_1-\alpha_2+2\zeta}{2}\right)\sum_j m_j\boldsymbol{v}_j/|\boldsymbol{r}-\boldsymbol{r}_j|$$

$$+\left(\frac{1+\alpha_2-2\zeta}{2}\right)\sum_j m_j(\boldsymbol{r}-\boldsymbol{r}_j)\cdot\boldsymbol{v}_j(\boldsymbol{r}-\boldsymbol{r}_j)/|\boldsymbol{r}-\boldsymbol{r}_j|^3 \qquad (5.8(b))$$

and

$$g_{kl}=-(1+2\gamma U)\delta_{kl}+\zeta\left(\sum_j m_j(\boldsymbol{r}-\boldsymbol{r}_j)_k(\boldsymbol{r}-\boldsymbol{r}_j)_l/|\boldsymbol{r}-\boldsymbol{r}_j|^3-U\delta_{kl}\right) \qquad (5.8(c))$$

for $k, l=x, y, z$. U is the Newtonian potential function:

$$U=\sum_j m_j/|\boldsymbol{r}-\boldsymbol{r}_j|.$$

A gauge (coordinate system) has been chosen in which g_{00} has no potential term of the form $\sum_j m_j\boldsymbol{a}_j\cdot(\boldsymbol{r}-\boldsymbol{r}_j)/|\boldsymbol{r}-\boldsymbol{r}_j|$ and no non-linear potentials proportional to the PPN coefficient ζ. Will (1971b) has shown that the preferred frame coefficients α_1 and α_2 must vanish if a centre of mass-energy can be defined for an isolated system which moves necessarily at constant velocity.

In other inertial frames moving at velocity $\boldsymbol{w}$ relative to the preferred inertial frame, new potentials are added to (5.8) of the form:

$$\delta g_{00}=(\alpha_1-\alpha_2)w^2U+\alpha_2\sum_j m_j[\boldsymbol{w}\cdot(\boldsymbol{r}-\boldsymbol{r}_j)]^2/|\boldsymbol{r}-\boldsymbol{r}_j|^3+\alpha_1\sum_j m_j\boldsymbol{w}\cdot\boldsymbol{v}_j/|\boldsymbol{r}-\boldsymbol{r}_j| \qquad (5.9(a))$$

and

$$\delta\boldsymbol{h}=\left(\frac{\alpha_1-2\alpha_2}{2}\right)U\boldsymbol{w}+\alpha_2\sum_j m_j\boldsymbol{w}\cdot(\boldsymbol{r}-\boldsymbol{r}_j)(\boldsymbol{r}-\boldsymbol{r}_j)/|\boldsymbol{r}-\boldsymbol{r}_j|^3. \qquad (5.9(b))$$

An additional gauge choice is used here to eliminate terms in g_{00} of the form:

$$\sum_j m_j\boldsymbol{w}\cdot(\boldsymbol{r}-\boldsymbol{r}_j)\boldsymbol{v}_j\cdot(\boldsymbol{r}-\boldsymbol{r}_j)/|\boldsymbol{r}-\boldsymbol{r}_j|^3.$$

The preferred frame potentials (5.9) do not alter the calculation of M_G/M_I. However, the preferred frame coefficients in (5.8(*b*)) do contribute to M_G/M_I as will be shown in §6. The results of that section are summarised here:

$$M_G/M_I = 1 + (4\beta - 3 - \gamma - \zeta/3 - \alpha_1 + 2\alpha_2/3)U_G/Mc^2 \tag{5.10}$$

for a spherically symmetric celestial body in (internal) equilibrium. For bodies lacking spherical symmetry (rotating bodies, etc) there are additional contributions to M_G/M_I including a non-isotropic term:

$$\delta(M_G/M_I)_{kl} = (\zeta + \alpha_2)/2(\sum m_i m_j (\boldsymbol{r}_i - \boldsymbol{r}_j)_k (\boldsymbol{r}_i - \boldsymbol{r}_j)_l / |\boldsymbol{r}_i - \boldsymbol{r}_j|^3 + 2U_G/3\delta_{kl})/Mc^2$$
$$- 3\zeta/4(\sum m_i m_j [(\boldsymbol{r}_i - \boldsymbol{r}_j)\cdot \hat{\boldsymbol{R}}]^2 / |\boldsymbol{r}_i - \boldsymbol{r}_j|^3 + 2U_G/3)\delta_{kl}. \tag{5.11}$$

$\hat{\boldsymbol{R}}$ is a unit vector toward the external body accelerating the celestial body. The ratio (5.10) results from separate expressions for M_G and M_I:

$$M_{GP} = E + (4\beta - 3 - \gamma - \zeta/3)U_G \tag{5.12}$$

and

$$M_I = E + (\alpha_1 - 2\alpha_2/3)U_G \tag{5.13}$$

where E is the mass-energy content of the body:

$$E = \sum_j m_j(1 + \tfrac{1}{2}v_j^2) + \tfrac{1}{2}\sum_{i,j} e_i e_j / |\boldsymbol{r}_i - \boldsymbol{r}_j| - \tfrac{1}{2}\sum_{i,j} m_i m_j / |\boldsymbol{r}_i - \boldsymbol{r}_j|. \tag{5.14}$$

In the conservative metric theories to which we restrict our consideration, the active gravitational mass M_{GA} is found to be equal to the passive gravitational mass given by (5.13). M_{GA} is operationally defined as

$$M_{GA} = \lim\,(g(r)r^2/G) \tag{5.15}$$

where $g(r)$ is the gravitational acceleration field of the body. $M_{GA} = M_{GP}$ guarantees equality of action and reaction in the gravitational force.

6. Calculation of M_G/M_I in metric theories of gravity

The equation of motion for each particle of mass m_j and charge e_j in the celestial body (see figure 3) is obtained from the variation of the action integral:

$$A = \int [-m_j(g_{\mu\nu}(r, t)\,\mathrm{d}x^\mu/\mathrm{d}t\,\mathrm{d}x^\nu/\mathrm{d}t)^{1/2} + e_j A_\mu\,\mathrm{d}x^\mu/\mathrm{d}t]\,\mathrm{d}t. \tag{6.1}$$

The electromagnetic potentials appearing in (6.1) satisfy the equations

$$F_{\mu\nu} = \partial/\partial x^\mu A_\nu - \partial/\partial x^\nu A_\mu \tag{6.2(a)}$$

and

$$F^{\mu\nu}{}_{;\nu} = 4\pi J^\mu \tag{6.2(b)}$$

the latter equation following from the electromagnetic free-field action integral:

$$A' = -\int \mathrm{d}^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}/16\pi.$$

Symbolically writing the gravitational metric field as

$$g_{\mu\nu} = \left[\begin{array}{c|c} 1 - 2U + h_{00}^{(2)} & \boldsymbol{h} \\ \hline \boldsymbol{h} & -\delta_{kl} + h_{kl}^{(1)} \end{array}\right] \tag{6.3}$$

(see (5.9) for the full expressions). The particle equation of motion becomes

$$\mathrm{d}/\mathrm{d}t(1+\tfrac{1}{2}v^2+U-h^{(1)})\boldsymbol{v}=\boldsymbol{\nabla}U+\partial\boldsymbol{h}/\partial t+\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{h})+\tfrac{1}{2}\boldsymbol{\nabla}[U^2-h^{(2)}_{00}]+\tfrac{1}{2}\boldsymbol{\nabla}[U-h^{(1)}]v^2$$
$$+(e/m)\boldsymbol{E}(r)(1-U). \tag{6.4}$$

We have kept only the necessary post-Newtonian terms. The electric field $\boldsymbol{E}(r)$ is obtained by a solution of (6.2) (Nordtvedt 1973):

$$\boldsymbol{E}(r)=e_j\sum_j(\boldsymbol{r}-\boldsymbol{r}_j)/|\boldsymbol{r}-\boldsymbol{r}_j|^3+\tfrac{1}{2}\sum_j e_j(\boldsymbol{g}-\boldsymbol{a}_j)/|\boldsymbol{r}-\boldsymbol{r}_j|$$
$$+\tfrac{1}{2}\sum_j e_j(\boldsymbol{r}-\boldsymbol{r}_j)(\boldsymbol{r}-\boldsymbol{r}_j)\cdot(\boldsymbol{g}-\boldsymbol{a}_j)/|\boldsymbol{r}-\boldsymbol{r}_j|^3 \tag{6.5}$$

where $\boldsymbol{g}$ is the gravitational acceleration field of the external body

$$\boldsymbol{g}=GM\boldsymbol{R}/R^3$$

and $\boldsymbol{a}_j$ is the acceleration of the jth particle. The electric-field expression (6.5) is valid in a spatial coordinate system in which spatial geometry distortions due to the external mass have been eliminated by the appropriate coordinate transformation (Nordtvedt 1973). In such coordinates a truly spherical body has a spherical coordinate representation.

When collecting the various terms of (6.4), those proportional to the acceleration of the celestial body are identified as contributing to its inertial mass, and those terms proportional to $\boldsymbol{g}$ are identified as the body's gravitational mass contributions. Let us look at the terms in (6.4) one by one.

(i) On the left-hand side of (6.4) the $\mathrm{d}/\mathrm{d}t$ operating on $\boldsymbol{v}$ yields $\boldsymbol{a}$; the coefficient of $\boldsymbol{a}$ includes inertial mass modifications due to kinetic motion, $\tfrac{1}{2}v^2$, and gravitational potential energy, $U-h^{(1)}$, due to the other matter in the celestial body. The $\mathrm{d}/\mathrm{d}t$ operating on $\tfrac{1}{2}v^2$ creates another inertial mass term. When $\mathrm{d}/\mathrm{d}t$ operates on $U-h^{(1)}$ the significant contribution comes from the external mass contribution to U. The $\mathrm{d}/\mathrm{d}t$ becomes $\boldsymbol{v}\cdot\boldsymbol{\nabla}$, producing a gravitational mass contribution proportional to $\boldsymbol{g}$.

(ii) Turning to the right-hand side of (6.4), U is simply the Newtonian gravitational acceleration which includes a $\boldsymbol{g}$ contribution.

(iii) Since $\boldsymbol{h}$ is proportional to $\boldsymbol{v}$ of the mass sources in the celestial body, $\partial\boldsymbol{h}/\partial t$ is proportional to the $\boldsymbol{a}_j$ (accelerations) of celestial-body mass elements and contributes to the body's inertial mass.

(iv) $\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{h})$ makes no contributions to M_{GP} or M_{I}.

(v) $U^2-h^{(2)}_{00}$ is a non-linear gravitational potential. Cross terms proportional to Mm_j (m_j being any mass elements inside the celestial body) contribute gravitational mass contributions proportional to $\boldsymbol{g}$.

(vi) The last gravitational term in (6.4) proportional to v^2 is a motional correction to Newtonian gravity and contributes to gravitational mass.

(vii) The electrical accelerations in (6.4) contribute to both gravitational and inertial mass of the celestial body (from the $\boldsymbol{g}$ and $\boldsymbol{a}_j$ terms, respectively, in (6.5)). There is a further gravitational mass contribution from the $-U\boldsymbol{E}(r)$ term.

The acceleration of each mass element of the celestial body (see figure 3) consists of an internal part relative to the body and the acceleration $\boldsymbol{a}$ of the body as a whole in the external field. Collecting all terms proportional to $\boldsymbol{a}$ and $\boldsymbol{g}$ and summing over the mass elements, appropriately weighted, gives

$$M_{\mathrm{GP}}\boldsymbol{g} = M_{\mathrm{I}}\boldsymbol{a} \tag{6.6}$$

with

$$\{M_{\mathrm{I}}\}_{kl} = [E + (\alpha_1 - 2\alpha_2/3)U_{\mathrm{G}}]\delta_{kl} - \alpha_2/2\Big(\sum_{i,j} m_i m_j (r_i - r_j)_k (r_i - r_j)_l / |\boldsymbol{r}_i - \boldsymbol{r}_j|^3 + \tfrac{2}{3}U_{\mathrm{G}}\delta_{kl}\Big) \tag{6.7(a)}$$

$$+\Big(\sum_i m_i (v_i)_k (v_i)_l + \tfrac{1}{2}\sum_{i,j}(e_i e_j - m_i m_j)(r_i - r_j)_k (r_i - r_j)_l / |\boldsymbol{r}_i - \boldsymbol{r}_j|^3\Big)$$

and

$$\{M_{\mathrm{GP}}\}_{kl} = [E + (4\beta - 3 - \gamma - \zeta/3)U_{\mathrm{G}}]\delta_{kl}$$

$$+\tfrac{1}{2}\zeta\Big(\sum_{i,j} m_i m_j (r_i - r_j)_k (r_i - r_j)_l / |\boldsymbol{r}_i - \boldsymbol{r}_j|^3$$

$$+\tfrac{3}{2}\sum_{i,j} m_i m_j [(r_i - r_j)\cdot\hat{\boldsymbol{R}}]^2 / |\boldsymbol{r}_i - \boldsymbol{r}_j|^3 \delta_{kl} - \tfrac{1}{3}U_{\mathrm{G}}\delta_{kl}\Big)$$

$$-\Big(\sum_i m_i (v_i)_k (v_i)_l + \tfrac{1}{2}\sum_{i,j}(e_i e_j - m_i m_j)(r_i - r_j)_k (r_i - r_j)_l / |\boldsymbol{r}_i - \boldsymbol{r}_j|^3\Big) \tag{6.7(b)}$$

with

$$E = \sum_i m_i(1 + \tfrac{1}{2}v_i^2) + \tfrac{1}{2}\sum_{i,j} e_i e_j / |\boldsymbol{r}_i - \boldsymbol{r}_j| - \tfrac{1}{2}\sum_{i,j} m_i m_j / |\boldsymbol{r}_i - \boldsymbol{r}_j|. \tag{6.8}$$

For bodies in equilibrium (or averaged over time scales of internal variations of structure) and free from external forces (celestial bodies in gravitational free-fall fulfil this condition neglecting gravitational tidal forces) the last bracket of terms in both (6.7(a)) and (6.7(b)) is seen to be virial tensor summations which vanish:

$$\sum_i m_i (v_i)_k (v_i)_l + \tfrac{1}{2}\sum_{i,j}(f_{ij})_k (r_i - r_j)_l = 0 \tag{6.9}$$

with

$$f_{ij} = e_i \sum_j e_j (\boldsymbol{r}_i - \boldsymbol{r}_j)/|\boldsymbol{r}_i - \boldsymbol{r}_j|^3 - m_i \sum_j m_j (\boldsymbol{r}_i - \boldsymbol{r}_j)/|\boldsymbol{r}_i - \boldsymbol{r}_j|^3.$$

With the virial condition (6.9) fulfilled, division of M_{GP} by M_{I} as given above yields the previously quoted ratios (5.11) and (5.12).

7. Dependence of Newton's gravitational constant on motion and proximity of matter

Another outcome of the previous calculation is a general expression for the effect of proximate matter on Newton's gravitational constant in metric theories. It is found that an external body M at distance $|\boldsymbol{R}|$ alters G by the relation:

$$(G)_{kl} = G_0[1 - (4\beta - 3 - \gamma - \tfrac{1}{2}\zeta)G_0 M/c^2 R]\delta_{kl} - \tfrac{1}{2}\zeta G_0^2 M R_k R_l / c^2 R^3. \tag{7.1}$$

G becomes anisotropic in metric theories with $\zeta \neq 0$ (Will 1971c). Averaging over all directions in a spherically symmetric body gives

$$\langle G(R)\rangle = G_0[1-(4\beta-3-\gamma-\zeta/3)G_0M/c^2R]. \tag{7.2}$$

On the other hand, (5.9(a)) indicates that preferred frame metric theories of gravity ($\alpha_1 \neq 0$ and/or $\alpha_2 \neq 0$) produce a velocity dependence and anisotropy of Newton's gravitational constant (Nordtvedt and Will 1972):

$$(G)_{kl} = G_0\{[1-\tfrac{1}{2}(\alpha_1-\alpha_2)w^2/c^2]\delta_{kl}-\tfrac{1}{2}\alpha_2 w_k w_l/c^2\} \tag{7.3}$$

where $\boldsymbol{w}$ is the velocity relative to the preferred inertial frame. Averaging over directions in a spherically symmetric body yields

$$\langle G(w)\rangle = G_0[1-\tfrac{1}{2}(\alpha_1-2\alpha_2/3)w^2/c^2]. \tag{7.4}$$

An anomalous gravitational or inertial mass for a celestial body can be understood in terms of the dependences of G on the proximity of matter (7.2) or on motion (7.4). Dicke (1969) first suggested this interpretation with regard to the position dependence of G (see also Haugan 1979).

Let E be the mass-energy of a celestial body in the absence of motion relative to the preferred inertial frame ($\boldsymbol{w} = 0$) and in the absence of proximate matter ($M = 0$). The effective Lagrangian for a slowly moving celestial body with a proximate mass M at distance $|\boldsymbol{R}|$ is then

$$L = E+\tfrac{1}{2}Ew^2/c^2+G_0ME/c^2R-[G(R, w)/G_0-1]U_G. \tag{7.5}$$

The celestial body's equation of motion is then

$$\mathrm{d}/\mathrm{d}t(\boldsymbol{\nabla}_w L)-\boldsymbol{\nabla}_R L = 0$$

with

$$\boldsymbol{\nabla}_w L = E\boldsymbol{w} - U_G(\boldsymbol{\nabla}_w G/G_0) \tag{7.6(a)}$$

and

$$\boldsymbol{\nabla}_R L = E\boldsymbol{g} - U_G(\boldsymbol{\nabla}_R G/G_0). \tag{7.6(b)}$$

Using (7.2) and (7.4) in the above reproduces the results for M_G and M_I given in (5.12) and (5.13).

8. Implications for gravitational theory constraints

In §6 the PPN metric expression for the experimentally measured parameter η was obtained:

$$\eta = (4\beta-3-\gamma-\zeta/3-\alpha_1+2\alpha_2/3) \tag{8.1}$$

with

$$M_G/M_I = 1+\eta U_G/Mc^2.$$

Lunar laser-ranging experimental results presently limit η:

$$|\eta| < 1.4\times 10^{-2}$$

which (8.1) translates into constraints on PPN coefficients.

Both α_2 and ζ coefficients imply an anisotropic gravitational constant as given by (7.1) and (7.3). Gravimeter experiments on Earth can put stringent limits on α_2 and ζ, since the Earth's 24 h rotation translates an anisotropic G into 12 h period gravitational force variations at a gravimeter station whose orientation relative to the spatial directions $\boldsymbol{w}$ and $\boldsymbol{R}$ rotates. The dominant proximate mass for this effect is the galaxy—$GM/c^2R \simeq 10^{-6}$—and also $w^2/c^2 \simeq 10^{-6}$ (Will 1971c, Nordtvedt and Will 1972, Warburton and Goodkind 1976). The present experimental limits on these coefficients are

$$|\alpha_2| \lesssim 10^{-3} \tag{8.2(a)}$$

and

$$|\zeta| \lesssim 10^{-3}. \tag{8.2(b)}$$

The PPN coefficient γ has been most precisely measured by the radio signal time delay experiments discussed in §2 (Reasenberg 1979):

$$|1-\gamma| \lesssim 2 \times 10^{-3}. \tag{8.3}$$

The most sensitive experimental limit on α_1 comes from the perihelion precession of Mercury; α_1 makes a contribution to this precession (Nordtvedt and Will 1972) of

$$\omega = 35\alpha_1 \text{ arc-seconds/century.}$$

The uncertainty in that observation ($\pm 0.4''$/century) therefore imposes the constraint

$$|\alpha_1| \lesssim 10^{-3} \tag{8.4}$$

unless there is a correlated variation of α_1 and β from their values as predicted in general relativity. $\gamma = \beta = 1$, $\alpha_1 = \alpha_2 = \zeta = 0$ in Einstein's general theory of relativity of gravity.

The factor of 4 multiplying β in (8.1) makes the lunar laser-ranging experiment primarily a 'β' experiment, with the resulting experimental constraint

$$|1-\beta| \lesssim 4 \times 10^{-3}. \tag{8.5}$$

A different way of viewing the significance of the M_G/M_I measurement for celestial bodies is by noting the post-Newtonian gravitational potentials of general relativity which participate in determining M_G/M_I, but which play no role in the other previous historic tests of general relativity (Nordtvedt 1977a). In Einstein's general relativity post-Newtonian field expansion there is the gravitational vector potential:

$$\boldsymbol{h} = \tfrac{7}{2}\sum_j m_j \boldsymbol{v}_j/|\boldsymbol{r}-\boldsymbol{r}_j| + \tfrac{1}{2}\sum_j m_j(\boldsymbol{r}-\boldsymbol{r}_j)\cdot \boldsymbol{v}_j(\boldsymbol{r}-\boldsymbol{r}_j)/|\boldsymbol{r}-\boldsymbol{r}_j|^3$$

and there is a second type of non-linear gravitational potential not present in the Eddington–Robertson metric:

$$g_{00}^{(2)} = 2\sum_{i,j} m_j m_i/|\boldsymbol{r}-\boldsymbol{r}_i||\boldsymbol{r}_i-\boldsymbol{r}_j|.$$

If the gravitational vector potential were absent from the metric field, the lunar orbit polarisation toward the Sun would be 70 m! If the second type of non-linear potential in g_{00} were absent, the lunar orbit polarisation would be 3 m! The experimental upper limit of 15 cm for this polarisation amplitude clearly indicates the presence of these general relativistic post-Newtonian potentials as predicted from calculation.

Anyone proposing an alternative theory of gravity must subject his theory to all the experimental tests of post-Newtonian gravity—including the result for M_G/M_I for celestial bodies. The full PPN metric for any theory must therefore be calculated and its values of γ, β, ζ, α_1 and α_2 compared with general relativity's values which are now confirmed by observations to a few parts in a thousand.

Application of general relativity to cosmology, astrophysics, gravitational radiation, or to probe the interior mass structure of our Sun by measuring its angular momentum and quadrupole moment (Nordtvedt 1977b), should now be possible with higher confidence, because of the additional post-Newtonian empirical tests that general relativity has successfully met during the last decade.

Gravitational theories can be classified by the long-range fields which produce the gravitational interaction among the massive bodies. 'Pure' metric gravity has only the metric field $g_{\mu\nu}$. Einstein's general theory of relativity is a specific 'pure' metric theory with specific field equations for $g_{\mu\nu}$.

Scalar-metric theories have a scalar field φ in addition to the metric field; vector-metric theories contain a vector field k_μ supplementing the metric field, while two-tensor theories have a second tensor field $k_{\mu\nu}$ in addition to the $g_{\mu\nu}$.

If, in each of the above cases, the augmenting fields (φ, k_μ, $k_{\mu\nu}$, . . . ,) do not couple to matter, but only to $g_{\mu\nu}$, then the gravitational theory is a metric theory and is subject to a PPN expansion.

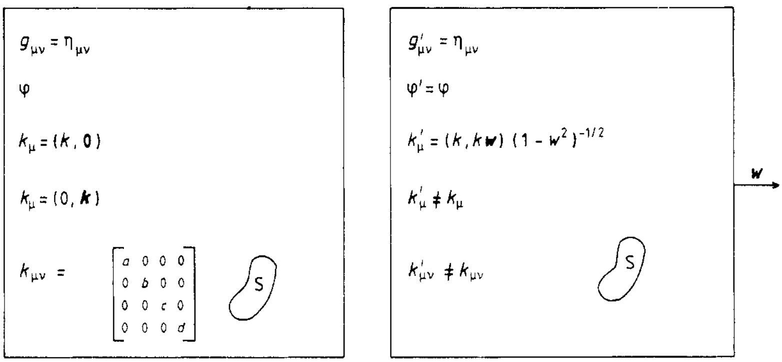

Figure 5. Two identical sources (S) of gravitational fields move relative to each other with velocity $\boldsymbol{w}$. The asymptotic background fields (far from source) are shown for the two rest frames and are related to each other by a Lorentz transformation. Pure metric gravity—theories in which only the field $g_{\mu\nu}$ plays a role in gravity—and scalar-metric theories are seen to lead to frame-independent gravitational physics because of the invariance of the asymptotic fields. Vector-metric theories would have a preferred frame for the gravitational physics, while two-tensor theories of gravity will generally have anisotropies in the gravitational fields as well as preferred-frame effects.

In a general way one can relate the PPN coefficients to the kind of fields present in the gravitational theory. Figure 5 illustrates two identical, localised gravitational sources (S) (e.g. the solar system), except that one system is in motion relative to the other. In the rest frame of each system there are the asymptotic (background, cosmological) values of the various gravitational fields ($g_{\mu\nu}$, φ, k_μ, $k_{\mu\nu}$, . . . ,) which are straightforwardly related in the two frames by Lorentz transformations.

In one preferred inertial frame it is possible by coordinate transformations (including Lorentz transformations) to make the asymptotic $g_{\mu\nu}$ equal to the Minkowski metric $\eta_{\mu\nu}$; and simultaneously to make a vector field either pure time components or pure space components; or to diagonalise a second tensor field into four diagonal elements which will generally not be proportional to the Minkowski metric $\eta_{\mu\nu}$.

The solution for $g_{\mu\nu}$, , the gravitational metric field, and any supplementary fields, is determined by the sources and the boundary conditions on the fields. Since the two sources are identical, Lorentz invariance of $\eta_{\mu\nu}$ and φ indicates that there can be no preferred frame PPN coefficients in 'pure' metric or in scalar-metric theories of gravity. φ could, however, depend on cosmological time, and thereby produce time variation of the gravitational 'constant'. Pure metric-field theories can not have a time variation of G since $\eta_{\mu\nu}$ has no time dependence (i.e. an ability to perform arbitrary coordinate transformations always allows the asymptotic $g_{\mu\nu}$ to be the Minkowski metric $\eta_{\mu\nu}$).

For vector-metric theories of gravity in which the cosmological background vector field is time-like, there is a preferred frame in which only the time component of k_μ is non-zero, but in other inertial frames k_μ signals a preferred direction in space resulting generally in preferred frame and anisotropic gravity effects (α_1 and/or $\alpha_2 \neq 0$).

For a space-like vector field in a vector-metric theory or in two-tensor theories it is seen from figure 5 that frame-dependent and anisotropic gravity will generally result.

The fourth test of general relativity—measurement of the gravitational to inertial mass ratio for celestial bodies—confirms predictions of general relativity's post-Newtonian structure for moving, non-linear and non-spherically symmetric sources; a domain of the field equations not reached in the other tests. Taken with the other tests, it is unlikely that any other fields augment the metric field $g_{\mu\nu}$ in producing gravity unless the supplementary fields play a very small role (one part in a thousand or less) in determining the metric field $g_{\mu\nu}$ to which matter responds.

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