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Initial-value problem of general relativity. II. Stability of solutions of the initial-value equations

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We discuss the completeness and "linearization stability" of the initial-value constraints. We show that all closed solutions for which the intrinsic geometry possesses a conformal symmetry are incomplete, but stable. All closed, vacuum, moment-of-time symmetry solutions are incomplete, but only the flat case is unstable. This particular incompleteness vanishes on the addition of any source field. All other closed solutions to the initial-value constraints for which the trace of the momentum is a covariant constant are complete and stable except those solutions where the metric and the momentum have the same exact symmetry. All such closed, vacuum solutions are unstable. All asymptotically flat maximal solutions are complete and stable. In this paper we treat only the linearization stability of the initial-value constraints and make no statements about the dynamical stability of the solutions.

I. INTRODUCTION

In the preceding paper[1] (hereafter referred to as I) we identified the independent initial data as the conformal metric $\bar{g}_{ij}$, the transverse, trace-free part of the momentum $\bar{\sigma}^{ij}$, and the contraction scalar τ. The dependent data, determined by the initial-value equations, are the conformal factor ϕ and the vector W^i, which generates the longitudinal part of the momentum μ^{ij}. Combining the independent and dependent quantities gives the complete initial data g_{ij} and π^{ij} on the initial spacelike manifold. These complete initial data then satisfy the initial-value constraints. We shall refer to g_{ij}'s and π^{ij}'s satisfying the constraints as "solutions." The evolution equations for $\dot{g}_{ij} = \mathcal{L}_t g_{ij}$ and $\dot{\pi}^{ij} = \mathcal{L}_t \pi^{ij}$ then determine the continuation of the data off the initial surface. This continuation of g_{ij} and π^{ij} depends on the choice of a timelike vector $\vec{t}$; however, the resulting coordinate-free spacetime structure does not depend on this choice.[2]

As we saw in I, with this definition of the independent data, the initial-value equations form a coupled elliptic system in the case of a general choice of $\tau(x)$. However, the coupling is broken by the special choice $\tau = \text{constant}$, where we find that $(LW)^{ij} \equiv 0$. The equations reduce to a single quasilinear elliptic equation for the conformal factor ϕ. We have shown elsewhere[3] that in this case a solution ϕ almost always exists, and that whenever it exists it is unique, except vacuum moment-of-time symmetry solutions (see I).

In view of the complexity of the constraints we expect that most solutions will be obtained by perturbing an exact solution. However, since the constraints are nonlinear we have no *a priori* guarantee that solutions to the first-order perturbation equations do in fact approximate solutions to the exact equations. It is important to identify those cases in which this guarantee can and cannot be given.

Technically, the property we are investigating is known as "stability" or "linearization stability"[4] of the initial data. Stability is defined by the property that all solutions to the first-order perturbation equations are tangent to the space of exact nearby solutions. The tangent to any continuous curve of exact solutions obeys the first-order perturbation equations. A solution is stable if we can invert this statement. In other words, a solution is said to be stable if for *every* solution of the first-order perturbation equations there exists some curve of exact solutions to which it is tangent.

Any curve of exact solutions generates solutions, not only to the first-order perturbation equations, but to the perturbation equations of all orders. Therefore, unstable points are places where a solution to the first-order perturbation equations exists but this first-order solution does not lead to any solution of the higher-order perturbation equations. This effect can only arise when the exact equations are nonlinear. It is frequently caused by the existence of some constraint which must be satisfied to permit an exact solution. An unstable point would be one where the first-order constraint can be satisfied, quite frequently trivially satisfied, but one or more of the higher-order constraints leads to a contradiction.

The solutions to the initial-value constraints form a subset C of the space of all metrics g_{ij}

and all symmetric tensor densities $\pi^{ij}(\{g_{ij}\}\times\{\pi^{ij}\})$. The requirement that a particular solution $p\in C$ be stable is entirely equivalent to the requirement that C be a smooth submanifold of $(\{g_{ij}\}\times\{\pi^{ij}\})$ in the neighborhood of p.[4] If C is a smooth submanifold at p, then the tangent space to C at p $(T_p C)$ "models" the space of exact nearby solutions, and every vector in $T_p C$ is tangent to a curve in C. Therefore, finding the unstable points of C is the same as finding those points where it ceases to be a smooth submanifold. Hence, the investigation of completeness and stability helps determine the structure of the gravitational phase space.

Let us define the space E as the space of all conformal metrics $\tilde{g}_{ij}$, the space of all symmetric, transverse, trace-free tensors $\tilde{\sigma}^{ij}$ on these metrics and all scalar functions τ on the manifold, i.e.,

$$E=\{\{\tilde{g}_{ij}\}\times\{\tilde{\sigma}^{ij}\}\times\{\tau\}\}\,.$$

Following our identification of the independent initial data as being points in E, there exists a well-defined, continuous map

$$\mathcal{G}:\ C\rightarrow E\,,$$

as described in I. Then independent-data space D is defined as being the range of $\mathcal{G}$, and is a subset of E.

In our previous papers, and in the present one, we are concerned with the mapping $\mathcal{K}=\mathcal{G}^{-1}$: $D\rightarrow C$. A point in E is a point in D if and only if the four elliptic equations have a solution (ϕ, W^i). $\mathcal{K}$ is a mapping if and only if this solution is unique. We have already identified a subset of points in D, i.e., almost all points with $\tau=$ constant. For this subset $\mathcal{K}$ is a mapping and generates all the solutions to the initial-value constraints where the trace of the momentum is a covariant constant. In this paper we will investigate the completeness and stability of these special points of C.

The existence of the mapping $\mathcal{K}$ simplifies the analysis considerably. We will show that in the neighborhood of these special points $\mathcal{K}$ is a diffeomorphism. Also, $d\mathcal{K}=\mathcal{K}_*$ is an injection and so locally C may be regarded as an embedding of D in $(\{g_{ij}\}\times\{\pi^{ij}\})$. Hence it is locally a smooth submanifold. Therefore, the only solutions with $\tau=$ constant which may not be stable are those which correspond to a "boundary point" of D.

Equivalently, we will show that, in general, each of these $\tau=$ constant points of D belongs to a complete neighborhood of E over which $\mathcal{K}$ is well defined. In turn, each of the $\tau=$ constant solutions in C have a neighborhood which is the smooth image of the neighborhood in D (also E) and so they are stable. Each tangent vector to C is equivalent to a perturbation in E, but each perturbation in E generates a curve of exact solutions in C, because $\mathcal{K}$ is well defined in a complete neighborhood. We will show that the original tangent vector is tangent to the curve of exact solutions we generate. This analysis is carried out in Sec. IV. Thus, completeness of neighborhoods implies stability. However, incompleteness does not necessarily lead to instability, as will be demonstrated below.

This analysis breaks down at the $\tau=$ constant points which do not have a complete neighborhood over which $\mathcal{K}$ is well defined. This happens whenever the $\tau=$ constant points are boundary points of D. In this case there exists some perturbation in E for which the elliptic equations cannot be solved. In effect, in examining completeness we are looking for "forbidden directions" in the space E. Moreover, if we think of a Hamiltonian formalism based on the choice of $\tilde{g}_{ij}$ and $\tilde{\sigma}^{ij}$ as dynamical variables, i.e., as "coordinates" and "momenta" and τ as determining the "time," then the present work may be viewed as an attempt to map out the "extended" phase space of the gravitational field $\{\text{coordinates}\}\times\{\text{momenta}\}\times\{\text{time}\}$.

The form of stability that is being investigated here concerns only the question of constructing initial data. This is not the same as the question whether the dynamical equations are stable, for example, whether a singularity-free initial geometry will remain singularity-free at later times. The latter problem is outside the scope of the present paper.

The ellipticity of the equations does not guarantee either completeness or stability on closed manifolds (compact, without boundary). For closed manifolds we will show that there are solutions with incomplete neighborhoods and that these fall into two classes: (1) cases in which the intrinsic geometry admits a conformal Killing vector and (2) *vacuum* solutions corresponding to a "moment-of-time symmetry" ($\tilde{\sigma}^{ij}=0$, $\tau=0$). All these special solutions are incomplete, all other $\tau=$ constant solutions are complete and hence stable. These results are demonstrated in Sec. II. The solutions with conformal Killing vectors give rise to incompleteness because in this case there is an extra global integrability condition that must be satisfied, as pointed out in I.

Since $\mathcal{K}$ is well defined on the $\tau=$ constant subspaces of E, the $\tau=$ constant subspaces of C are smooth submanifolds of $(\{g_{ij}\}\times\{\pi^{ij}\})$. Any first-order perturbation which stays on this submanifold ($\tau=$ constant) obviously is well behaved. This limited form of completeness breaks down only at the moment-of-time symmetry solutions. The existence of conformal Killing vectors has no effect in this case. This particular result follows from the present analysis and also from a previous paper,[3] where we considered the *exact* equations for arbitrary $\tilde{g}_{ij}$, $\tilde{\sigma}^{ij}$, τ = constant. In this paper, however, we do *not* limit ourselves to $\delta\tau$ = constant but consider *all* perturbations around a τ = constant solution. In this case the existence of a conformal Killing vector does affect the completeness.

Not all points on the $\tau=0$ subspace of E belong to D. Therefore, there may be some boundary points and hence incomplete initial-data sets on the subspace. We get this nonexistence result because a globally uniform sign of the scalar curvature is a conformal invariant on closed manifolds. Almost all closed Riemannian manifolds can be conformally mapped into a manifold with constant scalar curvature (Yamabe's "theorem").[5] Since the sign of this scalar curvature is a conformal invariant, it is not unreasonable to expect that the incomplete solutions are those for which the global sign of the scalar curvature vanishes, i.e., vacuum moment-of-time symmetry solutions.

As we described above, points corresponding either to a vacuum moment-of-time symmetry solution or to solutions possessing an exact conformal Killing vector have the incompleteness-of-neighborhoods property. This means that these neighborhoods have "fewer dimensions" than the generic ones, but not that there are necessarily instabilities associated with these points. At these points, the tangent directions (first-order perturbations) are restricted (incompleteness), but in almost every instance one can show that there do exist curves of exact solutions to the constraints having the given first-order perturbation as tangent. The unstable cases we find here are (1) the closed *flat* vacuum moment-of-time symmetry solutions, an instability found by Brill and Deser,[6] and (2) all vacuum solutions which have both an exact Killing vector and for which the momentum has the same symmetry as the metric. Fischer and Marsden[4] have proved previously, using different techniques, that one obtains stability upon excluding (1) and (2). They also stated[4] that cases (1) and (2) may be unstable.

On open manifolds, the problem is quite different. In this case, for convenience and for the sake of physical interpretation, we use the physically natural boundary condition that the data set is asymptotically flat. We do this so that the spacetime is asymptotically flat and then we can use Lorentzian observers at infinity. This automatically excludes the incompleteness associated with conformal Killing vectors because they do not vanish at infinity. As distinct from the closed manifold case, we have no restriction on the global sign of the scalar curvature and all points on the $\tau=0$ subspace generate solutions. Therefore we have no difficulty with the moment-of-time symmetry solutions either and can show that all τ = constant solutions on open manifolds are complete and hence stable. Of course, we need only consider maximal solutions, i.e., $\tau=0$, because we wish τ to vanish at infinity.

Finally, in Sec. V we discuss completeness and incompleteness of nonvacuum and nonmaximal solutions. The only major change that occurs is that the addition of any source field transforms the closed, moment-of-time symmetry solutions from being incomplete to being complete. However, the addition of sources has no effect on the incompleteness due to the conformal symmetries of the manifold.

This paper, in proving the existence of complete neighborhoods, uses the implicit function theorem on Banach spaces, following Choquet-Bruhat and Deser, who proved the completeness and stability of open flat initial data.[7] The "smooth submanifold" description of such results is due to Fischer and Marsden.[4] However, we apply it to the exact formulation of the initial-value equations given in I. By this means we are able to give a unified treatment of the completeness and stability problem on open and closed manifolds, with no restrictions on the strength of the unperturbed gravitational field.

II. EXISTENCE OF SOLUTIONS ON CLOSED MANIFOLDS

The conformal Killing form of a vector W^i is defined as

$$(LW)^{ij} = \nabla^i W^j + \nabla^j W^i - \tfrac{2}{3} g^{ij} \nabla_k W^k . \tag{1}$$

A conformal Killing vector T^i is one for which the conformal Killing form vanishes identically, i.e.,

$$(LT)^{ij} = 0 . \tag{2}$$

Given a point in independent-data space ($\tilde{g}_{ij}$, $\tilde{\sigma}^{ij}$, τ) we construct a solution ($\bar{g}_{ij}, \bar{\pi}^{ij}$) to the initial-value constraints as

$$\bar{g}_{ij} = \phi^4 g_{ij} , \tag{3a}$$

$$\pi^{ij} = \phi^{-4}\sigma^{ij} + \phi^2 \sqrt{g}\,[(LW)^{ij} + \tfrac{1}{2} g^{ij}\tau] , \tag{3b}$$

where (ϕ, W^i) form a solution to

$$8\nabla^2\phi + \frac{1}{g}\sigma^{ij}\sigma_{ij}\phi^{-7} + 2\frac{1}{\sqrt{g}}\sigma^{ij}(LW)_{ij}\phi^{-1} - R\phi - [\tfrac{3}{8}\tau^2 - (LW)^{ij}(LW)_{ij}]\phi^5 = 0 , \tag{4a}$$

$$\nabla_b[\phi^6(LW)^{ab}] + \tfrac{1}{2}\phi^6\nabla^a\tau = 0 . \tag{4b}$$

We also require $\phi > 0$.

We define independent-data space D as being the subset of $\{\tilde{g}_{ij}\}\times\{\tilde{\sigma}^{ij}\}\times\{\tau\}$ for which Eqs. (4) possess a solution. In seeking solutions to Eqs. (4) on a closed manifold we have to place suitable continuity conditions on the initial data. Reasonable ones are

$$g_{ij}, \sigma^{ij}, \tau \in C^{2,\alpha}, \tag{5}$$

where $C^{2,\alpha}$ stands for Hölder continuity.[8] This will guarantee

$$\phi, (LW)^{ij} \in C^{2,\alpha}, \tag{6}$$

and hence

$$\bar{g}_{ij}, \bar{\pi}^{ij} \in C^{2,\alpha}. \tag{7}$$

To show that a given point in independent-data space is an interior point we will have to prove that Eqs. (4) have a solution in a neighborhood of a given solution. To do this we will make use of the implicit function theorem on Banach spaces.

The implicit function theorem on Banach spaces is exactly analogous to the standard implicit function theorem of elementary analysis. In the standard theorem let us have n independent variables $(x_1, \ldots, x_n)$, m dependent variables $(y_1, \ldots, y_m)$, and m implicit functions relating them:

$$f_l(x_1 \ldots x_n; y_1 \ldots y_m) = 0, \tag{8}$$

where l runs from 1 to m. The implicit function theorem says that these functions can be solved for y in terms of x in a neighborhood of x,

$$y_l = u_l(x_1, \ldots, x_n) \quad (1 \leq l \leq m), \tag{9}$$

so long as the Jacobian determinant of f with respect to y does not vanish. Now

$$\frac{\partial f}{\partial y} dy = \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \dfrac{\partial f_k}{\partial y_l} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} dy_1 \\ \vdots \\ dy_l \end{pmatrix}. \tag{10}$$

The nonvanishing of the Jacobian implies that the matrix $(\partial f_k / \partial y_l)$ can be inverted, and so $(\partial f/\partial y)\, dy$ is an isomorphism from the set of m variables dy to itself.

The implicit function theorem on Banach spaces is as follows. Let us have three Banach spaces E, F, G and a continuous mapping $\mathfrak{F}$: $E \times F \to G$. Given $u_0 \in E$, $v_0 \in F$ such that $\mathfrak{F}(u_0, v_0) = 0$, and if also $\mathfrak{F}'_v dv$ at (u_0, v_0) is an isomorphism from F to G, then there exist neighborhoods δu of u_0 and δv of v_0, such that for every $u \in \delta u$ there exists $v \in \delta v$ such that $\mathfrak{F}(u, v) = 0$.[9]

To use this theorem on the stability problem we define the Banach space E to be the triplet $(\{g_{ij}\}, \{\sigma^{ij}\}, \tau)$ and the Banach space F to be $(\{\phi\}, \{W^i\})$. They will have to obey the continuity requirements of Eqs. (5) and (6) and the norms are the natural ones, i.e.,

$$\| \cdot \|_E = \| g_{ij} \|_{2,\alpha} + \| \sigma^{ij} \|_{2,\alpha} + \| \tau \|_{2,\alpha}. \tag{11}$$

The Banach space G will contain scalars and vectors which belong to $C^{0,\alpha}$. The mapping $\mathfrak{F}$ will be the one defined by Eqs. (4). One can see by inspection that it is a continuous mapping from $E \times F$ to G.

Now $\mathfrak{F}'_v dv$ is of the form

$$8\nabla^2 \delta\phi - 7\frac{1}{g}\sigma^{ij}\sigma_{ij}\phi^{-8}\delta\phi - 2\frac{\sigma^{ij}}{\sqrt{g}}(LW)_{ij}\phi^{-2}\delta\phi + \frac{2}{\sqrt{g}}\sigma^{ij}(L\delta W)_{ij}\phi^{-1} - R\delta\phi$$
$$- 5[\tfrac{3}{8}\tau^2 - (LW)^{ij}(LW)_{ij}]\phi^4\delta\phi + 2\phi^5(LW)^{ij}(L\delta W)_{ij}, \tag{12a}$$

$$6\nabla_b(L\delta W)^{ab} + 6\frac{\delta\phi}{\phi}\nabla_b(LW)^{ab} + \frac{6}{\phi}(LW)^{ab}\nabla_b\delta\phi + 30\frac{\delta\phi}{\phi^2}(LW)^{ab}\nabla_b\phi + 3g^{ab}\frac{\delta\phi}{\phi}\nabla_b\tau. \tag{12b}$$

We wish to consider $\mathfrak{F}'_v dv$, when evaluated at some solution (u_0, v_0) with $\tau = \text{constant}$, i.e.,

$$\phi_0 = 1, \quad (LW)_0^{ij} = 0, \quad R = \frac{1}{g}\sigma^{ij}\sigma_{ij} - \tfrac{3}{8}\tau^2; \tag{13}$$

then $\mathfrak{F}'_v dv$ becomes

$$\left(8\nabla^2\delta\phi - 7\frac{1}{g}\sigma^{ij}\sigma_{ij}\delta\phi + \frac{2\sigma^{ij}}{\sqrt{g}}(L\delta W)_{ij} - R\delta\phi \right.$$
$$\left. - \tfrac{15}{8}\tau^2\delta\phi, \nabla_b(L\delta W)^{ab}\right). \tag{14}$$

This can be rewritten as

$$\{ 8\nabla^2\delta\phi - (8g^{-1}\sigma^{ij}\sigma_{ij} + \tfrac{3}{2}\tau^2)\delta\phi$$
$$+ 2g^{-1/2}\sigma^{ij}(L\delta W)_{ij};\ \nabla_b(L\delta W)^{ab}\}. \tag{15}$$

Obviously $8g^{-1}\sigma^{ij}\sigma_{ij} + \frac{3}{2}\tau^2 \geq 0$.

So long as the manifold does *not* possess a conformal Killing vector T^i, i.e., $(LT)^{ij} = 0$, $\nabla_b(L\delta W)^{ab}$ is an isomorphism from vectors to vectors.[10] This is because $\nabla \cdot L$ is a Hermitian operator, whose only harmonic functions are conformal Killing vectors.[10] Therefore, so long as the source is orthogonal to its harmonic functions it can be inverted. Hence it is an isomorphism except when the manifold possesses a conformal symmetry. Similarly,

$$8\nabla^2\delta\phi - (8g^{-1}\sigma^{ij}\sigma_{ij} + \tfrac{3}{2}\tau^2)\delta\phi = S \tag{16}$$

can always be solved for a unique $\delta\phi$ *so long as* $(1/g)\sigma^{ij}\sigma_{ij} + \frac{3}{8}\tau^2$ does not identically vanish. This is equivalent to demanding that the original solution not have vanishing momentum π^{ij}, i.e., not be a moment-of-time symmetry solution. $\mathfrak{F}_v' dv$ is an isomorphism from F to G except (1) at a moment-of-time symmetry or (2) when the intrinsic geometry has a conformal Killing vector. Excluding those two cases, the implicit function theorem shows that every solution with $\tau=\text{constant}$ has a complete neighborhood of nearby solutions.

The reason for the incompleteness of the neighborhood of time-symmetric solutions arises from the fact that Eq. (4a) has an integrability condition on closed manifolds due to the fact that

$$\int_V \sqrt{g}\, \nabla^2 \phi d^3x = 0\,. \tag{17}$$

Therefore

$$\int_V \sqrt{g}\{\, g^{-1}\sigma^{ij}\sigma_{ij}\phi^{-7} + 2g^{-1/2}\sigma^{ij}(LW)_{ij}\phi^{-1} - R\phi - [\tfrac{3}{8}\tau^2 - (LW)^{ij}(LW)_{ij}]\phi^5\}\, d^3x = 0\,. \tag{18}$$

This shows that the choice of independent data cannot be made completely arbitrarily in this case. The fact that the general solution with $\tau=\text{constant}$ has a complete neighborhood of nearby solutions means that at a $\tau=\text{constant}$ solution this constraint can be satisfied to all orders in perturbation theory. However, at a moment-of-time symmetry solution this constraint places a restriction on perturbations of the independent data. Consider the following perturbation:

$$(\delta^{(1)}g_{ij}^{\mathrm{TT}} = \delta^{(2)}g_{ij}^{\mathrm{TT}} = 0\,;\ \delta^{(1)}\sigma_{\mathrm{TT}}^{ij} \neq 0\,,\ \delta^{(2)}\sigma_{\mathrm{TT}}^{ij} = 0\,;\ \delta^{(1)}\tau = \delta^{(2)}\tau = 0)\,. \tag{19}$$

The first-order perturbation equations have as solution

$$(\delta^{(1)}\phi = 0\,,\ \delta^{(1)}W = 0)\,, \tag{20}$$

and condition (18) has no first-order part. This condition does have a second-order part however. Because of (19) and (20), this condition is

$$\int_V \sqrt{g}\, g^{-1}\delta^{(1)}\sigma_{\mathrm{TT}}^{ij}\delta^{(1)}\sigma_{ij}^{\mathrm{TT}}\, d^3x = 0\,, \tag{21}$$

which cannot hold for $\delta^{(1)}\sigma_{\mathrm{TT}}^{ij} \neq 0$. Therefore, all closed, vacuum moment-of-time symmetry solutions are incomplete.

On the other hand, in analyzing the stability of this system, one does not specify the second-order perturbations in advance as we did in (19). Instead one tries to pick them so as to satisfy the second-order equation. In the case when $\delta^{(2)}g_{ij}^{\mathrm{TT}} \neq 0$, the second-order equation has an additional term $\delta^{(2)}g_{ij}^{\mathrm{TT}}R^{ij}$. So long as $R^{ij} \neq 0$, we can pick $\delta^{(2)}g_{ij}^{\mathrm{TT}}$ so as to satisfy the second-order constraint. Therefore, *except for flat space*, this incompleteness does not lead to instability.[4,6] Because $R^{ij}=0$ for flat space, the term $\delta^{(2)}g_{ij}^{\mathrm{TT}}R^{ij}$ cannot contribute and hence flat space is unstable.

The reason for the incompleteness of solutions possessing a conformal Killing vector is that the momentum constraint has a global integrability condition. This result is shown in paper I. However, this incompleteness implies instability only in a certain special case in which there is a "double" symmetry defined by a *Killing* vector T^i such that[14]

$$\mathfrak{L}_T g_{ij} = 0\,,\quad \mathfrak{L}_T \pi^{ij} = 0\,. \tag{22}$$

Nontrivial conformal Killing vectors ($\nabla_i T^i \neq 0$) do not cause instability problems. These results may be seen in a straightforward manner by examining the integrability conditions of the perturbed momentum constraints. In first order, one finds that

$$0 = \int \sqrt{g}\,[\,(\tfrac{1}{2}\delta\tau + 8\tau\delta\phi)\nabla_i T^i + 3\delta\phi\,\tau_{;i}T^i + \tfrac{1}{2}\delta g_{ij}^{\mathrm{TT}}\mathfrak{L}_T P^{ij}]\, d^3x\,, \tag{23}$$

where

$$\pi^{ij} = \sqrt{g}\, P^{ij}\,.$$

If $\nabla_i T^i \neq 0$, this condition may be readily satisfied by choice of the variations $\delta\tau$ and $\delta g_{ij}^{\mathrm{TT}}$, and similarly in higher orders. When $\tau=\text{constant}$, $\nabla_i T^i = 0$ (Killing vector), (23) becomes

$$0 = \tfrac{1}{2}\int \sqrt{g}\, \delta g_{ij}^{\mathrm{TT}}\mathfrak{L}_T P^{ij} d^3x\,. \tag{24}$$

If $\mathfrak{L}_T P^{ij} \neq 0$, (24) implies a restriction on $\delta g_{ij}^{\mathrm{TT}}$, i.e., incompleteness. However, $\mathfrak{L}_T P^{ij} = 0$, as in (22), implies that there is no first-order integrability condition. In general, the second-order integrability condition has the form

$$0 = \int \sqrt{g}\,[\,(\tfrac{1}{2}\delta^{(2)}\tau + 8\tau\delta^{(2)}\phi)\nabla_i T^i + 3\delta^{(2)}\phi\,\tau_{;i}T^i + \tfrac{1}{2}\delta^{(2)}g_{ij}^{\mathrm{TT}}\mathfrak{L}_T P^{ij} + f^i T_i]\, d^3x\,, \tag{25}$$

where

$$f^i = f^i[\text{second-order products of first-order variations}]\,. \tag{26}$$

Again the terms similar to those in (23) vanish if T^i satisfies (22). In this case (25) reduces to

$$\int f^i T_i \sqrt{g}\, d^3x = 0\,, \tag{27}$$

which will not be satisfied in general. Note that we make no use of the condition $\tau =$ constant in demonstrating this instability.

It is interesting that the incompleteness of the neighborhood of solutions with conformal Killing vectors can be removed by reducing the size of the manifold E. We do this by restricting our choice of $\delta\tau$ by demanding that in the presence of a conformal Killing vector the gradient of τ be locally orthogonal to the symmetry vector, i.e.,

$$(\nabla_a \tau) T^a = 0\,. \tag{28}$$

In the absence of symmetry we have no restriction on τ, but in the highly symmetric case where we have three linearly independent conformal Killing vectors, for example, in a homogeneous cosmological space, Eq. (28) limits us to $\tau =$ constant.

We can prove that any solution with constant τ to Eqs. (4) has a complete neighborhood of nearby solutions in the Banach space $\overline{E}$, which is the same as the original E, but with the extra requirement that τ must obey Eq. (28). To show this we set up our Banach spaces as before, but with the following changes: $\overline{E}$ is defined as above; $\overline{F}$ is the same as F, but instead of vectors W^i we have equivalence classes of vectors $\tilde{W}^i = W^i/\mathrm{Ker}(L)$, i.e., identify vectors that differ by constant multiples of T^i. $\overline{\mathfrak{F}}$: $\overline{E}\times\overline{F}\to\overline{G}$ is again defined by Eq. (4), but the range of $\overline{\mathfrak{F}}$, i.e., $\overline{G}$, is slightly different from G. The vector part of $\overline{G}$ is defined by

$$V^a = \nabla_b[\phi^6 (LW)^{ab}] + \tfrac{1}{2}\phi^6 \nabla^a \tau\,. \tag{29}$$

Then

$$\int \sqrt{g}\, T_a V^a d^3x = \int \sqrt{g}\, T_a \nabla_b \{\phi^6 (LW)^{ab}\} + \tfrac{1}{2}\int \sqrt{g}\, \phi^6 \tau_{,a} T^a d^3x\,. \tag{30}$$

The first term on the right-hand side can be shown to be zero by partial integration; the second term vanishes since τ belongs to $\overline{E}$. Therefore, the vector part of $\overline{G}$ is globally orthogonal to the conformal Killing vector. Now $\overline{\mathfrak{F}}'_v dv$ is just as in Eq. (15), but now $\nabla_b (L\delta W)^{ab}$ *is* an isomorphism between $\tilde{W}^i \in \overline{F}$ and the vector part of $\overline{G}$. Hence $\overline{\mathfrak{F}}'_v dv$ is an isomorphism (except at a moment-of-time symmetry, of course) between the new domain and range. Therefore we can use the implicit function theorem to show that a complete neighborhood of solutions exists (in $\overline{E}$) to any $\tau =$ constant solution.

III. EXISTENCE OF SOLUTIONS ON OPEN MANIFOLDS NEAR MAXIMAL SOLUTIONS

The problem that is posed in this section and its resolution are almost identical to that of Sec. II. Again we seek solutions to Eq. (4) but now we wish to consider the problem when posed on an open topologically Euclidean manifold. In this case we will have to introduce boundary conditions at infinity. For convenience and so that we can discuss the gravitational field in terms of Lorentzian observers at infinity we will only consider those gravitational fields which are asymptotically flat. We will also limit ourselves to considering only those spaces which have a well-defined, finite mass at infinity. This requirement and its consequences are discussed at length elsewhere.[11] Here we will satisfy ourselves with writing down a suitable set of asymptotic conditions on the initial data. We seek solutions to Eq. (4) that satisfy

$$\phi \to 1\,,\quad W^i \to 0 \tag{31}$$

at infinity. The finiteness of energy also requires

$$\phi - 1 \sim O\left(\frac{1}{r}\right),\quad \nabla\phi \sim O\left(\frac{1}{r^2}\right),\quad \nabla^2\phi \sim O\left(\frac{1}{r^3}\right) \tag{32}$$

at infinity. We require $g_{ij}\to f_{ij}$ at infinity, where f_{ij} is a flat metric. It has been shown elsewhere[10] that any symmetric tensor may be decomposed into a transverse, trace-free part, a longitudinal part of the form $(LV)^{ij}$, and a trace. We perform this decomposition on $h_{ij} = g_{ij} - f_{ij}$, using f_{ij} as a base metric:

$$h_{ij} = h_{ij}^{\mathrm{TT}} + (LV)_{ij} + \tfrac{1}{3} h f_{ij}\,. \tag{33}$$

Then we require

$$\begin{aligned} &\nabla_k h_{ij}^{\mathrm{TT}} \sim O(r^{-(3/2+\epsilon)})\,;\\ &h_{\mathrm{T}} = h - 2\nabla_k V^k \sim O\left(\frac{1}{r}\right);\\ &\nabla h_{\mathrm{T}} \sim O\left(\frac{1}{r^2}\right),\quad \nabla^2(h_{\mathrm{T}}) \sim O\left(\frac{1}{r^3}\right) \end{aligned} \tag{34}$$

at infinity. Note that we need not require any particular asymptotic behavior on the part of h_{ij} given by $\nabla_i V_j + \nabla_j V_i$. This, of course, is purely a coordinate transformation and cannot have any effect on the physics of the situation. Finally we require

$$\sigma^{ij} \sim O\left(\frac{1}{r^{3/2+\epsilon}}\right),\quad \tau \sim O\left(\frac{1}{r^{2+\epsilon}}\right) \tag{35}$$

at infinity. Asymptotic requirements (34) and (35) are sufficient to guarantee (31) and (32), and if Eq. (4) has a solution, $(\overline{g}_{ij}, \overline{\pi}^{ij})$ is asymptotically flat, with finite energy and momentum.

To show the existence of a complete neighborhood of solutions we will again use the implicit function theorem on the same three Banach spaces E, F, G. The functions that belong to E and F must now satisfy the asymptotic requirements listed above, and we alter the norm to reflect this. For example

$$\|\cdot\|_E = \|g_{ij}\|_{2,\alpha} + \max|\nabla_k h^{TT}_{ij} r^{3/2}| + \max|h_T r| + \cdots . \quad (36)$$

Again the mapping $\mathfrak{F}$: $E\times F \to G$ will be defined by Eq. (4) and $\mathfrak{F}'_v dv$ as in Eq. (15). In this case, however, $\mathfrak{F}'_v dv$ will always be an isomorphism $F\to G$, because (1) $W^i \to 0$ at infinity excludes all conformal Killing vectors, and (2) $\nabla^2\delta\phi$ ($\delta\phi = 0$ at infinity) can always be inverted. Therefore every maximal solution on an open manifold has a complete neighborhood of nearby solutions in independent-data space.

IV. STABILITY OF SOLUTIONS WITH τ=CONSTANT

The tangent to every continuous curve of solutions to the initial-value constraints obeys the varied initial-value equations. The inverse question is whether for every solution to the varied initial-value equations there exists a curve of real solutions to which it is tangent. This is the criterion for stability. We have shown in the previous sections that almost all the solutions we have considered, i.e., those without a conformal Killing vector, have a complete neighborhood of nearby solutions. In these cases one would find it difficult to imagine that these points are not stable. This section will contain a constructive proof of their stability.

The varied constraints take the form

$$-\frac{\delta g}{g}(\pi^{ij}\pi_{ij} - \tfrac{1}{2}\pi^2) + \frac{2}{g}\pi^{ij}\pi_j{}^k\,\delta g_{ik} + \frac{2}{g}\pi^{ij}\,\delta\pi_{ij} - \frac{\pi}{g}\pi^{ij}\,\delta g_{ij} - \frac{\pi}{g}\,\delta\pi - \delta_g R = 0 , \quad (37)$$

$$\nabla_i \delta\pi^{ij} + \delta_g(\Gamma^j_{ab})\pi^{ab} = 0 . \quad (38)$$

When τ = constant, i.e., $\pi^{ij} = \sigma^{ij} + \frac{1}{2}\sqrt{g}\,g^{ij}\tau$, where σ^{ij} is TT, Eq. (37) simplifies to

$$-\frac{1}{g}\sigma^{ij}\sigma_{ij}\delta g + \frac{\tau^2}{8}\,\delta g + \frac{2}{g}\sigma^{ij}\sigma_j{}^k\delta g_{ik} + \frac{1}{2\sqrt{g}}\,\tau\sigma^{ij}\delta g_{ij} + \frac{2}{g}\,\sigma^{ij}\delta\pi_{ij} - \frac{\tau}{2\sqrt{g}}\,\delta\pi - \delta_g R = 0 . \quad (39)$$

Given any solution (δg_{ij}, $\delta\pi^{ij}$) to Eqs. (38), (39), the technique is to write down a curve in independent-data space, use the existence of solutions to show that this curve generates a curve in solution space, and show that the tangent to this curve is the original (δg_{ij}, $\delta\pi^{ij}$) we started from.

First of all, decompose $\delta\pi^{ij}$ with respect to $g^{(0)}_{ij}$:

$$\delta\pi^{ij} = \delta\pi^{ij}_{TT} + (L\delta\vec{\pi})^{ij} + \tfrac{1}{3}\delta\pi g^{(0)}_{ij} . \quad (40)$$

Consider the following curve in the space of independent variables:

$$g_{ij}(t) = g^{(0)}_{ij} + \delta g_{ij} t , \quad (41)$$

$$\tau(t) = \tau_0 + \frac{2}{3\sqrt{g}}(\delta\pi + \sigma^{ij}\delta g_{ij} - \tfrac{1}{4}\sqrt{g}\,\tau\delta g)t . \quad (42)$$

To define $\sigma^{ij}(t)$, first solve

$$\nabla_b(L\delta W)^{ab} + \tfrac{1}{2}g^{ab}\nabla_b\delta\tau = 0 \quad (43)$$

for δW^i, where

$$\delta\tau = \frac{\partial\tau(t)}{\partial t} = \frac{2}{3\sqrt{g}}(\delta\pi + \sigma^{ij}\delta g_{ij} - \tfrac{1}{4}\sqrt{g}\,\tau\delta g).$$

Now define $\sigma^{ij}(t)$ = TT part {with respect to $g_{ij}(t)$} of

$$\sigma_0^{ij} + [-\tfrac{1}{3}g^{ij}\sigma^{ab}\delta g_{ab} + \sqrt{g}\,L(\delta\vec{\pi} - \delta W)^{ij} + \delta\pi^{ij}_{TT} + \tfrac{1}{2}\sqrt{g}\,\tau\delta g^{ij} - \tfrac{1}{6}\sqrt{g}\,g^{ij}\tau\delta g]t . \quad (44)$$

The existence theorems we have proven in Secs. II and III show that there exists some $t_0 > 0$ such that for all t $(0 \le t \le t_0)$ $\{g_{ij}(t), \sigma^{ij}(t), \tau(t)\}$ generates a solution ($\phi(t), W^i(t)$) to Eqs. (4) and hence generates a curve of solutions ($\bar{g}_{ij}(t), \bar{\pi}^{ij}(t)$) to the initial-value constraints. Now

$$\bar{g}_{ij}(t) = \phi^4(t)g_{ij}(t) , \quad (45)$$

$$\bar{\pi}^{ij}(t) = \phi^{-4}(t)\sigma^{ij}(t) + \phi^2[g(t)]^{1/2}[(LW)^{ij} + \tfrac{1}{2}g^{ij}(t)\tau(t)] , \quad (46)$$

where $\{\phi(t), W^i(t)\}$ are solutions to

$$8\nabla^2\phi + \frac{1}{g}\sigma^{ij}\sigma_{ij}\phi^{-7} + \frac{2}{\sqrt{g}}\sigma^{ij}(LW)_{ij}\phi^{-1} - R(t)\phi - [\tfrac{3}{8}\tau^2 - (LW)^{ab}(LW)_{ab}]\phi^5 = 0 , \quad (47a)$$

$$\nabla_b[\phi^6(LW)^{ab}] + \tfrac{1}{2}\phi^6 g^{ab}(t)\nabla_b\tau(t) = 0 . \quad (47b)$$

At $(g^{(0)}_{ij}, \sigma_0^{ij}, \tau_0)$, i.e., $(\phi = 1, W^i = 0)$ we get

$$\frac{d\bar{g}_{ij}}{dt} = 4g^{(0)}_{ij}\frac{\partial\phi}{\partial t} + \frac{\partial g_{ij}}{\partial t} , \quad (48a)$$

$$\frac{d\bar{\pi}^{ij}}{dt} = -4\sigma_0^{ij}\frac{\partial\phi}{\partial t} + \sqrt{g}\,g^{ij}\tau_0\frac{\partial\phi}{\partial t} + \frac{\partial\sigma^{ij}}{\partial t} + \sqrt{g}\left[\left(L\frac{\partial W}{\partial t}\right)^{ij} + \tfrac{1}{2}g^{ij}\frac{\partial\tau}{\partial t}\right] + \tfrac{1}{4}\sqrt{g}\,g^{ij}\tau\frac{\partial g}{\partial t} - \tfrac{1}{2}\sqrt{g}\,\tau\frac{\partial g^{ij}}{\partial t} , \quad (48b)$$

where $(\partial\phi/\partial t, \partial W^i/\partial t)$ are solutions to

$$8\nabla^2\frac{\partial\phi}{\partial t}-\frac{7}{g}\sigma^{ij}\sigma_{ij}\frac{\partial\phi}{\partial t}-R\frac{\partial\phi}{\partial t}-\tfrac{15}{8}\tau^2\frac{\partial\phi}{\partial t}-\frac{1}{g}\sigma^{ij}\sigma_{ij}\frac{\partial g}{\partial t}-\frac{2}{g}\sigma^{ij}\sigma_j{}^k\frac{\partial g_{ik}}{\partial t}+\frac{2}{g}\sigma^{ij}\frac{\partial\sigma_{ij}}{\partial t}+\frac{2}{\sqrt{g}}\sigma^{ij}\left(L\frac{\partial W}{\partial t}\right)_{ij}-\delta_t R-\tfrac{3}{4}\tau\frac{\partial\tau}{\partial t}=0\,, \quad (49a)$$

$$\nabla_b\left(L\frac{\partial W}{\partial t}\right)^{ab}+\tfrac{1}{2}g^{ab}\nabla_b\frac{\partial\tau}{\partial t}=0\,. \quad (49b)$$

Equation (43) and Eq. (49b) are identical. Therefore

$$\delta W^i=\frac{\partial W^i}{\partial t}\,. \quad (50)$$

Next we will show that

$$\frac{\partial\sigma^{ij}}{\partial t}=[-\tfrac{1}{3}g^{ij}\sigma^{ab}\delta g_{ab}+\sqrt{g}\,L(\delta\,\vec{\pi}-\delta\,\vec{W})^{ij}+\delta\pi^{ij}_{TT}+\tfrac{1}{2}\sqrt{g}\,\tau\delta g^{ij}-\tfrac{1}{6}\sqrt{g}\,g^{ij}\tau\delta g]\,. \quad (51)$$

This is the same as showing that Eq. (44) is already TT to first order in t. The trace condition is obviously satisfied:

$$\sigma^{ij}\frac{\partial g_{ij}}{\partial t}+g_{ij}\frac{\partial\sigma^{ij}}{\partial t}=0\,. \quad (52)$$

The transversality condition requires

$$\nabla_j\frac{\partial\sigma^{ij}}{\partial t}+\partial_g(\Gamma^i_{ab})\sigma^{ab}=0\,. \quad (53)$$

Substitution from Eqs. (38) and (43) is sufficient to demonstrate this. Substitution of $\partial\sigma^{ij}/\partial t$ and $\partial\tau/\partial t$ into Eq. (49a) and use of Eq. (39) reduces it to

$$8\nabla^2\partial\phi/\partial t-(7g^{-1}\sigma^{ij}\sigma_{ij}+R+\tfrac{15}{8}\tau^2)\frac{\partial\phi}{\partial t}=0\,, \quad (54)$$

$$\Rightarrow 8\nabla^2\frac{\partial\phi}{\partial t}-(8g^{-1}\sigma^{ij}\sigma_{ij}+\tfrac{3}{2}\tau^2)\frac{\partial\phi}{\partial t}=0\,. \quad (55)$$

This has the unique solution

$$\frac{\partial\phi}{\partial t}=0\,. \quad (56)$$

Now we can immediately evaluate Eq. (49) to give

$$\frac{d\bar{g}_{ij}}{dt}=\delta g_{ij} \quad (57a)$$

and

$$\frac{d\bar{\pi}^{ij}}{dt}=\delta\pi^{ij}\,, \quad (57b)$$

exactly as wanted.

The stability analysis works for both open and closed manifolds. Of course $(\delta g_{ij},\delta\pi^{ij})$ must obey the same continuity and asymptotic requirements that we need for (g_{ij},π^{ij}). All the operators used are well behaved except in the presence of conformal Killing vectors.

V. STABILITY OF NONVACUUM AND NONMAXIMAL SOLUTIONS

When we consider solutions with sources we have to add (T^*_*, S^i) to our set of independent data, where $T^*_*\geq 0$ is the energy density of the sources and S^i is the current density. Under conformal transformations they scale as

$$\bar{T}^*_*=\phi^{-8}T^*_*\,, \quad (58a)$$

$$\bar{S}^i=\phi^{-10}S^i\,. \quad (58b)$$

The way in which these quantities transform under a conformal transformation can be strongly justified by physical as well as mathematical arguments. This question will be discussed in detail in a future paper.[12] In this case the equations for (ϕ, W^i) take on the following form:

$$8\nabla^2\phi+g^{-1}\sigma^{ij}\sigma_{ij}\phi^{-7}+16\pi T^*_*\phi^{-3}+2g^{-1/2}\sigma^{ij}(LW)_{ij}\phi^{-1}-R\phi-[\tfrac{3}{8}\tau^2-(LW)^{ij}(LW)_{ij}]\phi^5=0\,, \quad (59a)$$

$$\nabla_b[\phi^6(LW)^{ab}]+\tfrac{1}{2}\phi^6\nabla^a\tau+8\pi S^a=0\,. \quad (59b)$$

If we limit ourselves to the case where $\tau=$ constant, $S^i=0$, we have already shown that a solution almost always exists to these equations. These solutions to the initial-value equations have simple completeness properties also. In this case we get

$$\mathfrak{F}'_v\,dv=\{\,8\nabla^2\delta\phi-(8g^{-1}\sigma^{ij}\sigma_{ij}+64T^*_*+\tfrac{3}{8}\tau^2)\delta\phi+2\frac{\sigma^{ij}}{\sqrt{g}}(L\delta W)_{ij};\nabla_b(L\delta W)^{ab}\}\,. \quad (60)$$

The only situations where this is not an isomorphism are the situations we have already investigated, i.e., closed *mass-free* moment-of-time symmetry solutions and closed solutions possessing a conformal Killing vector. The only important new feature is that all closed *nonvacuum* moment-of-time symmetry solutions are complete and hence stable. All open solutions are stable. Of course, these solutions are stable under *all* perturbations, including the addition of infinitesimal mass distributions and currents.

So far in this paper we have only discussed special solutions, i.e., those with $\tau=$constant. These are especially interesting because we have demonstrated that these solutions do exist. If we relax the condition that $\tau =$ constant (or equivalently consider $S^i \neq 0$), the problem becomes more complicated because we have to solve four rather than one quasilinear equation. However, we expect that the existence and uniqueness results of the simpler problem will almost completely carry over to the more complicated problem and permit us to construct a large number of nonmaximal solutions to the initial-value constraints.

For these solutions, as in the earlier cases, the question of completeness reduces to whether or not $\mathfrak{F}'_v dv$ is an isomorphism. In this case $\mathfrak{F}'_v dv$ becomes

$$8\nabla^2\delta\phi + 2[g^{-1/2}\sigma^{ij} + (LW)^{ij}]\ (L\delta W)_{ij} - [\ 8g^{-1}\sigma^{ij}\sigma_{ij} + 4g^{-1/2}\sigma^{ij}(LW)_{ij} - \tfrac{3}{2}\tau^2 + 4(LW)^{ij}(LW)_{ij}]\delta\phi\ , \qquad (61)$$

$$\nabla_b(L\,\delta W)^{ab} + 6(LW)^{ab}\nabla_b\delta\phi\ .$$

This set forms a system of strongly elliptic, second-order, linear differential operators. Happily, the standard theorems for a single elliptic operator carry over to the case of an elliptic system of operators. In particular, the standard theorem, that if the homogeneous equations have only the trivial solution, then the system of operators defines an isomorphism, holds, i.e., "uniqueness implies existence."[13]

Just from inspection it is obvious that if the manifold has a conformal Killing vector T^i, then the homogeneous equations do have a nontrivial solution $(\delta\phi, \delta W^i) = (0, T^i)$. Therefore all closed solutions to the initial-value constraints with a conformal symmetry are incomplete.

From an extension of the well-known maximum principle, we have that a sufficient condition for the existence of no other harmonic functions to the system of operators is that

$$8g^{-1}\sigma^{ij}\sigma_{ij} + 4g^{-1/2}\sigma^{ij}(LW)_{ij} + 4(LW)^{ij}(LW)_{ij} - \tfrac{3}{2}\tau^2 \geq 0\ , \qquad (62)$$

or

$$^{(3)}R + \frac{1}{g}\sigma^{ij}\sigma_{ij} - g^{-1/2}\sigma^{ij}(LW)_{ij} \geq 0\ . \qquad (63)$$

In fact, drawing on our experience with single elliptic equations, it is probable that this condition can be much weakened, and that we really need this expression to be positive only "on the average," i.e.,

$$\int \sqrt{g}\left(^{(3)}R + \frac{1}{g}\sigma^{ij}\sigma_{ij} - g^{-1/2}\sigma^{ij}(LW)_{ij}\right) d^3x > 0\ . \qquad (64)$$

Since

$$\int_V \sigma^{ij}(LW)_{ij}\, d^3x = 0\ , \qquad (65)$$

we then have

$$\int_V \sqrt{g}\,(^{(3)}R + \frac{1}{g}\sigma^{ij}\sigma_{ij})\, d^3x > 0\ . \qquad (66)$$

Unfortunately, it is possible to construct counterexamples to the result (66), and therefore it may only be treated as a guide to completeness. However, it is interesting in that it relates the average sign of the scalar curvature to the question of completeness. On open manifolds the sign of the scalar curvature is closely related to the sign of the total energy, and this condition lends credence to the conjecture that even if negative energy solutions exist, they may be incomplete as solutions of the initial-value equations.

Since the addition of any source field, no matter how small, completes a closed solution at a moment-of-time symmetry, it is possible to argue that this incompleteness is in some sense unimportant, and does not place any real restrictions on investigations of realistic solutions to the initial-value constraints. On the other hand, the incompleteness associated with closed solutions which possess a conformal Killing vector appears to be much more important because it is not a property confined to a small number of "artificial" solutions, but is common to all closed solutions, be they vacuum or nonvacuum, maximal or nonmaximal, which possess a conformal symmetry. This fact could have important implications for a number of topics in general relativity, inasmuch as all closed solutions with symmetries correspond to atypical points of gravitational phase space.

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[7]Y. Choquet-Bruhat and S. Deser, Ann. Phys. (N.Y.) 81, 165 (1973). Professor Choquet-Bruhat suggested the stability problem to us, and the techniques we have used are adapted from this paper.

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[13]See, for example, O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasi-Linear Elliptic Equations* (Academic, New York, 1968), Chap. 7.

[14]It is interesting to note that $\mathcal{L}_T g_{ij} = 0$ and $\mathcal{L}_T \pi^{ij} = 0$ are the defining conditions for "mini-phase-space." See K. Kuchař, in *Relativity, Astrophysics, and Cosmology*, edited by W. Israel (Dordrecht, Reidel, 1973).