METRIC OF A CLOSED FRIEDMAN WORLD PERTURBED BY AN ELECTRIC CHARGE

(THEORY OF ELECTROMAGNETIC "FRIEDMONS")

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Tolmans' well-known problem is generalized to the case of electrically charged dust-like matter of a centrally symmetric system. First integrals are found of the corresponding Einstein – Maxwell equations. The problem is then specialized in such a way that the metric of a closed Friedman world is obtained when the total charge of the system tends to zero. Such a system is considered at the initial instant, the time of maximum expansion. For any arbitrarily small electric charge, the metric is not closed. The metric of the almost Friedman part of the world is continued through a narrow throat (for a small charge) by the Nordström – Reissner metric with parameters satisfying $\sqrt{\varkappa}m_0 = e_0$. The expression for the electric potential in the throat $\varphi_h = c^2/\sqrt{\varkappa}$ does not depend on the magnitude of the electric charge. With increasing charge, the radius of the throat increases ($r_h = e_0\sqrt{\varkappa}/c^2$). The state of the throat in the classical description is essentially unstable from the point of view of quantum physics. The generation of all kinds of pairs in the tremendously strong electric fields of the throat polarize the latter to an effective charge $Z < 137e$, irrespective of the initial, arbitrarily large charge of the material system.

1. Generalization of Tolman's Solution to the Case of Electrically Charged Dust-Like Matter

The solution of Einstein's equations for the case of a centrally symmetric gravitational field in a comoving frame for dust-like matter (pressure p = 0) was found by Tolman [1].

In connection with a number of problems, interest attaches to a generalization of Tolman's solution to the case of electrically charged dust-like matter. It is well known that Friedman's closed world is described by particular solutions of Tolman's problem. It is also well known that the metric of the world cannot be closed if the matter is charged, even if the matter density exceeds the critical density.

The question arises of the manner in which the metric of a closed Friedman world is altered under the influence of, say, a weak perturbation resulting from the presence of an electric charge. The answer to this question must be found by solving simultaneously the Einstein – Maxwell system of equations

$$G_i^k \equiv R_i^k - \frac{1}{2}\delta_i^k R = \frac{8\pi\varkappa}{c^4}(T_i^k + E_i^k), \tag{1}$$

$$F^{ik}{}_{;k} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^k}(\sqrt{-g}F^{ik}) = -\frac{4\pi}{c}j^i, \tag{2}$$

$$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{kl}}{\partial x^i} + \frac{\partial F_{li}}{\partial x^k} = 0. \tag{3}$$

We take the energy tensor on the right-hand side of (1) in the form

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$$T_i^k + E_i^k = \begin{pmatrix} \varepsilon + \Lambda/8\pi & 0 & 0 & 0 \\ 0 & \Lambda/8\pi & 0 & 0 \\ 0 & 0 & -\Lambda/8\pi & 0 \\ 0 & 0 & 0 & -\Lambda/8\pi \end{pmatrix}, \tag{4}$$

where

$$\Lambda = e^2/r^4 = -F_{01}F^{01} \tag{5}$$

arises as a result of the solution of the Maxwell equation for the case of a spherically symmetric system. Here

$$T_i^k = \delta_0^k \delta_i^0 \varepsilon \tag{6}$$

is the matter tensor in the comoving frame ($x^1 = q$).

Written out in full, Eq. (1) has the form

$$-\frac{8\pi\varkappa}{c^4}(T_1^1 + E_1^1) = \frac{1}{2}e^{-\lambda}\left(\frac{\mu'^2}{2} + \mu'\nu'\right) - e^{-\nu}\left(\ddot{\mu} - \frac{1}{2}\dot{\mu}\dot{\nu} + \frac{3}{4}\dot{\mu}^2\right) - e^{-\mu} = -\frac{\varkappa}{c^4}\Lambda \equiv -\tilde{\Lambda}, \tag{I}$$

$$-\frac{8\pi\varkappa}{c^4}(T_2^2 + E_2^2) = \frac{1}{4}e^{-\lambda}(2\nu'' + \nu'^2 + 2\mu'' + \mu'^2 - \mu'\lambda' - \nu'\lambda' + \mu'\nu')$$
$$+ \frac{1}{4}e^{-\nu}(\dot{\lambda}\dot{\nu} + \dot{\mu}\dot{\nu} - \dot{\lambda}\dot{\mu} - 2\ddot{\lambda} - \dot{\lambda}^2 - 2\ddot{\mu} - \dot{\mu}^2) = \frac{\varkappa}{c^4}\Lambda \equiv \tilde{\Lambda}, \tag{II}$$

$$\frac{8\pi\varkappa}{c^4}(T_0^0 + E_0^0) = -e^{-\lambda}\left(\mu'' + \frac{3}{4}\mu'^2 - \frac{\mu'\lambda'}{2}\right) + \frac{1}{2}e^{-\nu}\left(\dot{\lambda}\dot{\mu} + \frac{\dot{\mu}^2}{2}\right) + e^{-\mu}$$
$$= \frac{8\pi\varkappa}{c^4}\varepsilon + \frac{\varkappa}{c^4}\Lambda \equiv \tilde{\varepsilon} + \tilde{\Lambda}, \tag{III}$$

$$\frac{8\pi\varkappa}{c^4}(T_0^1 + E_0^1) = \frac{1}{2}e^{-\lambda}(2\dot{\mu}' + \dot{\mu}\mu' - \dot{\lambda}\mu' - \nu'\dot{\mu}) = 0. \tag{IV}$$

Here, we have taken the metric in the form

$$ds^2 = e^{\nu}dx^{0^2} - e^{\lambda}dx^{1^2} - e^{\mu}d\sigma^2, \tag{7}$$

where $d\sigma^2 = dx^{2^2} + \sin^2 x^2 dx^{3^2}$. The dot denotes differentiation with respect to x^0 and the prime differentiation with respect to q.

Using the confirmation laws, we readily obtain [1]

$$\dot{\tilde{\varepsilon}} = -\tilde{\varepsilon}\left(\frac{\dot{\lambda}}{2} + \dot{\mu}\right), \tag{V}$$

$$2\frac{e'}{e}\tilde{\Lambda} = \frac{1}{2}\nu'\tilde{\varepsilon}. \tag{VI}$$

In our case, the comoving system is not a synchronous system ($\nu \neq 0$).

Integration of (V) with respect to x^0 yields

$$\varepsilon = 2\frac{\varkappa}{c^4}\frac{C(q)}{r^2}e^{-\lambda/2}, \tag{8}$$

where $r^2 \equiv e^{\mu}$. Equation (VI) yields

$$\nu' = \frac{2ee'}{r^2 C(q)}e^{\lambda/2}, \tag{9}$$

and Eq. (IV) can be rewritten in the form

$$2(\ln r')^{\cdot} - \dot{\lambda} - \nu'\dot{r}/r' = 0. \tag{10}$$

Integrating Eq. (10) with respect to x^0, we obtain

$$\ln(r')^2 = \lambda + \int\frac{\nu'\dot{r}}{r'}dx^0 + \ln(1+f), \tag{11}$$

where $f = f(q)$, $1 + f \geq 0$.

Denoting by φ the expression

$$\varphi = \int\frac{\nu'\dot{r}}{r'}dx^0 = \frac{2ee'}{C(q)}\int\frac{e^{\lambda/2}\dot{r}}{r'r^2}dx^0, \tag{12}$$

we rewrite Eq. (11) in the form

$$e^{\lambda}=\frac{r'^{2}}{1+f}e^{-\varphi}. \tag{13}$$

An expression for φ can be obtained as follows. Using (13), we rewrite Eq. (12) as an integral equation for φ:

$$\varphi=\frac{2ee'}{C(q)\sqrt{1+f}}\int\frac{e^{-\varphi/2}\dot{r}}{r^{2}}\,dx^{0}, \tag{14}$$

from which we obtain a differential equation for φ:

$$\dot{\varphi}=\delta(q)e^{-\varphi/2}\dot{r}/r^{2}, \tag{15}$$

where

$$\delta(q)\equiv 2ee'/C(q)\sqrt{1+f}, \tag{16}$$

and, hence,

$$2e^{\varphi/2}=-\delta/r+2\psi(q). \tag{17}$$

Substituting (17 into (13), we obtain

$$e^{\lambda}=\frac{r'^{2}}{(\sqrt{1+f}\psi-ee'/C(q)r)^{2}}, \tag{18}$$

or, writing

$$\sqrt{1+f}\psi\equiv\sqrt{1+f}, \tag{19}$$

$$\delta(q)\equiv\frac{2ee'}{\sqrt{1+f}C(q)}, \tag{20}$$

we obtain finally

$$e^{\lambda}=\frac{r'^{2}}{1+f}\frac{1}{(1-\delta/2r)^{2}}. \tag{21}$$

Equation (1) can be rewritten as follows:

$$e^{-\lambda}(r''+r'r\nu')-e^{-\nu}(2\ddot{r}r+\dot{r}^{2}-\dot{r}r\dot{\nu})-1=-\frac{\varkappa e^{2}}{c^{4}r^{2}}. \tag{22}$$

It is readily seen that

$$e^{-\lambda}(r'^{2}+r'r\nu')=(1+f)(1-\delta^{2}/4r^{2}),$$

$$e^{-\nu}(2\ddot{r}r+\dot{r}^{2}-\dot{r}r\dot{\nu})=\frac{1}{\dot{r}}(e^{-\nu}\dot{r}^{2}r)^{\cdot}.$$

Integrating (22) with respect to x^{0}, we obtain

$$e^{-\nu}\dot{r}^{2}=f+\frac{2m(q)}{r}-\frac{1}{r^{2}}\left(\frac{\varkappa}{c^{4}}e^{2}-\frac{\delta^{2}(1+f)}{4}\right), \tag{23}$$

where m(q) is the constant for the integration with respect to x^{0}.

We rewrite Eq. (III) in the form

$$-e^{-\lambda}(2r''r+r'^{2}-r'r\lambda')+e^{-\nu}(\dot{\lambda}\dot{r}r+\dot{r}^{2})+1=(\tilde{\Lambda}+\varepsilon)r^{2}. \tag{24}$$

Noting that

$$e^{-\lambda}(2r''r+r'^{2}-r'r\lambda')=\frac{(e^{-\lambda}r'^{2}r)'}{r'},$$

$$e^{-\nu}(\dot{\lambda}\dot{r}r+\dot{r}^{2})=\frac{(e^{-\nu}\dot{r}^{2}r)'}{r'};$$

substituting (21) and (23) into these expressions and setting

$$m_{1}(q)\equiv\frac{c^{2}}{\varkappa}\left[m(q)+\frac{\delta(1+f)}{2}\right], \tag{25}$$

we obtain the relation

$$m_{1}'(q)=\frac{1}{c^{2}}C(q)\sqrt{1+f}. \tag{26}$$

Equation (II) does not yield any new relations, for it is a consequence of the other equations we have used.

Three unknown functions occur in the first integrals we have obtained for Eqs. (I) and (III):

$$f(q),\quad m(q),\quad e(q). \tag{27}$$

The problem is made completely concrete when these functions, which must be determined by the initial conditions, are specified. We shall take the surface $x^0 = 0$ as the space-like hypersurface Σ on which the initial conditions are specified.

The condition (IV) $G_0^1 = 0$ is compatible with the relation

$$e^{\lambda(0,q)} = \frac{r''(0,q)}{1+f(q)}\,\frac{1}{(1-\delta(q)/2r(0,q))^2}. \tag{28}$$

On the surface Σ, Eq. (III) can be written in the form

$$(e^{-\nu}\dot{r}^2 r)' - (e^{-\nu} r'' r)' + r' = \frac{2\varkappa}{c^4} C(q) e^{-\lambda/2} r' + \frac{\varkappa e^2}{c^4 r^2} r'. \tag{29}$$

We take q to be a canonical coordinate, the distance from the center at the initial instant of time [$\exp\lambda(0, q) = 1$]; the relation (28) then becomes the definition of $f(q)$:

$$\sqrt{1+f} = r'(0,q) + \frac{ee'}{C(q)r(0,q)}. \tag{30}$$

In what follows, we shall specialize our problem principally to the case when a closed Friedman world is obtained as the electric charge of the system tends to zero.

2. Friedman World Deformed by the Presence of an Electric Charge

The Interior Solution. In what follows, we shall attempt to define the unknown functions $f(q)$, $m(q)$, and $e(q)$ in such a way that the metric of a closed Friedman world is obtained in the limiting case $e(q) \to 0$. Since the total electric charge vanishes in a closed world, it is a priori evident that the metric of such a world, even in the case of a small electric charge, cannot be completely closed and that a Friedman metric deformed by the charge must have a Nordström – Reissner continuation outside the matter. Our task is to find at least special examples for which one can describe the whole space of such a world continuously. We therefore expect that the interior solution, which is close to Friedman's solution for a closed world, must go through a throat into the well-known exterior Nordstöm – Reissner solution. For the interior solution, we shall therefore try to formulate the initial conditions at $x^0 = 0$, the moment of maximum expansion of the system, so that they are most nearly Friedman. Namely, suppose that 1) on $x^0 = 0$ the whole space belongs to the R region [2]; 2) the initial velocities of all particles vanish; 3) the energy density at the initial instant does not depend on q:

$$T_0^0 + E_0^0 = \varepsilon_0 = \text{const.}$$

We shall show below that, under these conditions, the problem has a solution in the case of electrically charged dust, i.e., there exists a function $C(q)$ or $M(q)$ which is compatible with the given conditions.

For the chosen initial conditions, Eq. (29) can be rewritten in the form

$$1 - \frac{(r''r)'}{r'} = \frac{\varkappa}{c^4} 8\pi\varepsilon_0 r^2. \tag{31}$$

We define

$$\frac{8\pi\varkappa\varepsilon_0}{c^4} = \frac{3}{4a_0^2} \tag{32}$$

and integrate Eq. (31); then,

$$1 - \frac{r^2}{4a_0^2} = r''. \tag{33}$$

or, since $r(q = 0) = 0$,

$$r = 2a_0 \sin q/2a_0, \tag{34}$$

so that the expression (29) can now be rewritten in the form

$$2\frac{\varkappa}{c^4} C(q) + \frac{\varkappa}{c^4}\frac{e^2}{r^2} = 3\sin^2\frac{q}{2a_0}. \tag{35}$$

Further, we prescribe the charge distribution. Let all the dust particles of the system have the same charge-to-mass ratio β. If

$$\frac{1}{c^2}\int_0^q C(q)\,dq = M(q), \tag{36}$$

the new condition can be written in the form

$$\varepsilon(q) = \beta M(q). \tag{37}$$

Equation (35) now takes the form of an equation for the determination of M(q):

$$2\bar{M}'(q)+\bar{\beta}^2\frac{\bar{M}^2(q)}{r^2}=3\sin^2\frac{q}{2a_0}, \tag{38}$$

where

$$\bar{M}=\frac{\varkappa}{c^2}M; \qquad \bar{\beta}=\frac{\beta}{\sqrt{\varkappa}}. \tag{39}$$

One can verify by substitution that the following expression satisfies Eq. (38):

$$\bar{M}=\frac{4a_0}{\bar{\beta}^2}\sin\chi\,(b\operatorname{ctg} b\chi\sin\chi-\cos\chi),$$

where

$$\chi=\frac{q}{2a_0}, \qquad b=\sqrt{1-\frac{3}{4}\bar{\beta}^2}.$$

It is readily seen that, as $\beta\to 0$, $\bar{M}$ goes over into

$$\bar{M}_0(q)=\frac{3}{2}a_0\left(\chi-\frac{\sin 2\chi}{2}\right), \tag{40}$$

i. e., into the expression for the "interior" mass [3] in the uncharged Friedman world.* Further, using (9), one can obtain

$$e^{\nu(0,q)}=\left(\frac{\sin b\chi}{b\sin\chi}\right)^4. \tag{41}$$

With these remarks, we conclude our consideration of the interior solution at the initial instant of time, the time of greatest expansion of the material system. In the following sections, we shall analyze the solution in vacuum (in the regions where $\varepsilon = 0$) and the problem of fitting the interior and exterior solutions.

Exterior Nordström – Reissner Solution. As is well known, the geometry of space outside a mass m_0 which has a spherically symmetric distribution and an electric charge e_0 is described by the Nordström – Reissner metric:

$$ds^2 = \Phi(r)\,dt^2 - dr^2/\Phi(r) - r^2 d\sigma^2, \tag{42}$$

where

$$\Phi(r)=1-\frac{2\varkappa m_0}{c^2 r}+\frac{\varkappa e_0^2}{c^4 r^2}. \tag{43}$$

In this problem, we must distinguish three cases:

$$1)\ \sqrt{\varkappa}m_0 > e_0, \quad 2)\ \sqrt{\varkappa}m_0 = e_0, \quad 3)\ \sqrt{\varkappa}m_0 < e_0. \tag{44}$$

In the first case, the metric is characterized by two pseudosingularities of the type of the Schwarzschild pseudosingularity: $\Phi(r_1) = \Phi(r_2) = 0$. For $r_2 < r < r_1$, the coordinate r is time-like. For this case, the whole of space-time can be described in Kruskal-type coordinates [4]. A test particle placed at $r = r_1$ on $x^0 = 0$ reaches $r = r_2$ after a time $T = \pi\varkappa m_0/c^3$; it then comes to rest instantaneously and returns to r_1.

At the initial instant (the instant of time symmetry), the geometry of space has the form of a "wormhole" ("Einstein – Rosen bridge"). The throat of the wormhole pulsates with a period of 2T and never closes

* The total mass, taking into account the gravitation mass defect in a closed world, vanishes [1].

(In contrast to the Schwarzschild case). Complete closing of the throat is prevented by the electric lines of force passing through the throat into the Euclidean infinity.*

The second case differs from the first in that there is no T region. At the point

$$r = \frac{\varkappa m_0}{c^2} = \frac{e_0\sqrt{\varkappa}}{c^2}$$

we have a zero of second order:

$$\Phi(r) = (1 - r_h/r)^2, \text{ where } r_h = \varkappa m_0/c^2 = \sqrt{\varkappa}e_0/c^2. \tag{45}$$

The following analysis will show that the geometry at the instant of time symmetry can have two forms in this case: an α^0-type "wormhole"; β^0-geometry with a monotonic variation of r. The second case is realized, in particular, in Papapetrou's model (a static charged-dust model with $\bar{\beta} = e\sqrt{\varkappa}M = 1$). If we are concerned with a semiclosed charged world, the exterior solution satisfying the condition of flatness at infinity is of α^0-type.

The third case ($e_0 > \sqrt{\varkappa}m_0$) goes over into the β^0 case as e_0 decreases. There are now no singularities and the whole of space is R-type. In this case, semiclosed worlds (with flat space at infinity) are not realized. The limit $\bar{\beta} = 1$ gives an everywhere static system in this case (Papapetrou's model).

We are interested in the problem of fitting an exterior Nordström – Reissner solution to an interior solution describing an almost closed world, i.e., a world whose metric goes over into the metric of a closed Friedman world as $e_0 \to 0$. For $e_0 \neq 0$, our problem is to find a maximal continuation of the interior Friedman solution (to decrease the size of the throat) as far as this is permitted by the presence of the electric field. From this point of view, it is expedient to consider a deformation of the Friedman metric by a small electric charge $\bar{\beta} \ll 1$.

Of all the cases considered above, only case 2 ($\alpha^0 < 1$) satisfies our conditions. None of the remaining cases leads to a closed world as $e_0 \to 0$.

Fitting of Interior and Exterior Solutions. In order to be able to use the boundary conditions of fitting more conveniently, we transform Eq. (7) to a form similar to (42), namely to

$$ds^2 = \beta dt^2 - \alpha dr^2 - r^2 d\sigma^2, \tag{46}$$

where x^1 (or the q coordinate) is taken to be the coordinate whose square appears as the coefficient of $d\sigma^2$. The transformation

$$dr = \dot{r}\,dx^0 + r'\,dx^1, \qquad dx^1 = \frac{dr - \dot{r}\,dx^0}{r'}$$

transforms the first two terms of Eq. (7) to the form

$$e^{\nu} dx^{0^2} - e^{\lambda} dx^{1^2} = \left(\sqrt{e^{\nu} - e^{\lambda}\dot{r}^2/r'^2}\,dx^0 + \frac{\dot{r}e^{\lambda}}{r'^2\sqrt{e^{\nu} - e^{\lambda}\dot{r}^2/r'^2}}\,dr\right)^2 - \frac{dr^2}{r'^2e^{-\lambda} - \dot{r}^2e^{-\nu}}. \tag{47}$$

The expression in the brackets can be transformed by means of an integrating multiplier $\tilde{\mu}(t, r)$ to the form

$$\left(e^{\nu} - e^{\lambda}\frac{\dot{r}^2}{r'^2}\right)^{1/2} dx^0 + \frac{\dot{r}}{r'^2}e^{\lambda}\left(e^{\nu} - e^{\lambda}\frac{\dot{r}^2}{r'^2}\right)^{-1/2} dr = \frac{1}{\tilde{\mu}(t,r)}\,dt,$$

which has a total differential on the right-hand side. For what follows, the expression obtained for α is important:

$$\alpha = \frac{1}{r'^2e^{-\lambda} - \dot{r}^2e^{-\nu}}. \tag{48}$$

From the conditions of fitting of the interior and exterior solutions at the interface Σ:

$$\alpha^{\text{in}} = \alpha^{\text{out}}|_{\Sigma}, \qquad r^{\text{in}} = r^{\text{out}}|_{\Sigma}, \tag{49}$$

* For $e \to 0$, the Schwarzschild solution $\Phi(r) = 1 - 2\varkappa m_0/c^2 r$ can be interpreted as the exterior solution for a semiclosed world [3, 5]. The Kruskal metric is interpreted physically in [5].

we obtain

$$r'^2 e^{-\lambda} - \dot{r}^2 e^{-\nu}|_{q=q_0} = 1 - \frac{2\varkappa m_0}{c^2 r} + \frac{\varkappa e_0^2}{c^4 r^2}. \tag{50}$$

Using (21), (22), and (25), we find

$$m_1(q_0) = m_0, \qquad e(q_0) = e_0. \tag{51}$$

We still have not obtained an answer to the fundamental question, namely, that of the value of q_0 at which we must fit the interior and exterior solutions if we wish to continue the Friedman world to the maximum possible degree of closure with the minimum size of the throat at the instant of time symmetry. In the third case, it is impossible to fit a semiclosed world to a space that is flat at infinity. Consequently, $\sqrt{\varkappa}m_1(q_0) \geq e(q_0)$, at the boundary and the desired value of q_0 can be found from the equation

$$\sqrt{\varkappa}m_1(q_0) = e(q_0). \tag{52}$$

The condition (52) can be written in the form*

$$r_0'^2 = (1 - r_h/r_0)^2, \tag{53}$$

where $r_h = \sqrt{\varkappa}e_0/c^2 = \varkappa m_0/c^2$, and r_0 and r_0' are the values of r and r' at the matter boundary. The condition (53) for a semiclosed world ($r' < 0$) fitted to a space that is flat at infinity leads to the relation $r_h < r_0$, which corresponds to the presence of a "wormhole", i.e., to case 2 (α^β). For a semiclosed world fitted to a space that is flat at infinity, the condition (53) may therefore be rewritten in the form

$$r_0' = \frac{r_h}{r_0} - 1. \tag{54}$$

Let us consider in more detail the model of a world with a small charge $\bar{\beta} \ll 1$ (or $\pi\bar{\beta} \ll 1$). In this case,

$$\bar{M} = \frac{3}{2}a_0(\chi_0 - \sin\chi_0\cos\chi_0) + O(\beta^2),\ r_h = \frac{3}{2}\beta a_0(\chi_0 - \sin\chi_0\cos\chi_0) + O(\beta^3),\ r_0 = 2a_0\sin\chi_0, \qquad r_0' = \cos\chi_0, \tag{55}$$

where $0 < \chi_0 < \pi$, $\chi_0 = q_0/2a_0$.

The condition (54 for $\pi/2 < \chi_0 < \pi$ and small values of $\bar{\beta}$ can be written in the form

$$1 + \left(1 + \frac{3}{4}\beta\right)\cos\chi_0 = \frac{3}{4}\beta\frac{\chi_0}{\sin\chi_0}. \tag{56}$$

For $\bar{\beta} = 0$, $\chi_0 = \pi$, i.e., χ_0 attains its maximum value, and the world becomes a completely closed Friedman world.

In the case of a small charge $\bar{\beta}$ ($\bar{\beta} \ll 1$), the desired boundary of the interior (Friedman) solution must be somewhere near π, i.e., $\chi_0 = \pi - \delta$, where δ is small. For it follows from the graph (Fig. 1) that Eq. (56) has a single solution χ_0. For $\bar{\beta} \ll 1$, the solution χ_0 is near π and, as $\bar{\beta} \to 1$, the solution χ_0 tends to $\pi/2$.

As the charge e_0 of the world increases, its exterior (Schwarzschild) mass also increases. The radius of the throat increases accordingly:

$$r_h = \sqrt{\varkappa}e_0/c^2. \tag{57}$$

It is important to realize that the potential of the electric field in the throat

$$\varphi_h = e_0/r_h \tag{58}$$

does not change when the charge e_0 increases, but remains equal to the constant value

$$\varphi_h = c^2/\sqrt{\varkappa}. \tag{59}$$

The quantity φ_h plays the role of a maximum potential in the theory; it is composed of universal constants and it is interesting that it does not contain an electric charge.

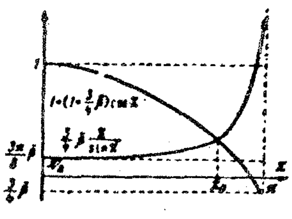

Fig. 1

* Equation (31) in this case can be rewritten in the form $[(1-r'^2)r]' = \frac{\varkappa}{c^4}\frac{e_0^2}{r^2}r'$; hence, $r'^2 = 1 - \frac{2\varkappa m_0}{c^2 r} + \frac{\varkappa e_0^2}{c^4 r^2}$. By virtue of the continuity of r and r' on the matter – vacuum boundary, Eq. (52) is valid in this region of r.

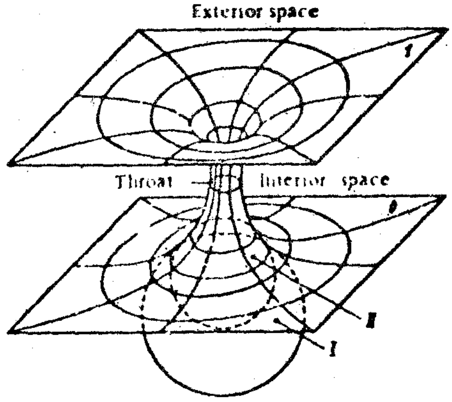

Fig. 2

The Throat

The condition $\sqrt{\varkappa} m_0 = e_0$ ensures that the throat is static. An exterior observer always sees a charged, maximally continued semiclosed world in the form of a solidified charged sphere.*

In this case, too, the dynamics of part of the almost closed uniformly charged world remains nonstationary. After the instant of maximum expansion, the charged cloud described by the interior solution contracts. However, the collapse of the system is halted by the electrical forces at the minimum radius determined by the dimensions of the throat, i.e., by the total electric charge of the system.

It should be emphasized that there is no matter in the throat. The nonstatic behavior of the material cloud does not affect the static behavior of the throat. In the throat, the bundle of electric lines of force are compressed by the maximum amount possible ($\varphi_h = c^2/\sqrt{\varkappa}$). From the throat, a bundle of lines of force diverge both outward, into the Euclidean infinity, and also into the almost Friedman world. Thus, the throat simulates a source of the electric field (a charge), although no material charge carriers are localized in the throat.

A more detailed consideration shows that the field in the exterior space and the field between the matter and the throat have opposite signs (Fig. 2):

$$F_{tr} = e/r^2 \text{ (in the exterior space, region 1).}$$
$$F_{tr} = -e/r^2 \text{ (between the matter and the throat, region 0).} \quad (60)$$

For the connection between F_{x^0q} and F_{tr} is given the transformation

$$F_{tr} = \frac{D(t,r)}{D(x^0,q)} F_{x^0q}. \quad (61)$$

Further, one can show† that

$$\operatorname{sign} \frac{D(t,r)}{D(x^0,q)} = \operatorname{sign} r', \quad (62)$$

i.e., the sign of $D(t, r)/D(x^0, q)$ is equal to the sign of r' and this implies (60).

In the throat itself, a test electric charge must always rest. In the regions 1 and 0, one can easily realize a static system of reference by using appropriately charged and weightless dust particles. This system coincides with the Reissner–Nordström system. As is well known, a complete description of the Reissner–Nordström metric (i.e., including the regions between its two pseudosingularities) can be given by Kruskal-type coordinates (a nonstatic system of reference).

In our case, the region (r_1, r_2) contracts to the single value $r_1 = r_2 = r_h$, the throat. The static frame of reference does not cover only this section immediately adjoining the throat.

* We recall that in the case $\sqrt{\varkappa} m_0 > e_0$ the throat oscillated between r_1 and r_2. In the limit $e_0 \to \sqrt{\varkappa} m_0$, we have $r_1 \to r_2$. For the case $[\sqrt{\varkappa} m_0 = e_0]$, the exterior (Schwartzschild) mass vanishes as $e_0 \to 0$. The world becomes completely closed, i.e., in the case $\sqrt{\varkappa} m_0 = e_0$, the entire mass is of electric origin. Under these conditions, any initial value of the interior mass of nonelectric origin is completely offset by the gravitational mass defect.

† $g_{x^0x^0} = \left(\frac{\partial t}{\partial x^0}\right)^2 g_{tt} + \left(\frac{\partial r}{\partial x^0}\right)^2 g_{rr}$, $g_{qq} = \left(\frac{\partial t}{\partial q}\right)^2 g_{tt} + \left(\frac{\partial r}{\partial q}\right)^2 g_{rr}$, but $g_{x^0x^0} > 0$, $g_{qq} < 0$, $g_{tt} > 0$, $g_{rr} < 0$. Consequently, $\left(\frac{\partial t}{\partial x^0}\right)^2 g_{tt} > \left(\frac{\partial r}{\partial x^0}\right)^2 (-g_{rr})$ and $\left(\frac{\partial r}{\partial q}\right)^2 (-g_{rr}) > \left(\frac{\partial t}{\partial q}\right)^2 g_{tt}$; hence $\left|\frac{\partial t}{\partial x^0}\frac{\partial r}{\partial q}\right| > \left|\frac{\partial r}{\partial x^0}\frac{\partial t}{\partial q}\right|$ and, consequently, the sign of $\frac{D(t,r)}{D(x^0,q)} = \frac{\partial t}{\partial x^0}\frac{\partial r}{\partial q} - \frac{\partial t}{\partial q}\frac{\partial r}{\partial x^0}$ is determined by the first term since $\partial t/\partial x^0 > 0$, i.e., the time always increases ("arrow of time").

Polarized Throats (The Need For a Quantum Description of the Throat)

On the basis of the relation (57) ($r_h = \sqrt{\varkappa} e_0/c^2$), we concluded that the radius of the throat increases proportionally to the total electric charge. This is the description of the throat that we obtain from the classical theory. However, from the point of view of quantum physics, such a state of the throat arises with the properties described above, a violent process of generation of all kinds of electrically charged pairs, such as proton – antiproton pairs, all kinds of meson pairs, and, finally, electron – positron pairs, would inevitably be initiated in the superstrong electric field of the throat. The charges of opposite sign would tend to decrease the effective charge of the throat, while the charges of the other components of the pairs would escape into the Euclidean infinity. In this process, the charge of the throat would gradually decrease; at the same time, the radius of the throat would also decrease and the interior metric of the system would become more and more closed. Let us consider this effect in more detail, not so much for the purpose of giving an exhaustive quantitative description of the generation of pairs in such a field, but rather in order to fix attention on a very curious, in our view, situation: the need to take into account quantum theory in the ultramacroscopic world, namely, to describe processes which one would imagine were only important in the microscopic world. Although the quantitative estimates are as yet far from satisfactory, they are not entirely devoid of interest in their own right.

The generation of pairs of electrically charged particles in a strong homogeneous electric field has been considered by Nikishov [6].*

If there is a homogeneous electrostatic field of intensity E filling the space of a cube of volume L^3, the probability of creation of pairs (say, electrons) in the field with given momentum (p) and spin (r) during the whole time is given by the expression

$$W_{pr} = \exp(-\pi\lambda), \quad \lambda = \frac{c^2(p_1^2 + p_2^2 + m_0^2c^2)}{eE\hbar c} \quad (\mathbf{E} = (0, 0, E)), \tag{63}$$

where m_0 is the mass of a particle of the pair and p is the value of the particle momentum of the generated pair after the field has been switched off. In such a problem, p must belong to a discrete spectrum, i.e., $Lp_n^{(i)} = 2\pi\hbar n$.

Equation (63) can be rewritten in the form

$$W_{n_1n_2n_3r} = \exp\left(-\frac{\pi m_0^2c^4}{eE\hbar c}\right) \exp\left[-\frac{\pi c^2}{eE\hbar c}\left(\frac{2\pi\hbar}{L}\right)^2 n_1^2\right] \exp\left[-\frac{\pi c^2}{eE\hbar c}\left(\frac{2\pi\hbar}{L}\right)^2 n_2^2\right]. \tag{64}$$

Here, the state of the generated particle is characterized by the numbers ($n_1n_2n_3r$). Summing $W_{n_1n_2n_3r}$ over all quantum numbers and then replacing the sum over n by an integral, we obtain

$$W = 4N \exp\left(-\frac{\pi m_0^2c^4}{eE\hbar c}\right) eE\hbar c \left(\frac{L}{2\pi\hbar c}\right)^2 \Phi^2(\xi_0), \tag{65}$$

where

$$\xi_0 = \sqrt{\frac{\pi}{eE\hbar c}}\frac{2\pi\hbar c}{L}N, \quad \Phi(\xi_0) = \frac{2}{\sqrt{\pi}}\int_0^{\xi_0} e^{-\xi^2} d\xi, \; N = n_{\max}; \quad p_{\max} = \frac{2\pi\hbar N}{L} = c\left(m_0 + \frac{e\varphi_{\max}}{c^2}\right). \tag{66}$$

For a large maximum momentum ($p_{\max}$), $\xi_0 \gg 1$ and $\Phi(\xi_0) \sim 1$. By virtue of (65) and (66), the probability of pair creation in unit volume is

$$w = \frac{4}{(2\pi\hbar)^3}\frac{p_{\max}}{c} eE\hbar \exp\left(-\frac{\pi m_0^2c^4}{eE\hbar c}\right)\Phi^2(\xi_0). \tag{67}$$

Further, in order to obtain an estimate, we apply (incorrectly) Eq. (67) to an inhomogeneous static field $E = Ze/r^2$ or $\varphi = Ze/r$. In calculating the total number of pairs (N_p) in the whole of space, we shall assume that $p_{\max}$ in (67) depends on r:

$$p_{\max} = Ze^2 / cr, \tag{68}$$

where Ze is the total charge of the material system. For the total number of pairs generated in the given field during the whole time, we obtain

$$N_p = \int_V w\, d^3v = \frac{16\pi Z^2e^4\hbar}{(2\pi\hbar)^3c^2}\int_{a_0}^{\infty}\frac{1}{r}\exp\left(-\frac{\pi m_0^2c^4r^2}{Ze^3\hbar c}\right)dr = \frac{16\pi Z^2e^4\hbar}{(2\pi\hbar)^3c^2}\int_{A_0}^{\infty}\frac{e^{-\xi^2}}{\xi}d\xi, \tag{69}$$

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* The authors are grateful to A.I. Nikishov for letting them know his results.

where $A_0 = \sqrt{\pi m_0^2 c^4 / Z e^2 \hbar c a_0}$, and $a_0 = \sqrt{\varkappa} Z e / c^2$ is the minimal racius. Since

$$\int_{A_0}^{\infty} \frac{e^{-\xi^2}}{\xi} d\xi = \frac{1}{2} \int_{A_0^2}^{\infty} \frac{e^{-x}}{x} dx = -\frac{1}{2} \mathrm{Ei}(-A_0^2),$$

we have

$$N_p = -\frac{1}{\pi^2} (Z\alpha)^2 \mathrm{Ei}(-A_0^2). \tag{70}$$

For small values of A_0^2, i.e., for

$$Z \ll \frac{1}{\pi\alpha} \left(\frac{e/\sqrt{\varkappa}}{m_0} \right)^2 \sim 10^{45}, \tag{71}$$

we have

$$\mathrm{Ei}(-A_0^2) \sim c + \ln A_0^2, \tag{72}$$

where c is Euler's constant and, consequently,

$$N_p \sim \frac{1}{\pi^2} (Z\alpha)^2 \left[\ln \left(\frac{e/\sqrt{\varkappa}}{m_0} \right)^2 - c - \ln \pi Z \alpha \right],$$

or, taking into account the condition (71),

$$N_p \sim \frac{1}{\pi^2} (Z\alpha)^2 \ln \left(\frac{e/\sqrt{\varkappa}}{m_0} \right)^2. \tag{73}$$

The condition $(Z - N_p) = \max = Z_f$ gives the value of the charge Z_f which cannot be suppressed by the effect of pair creation:

$$Z_f = \max \left\{ Z \left[1 - Z \frac{\alpha^2}{\pi^2} \ln \left(\frac{e/\sqrt{\varkappa}}{m_0} \right)^2 \right] \right\} = \frac{\pi^2}{4\alpha^2 \ln \left(\frac{e/\sqrt{\varkappa}}{m_0} \right)^2}, \tag{74}$$

Since $\left(\frac{e/\sqrt{\varkappa}}{m_0} \right)^2 \sim \frac{1}{\alpha}$, we have $Z_f \sim 137$. In other words, the effect of pair creation in such a strong electric field is to decrease the effective charge of the throat to a finite value $Z_f \sim 137$, irrespective of the value of the initial charge Z. *

The independence of the value of the final charge of the arbitrarily large initial charge also follows from Landau's well-known formula [9], which relates the value of a bare charge e_1 localized in a small region to the value of the physically effective charge e to which the effect of vacuum polarization reduces the original charge e_1:

$$e^2 = \frac{e_1^2}{1 + \frac{e_1^2}{3\pi} \ln \left(\frac{\Lambda}{m_0} \right)^2}. \tag{75}$$

For a large value of the charge e_1 or, more precisely, for

$$\frac{e_1^2}{3\pi} \ln \left(\frac{\Lambda}{m_0} \right)^2 \gg 1, \quad e^2 \sim \frac{3\pi}{\ln (\Lambda / m_0)^2}. \tag{76}$$

It is interesting to note that the crudely estimated expression (74) contains the very same characteristic logarithm as Landau's expression and that the argument of the logarithm in (73) gives

$$\Lambda = e / \sqrt{\varkappa} \sim 10^{28} \text{ ev} \tag{77}$$

for the expression introduced by Landau. This is precisely the quantity Λ discussed by Landau in his paper in connection with the possible role of gravitation in the theory of elementary particles. The image of such an object is, even from the point of view of a Schwarzschild observer, extremely complicated. The difficulty is that, at the initial instant of existence of such a system with a large electric charge, the exterior dimensions, which are proportional to the charge, may be very large: $r_h^1 = Z_1 e\sqrt{\varkappa}/c^2$.

* This result is hardly surprising, since, as is well known, the process of real creation of pairs commences for Z > 137 [7, 8].

The pair creation decreases the initial charge (Z_i) to $Z_f \sim 137$; hence,*

$$r_h{}^f \leqslant \frac{137\, e\sqrt{\varkappa}}{c^2} \approx 10^{-30}\ \text{cm}.$$

However, in this region of Z ($Z_f \sim 137$), the shells (around the source of the field) begin to be populated. The shells have radii $\sim \hbar/mc$, where m takes the values of the masses of the particles of the generated pairs. Now hadron particles (for example, protons) have their own intrinsic dimensions. It follows that our system is surrounded by a distinctive atmosphere, which increases its exterior dimensions by 20 orders of magnitude. Be it chance or no, an object whose exterior properties are characteristic of the physics of the microscopic world arises from an object of the cosmological world and the latter persists as the intrinsic content of the object.

The special name "friedmons" was introduced in [10] for objects with these properties.

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* $r_h{}^f < 10^{-30}$ cm, since we have not taken into account the possibility of creation of all types of particles in the foregoing estimates.