Minimal acceleration requirements for "time travel" in Gödel space-time

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It is demonstrated that the total integrated acceleration of any closed timelike curve in Gödel space-time must be at least $\ln(2+\sqrt{5})$. This answers a question posed by Geroch.

I. INTRODUCTION

In Gödel space-time, even though there exist closed timelike curves, there do not exist any closed timelike geodesics.[1] Thus any "time traveler" who would return to an "earlier" point on his own world line must undergo some acceleration, sometime during the trip. The question arises whether there is some minimal amount that is needed.

Let γ be a closed timelike curve.[2] We take its *total (integrated) acceleration* to be

$$\mathrm{TA}(\gamma)=\int_\gamma a\, ds,$$

where s is elapsed proper time along γ, and a is the magnitude of its acceleration. [Thus, if we let ξ^n be the unit tangent to γ, and let $\alpha^n=\xi^m\nabla_m\xi^n$ be its acceleration, then $a=(-\alpha^n\alpha_n)^{1/2}$.] Our question is this. Does there exist some number $k>0$ such that $\mathrm{TA}(\gamma)\geqslant k$ for all closed timelike curves γ in Gödel space-time? [Notice that $\mathrm{TA}(\gamma)$ is invariant under rescaling of the space-time metric.[3] It does not depend on our choice of units for space-time length.]

The simplest closed timelike curves in Gödel space-time ("Gödel circles") exhibit enormous total acceleration. (See Sec. II below.) But it is just possible that a would-be economical "time traveler" can make do with arbitrarily small quantities of total acceleration by properly choosing his navigational strategy. (For example, he might try using large bursts of acceleration for ultrashort periods of proper time, rather than sustaining acceleration over the entire trip. And he might try wandering over large regions of the space-time manifold, rather than staying close to home.)

Chakrabarti, Geroch, and Liang[4] have shown that this possibility can be ruled out *if* the "time traveler" is required to start out at rest relative to the major, field producing, mass points of the Gödel universe. In effect they show that "time travel" is not possible *at all* unless, during at least part of the trip, high relative speed is achieved. If one is starting from a state of relative rest, this is impossible without the accumulation of considerable total acceleration. Their argument establishes that $\mathrm{TA}(\gamma)\geqslant\frac{1}{2}\ln 2$ for all closed timelike curves γ in the restricted class.

We show below in Sec. IV that this bound holds (and can be raised) even if the "time traveler" is allowed arbitrarily large initial relative speed. Thus, $\mathrm{TA}(\gamma)\geqslant\ln(2+\sqrt{5})$ for *all* closed timelike curves in Gödel space-time [$\ln(2+\sqrt{5})$ is approximately 1.44]. This bound can probably be raised still further, but it is not clear by how much. In any case, the answer to our question above is certainly positive.

II. PRELIMINARIES

In this section we list several basic features of Gödel space-time that will be needed later, and then compute the total acceleration for a special class of closed timelike curves.

We start with the following characterization of Gödel space-time (M, g_{mn}). Here, M is the manifold R^4, and g_{mn} is such that for some point (and hence, by homogeneity, any point) p in M, there is a global adapted (cylindrical) coordinate system t, r, φ, y in which $t(p)=r(p)=y(p)=0$ and

$$\begin{aligned}g_{mn}=&(\nabla_m t)(\nabla_n t)-(\nabla_m r)(\nabla_n r)-(\nabla_m y)(\nabla_n y)\\&+(\mathrm{sh}^4 r-\mathrm{sh}^2 r)(\nabla_m\varphi)(\nabla_n\varphi)\\&+2\sqrt{2}\,\mathrm{sh}^2 r(\nabla_{(m}\varphi)(\nabla_{n)}t).\end{aligned}$$

(We shall use sh r and ch r, respectively, to abbreviate sinh r and cosh r.) Here $-\infty<t<\infty$, $-\infty<y<\infty$, $0\leqslant r<\infty$, and $0\leqslant\varphi\leqslant 2\pi$ with $\varphi=0$ identified with $\varphi=2\pi$.

Clearly $(\partial/\partial t)^n$ is a timelike Killing field of unit length. It represents the four-velocity of the major, field-producing, mass points of the universe, and determines a temporal orientation. The integral curves of $(\partial/\partial t)^n$, characterized by constant values for r, φ, and y, will be called *matter lines*.

Here, $(\partial/\partial\varphi)^n$ is a rotational Killing field with squared norm $(\mathrm{sh}^4 r-\mathrm{sh}^2 r)$. It will play an essential role in our argument. The (closed) integral curves of $(\partial/\partial\varphi)^n$, characterized by constant values for t, r, and y, will be called *Gödel circles.*

Given any two points p and q in M, we take the *(radial) distance* from p to q to be the r-coordinate value of q in any cylindrical coordinate system (of the sort above) adapted to p. This distance function is symmetric, and induces a natural geometric structure on all $t=$ const, $y=$ const submanifolds of Gödel space-time. Indeed, the following is true.

(1) Under the radial distance function every $t=$ const, $y=$ const submanifold is a model for the axioms of hyperbolic (i.e., Lobatchevskian) plane geometry.[5]

Given a point p, we take the *critical cylinder* associated with p to be the set of all points whose radial distance from p is less than $r_c=\ln(1+\sqrt{2})$. Since sh $r_c=1$, and the squared norm of $(\partial/\partial\varphi)^n$ is $(\mathrm{sh}^4 r-\mathrm{sh}^2 r)$, (2) follows immediately.

(2) Gödel circles with radius $r=r_c$ are closed null curves. Those with radius $r>r_c$ are closed timelike curves. Thus there exist closed timelike curves fully contained in any arbitrarily small radial expansion of a critical cylinder. But the expansion is essential.

(3) There are no closed timelike curves contained within any critical cylinder. Indeed, within a critical cylinder t is a universal time function (i.e., it increases along all future-di-

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rected timelike curves),[6] and so the cylinder considered as a space-time in its own right is stably causal.

(4) All timelike geodesics through a point p are confined to the critical cylinder associated with p. [Hence, by (3), there are no closed timelike geodesics.]

There is an easy proof of this statement which does not require a prior characterization of all geodesics in Gödel space-time. Since the argument will help to motivate our own proof in Sec. III, we present it here in detail.

Consider any timelike geodesic γ passing through p. Let ξ^n be its unit tangent, and let the function E_φ be defined by

$$E_\varphi = \xi^n\left(\frac{\partial}{\partial\varphi}\right)_n = (\mathrm{sh}^4 r - \mathrm{sh}^2 r)(\xi^n\nabla_n\varphi) + \sqrt{2}\,\mathrm{sh}^2 r(\xi^n\nabla_n t).$$

Here, E_φ must be constant along γ since

$$\xi^n\nabla_n E_\varphi = \xi^n\xi^m\nabla_n\left(\frac{\partial}{\partial\varphi}\right)_m + \xi^n\left(\frac{\partial}{\partial\varphi}\right)_m\nabla_n\xi^m = 0.$$

[The first term vanishes because $(\partial/\partial\varphi)^n$ is a Killing field; the second because γ is a geodesic.] Its constant value must be 0 since γ passes through p.[7]

Now suppose that γ escapes from the critical cylinder associated with p. Let q be the point where it reaches the critical radius r_c. Then at q we have $\mathrm{sh}\, r = 1$ and $E_\varphi = 0$. So $\xi^n\nabla_n t = 0$. But ξ^n is of unit length. So at all points

$$1 = \xi^n\xi_n = (\xi^n\nabla_n t)^2 - (\xi^n\nabla_n r)^2 - (\xi^n\nabla_n y)^2 + (\mathrm{sh}^4 r - \mathrm{sh}^2 r)(\xi^n\nabla_n\varphi)^2 + 2\sqrt{2}\,\mathrm{sh}^2 r(\xi^n\nabla_n\varphi)(\xi^m\nabla_m t).$$

Hence at q

$$1 = -(\xi^n\nabla_n r)^2 - (\xi^n\nabla_n y)^2,$$

which is impossible. ■

Now we do a simple calculation so as to have a numerical value for total acceleration in at least one case.

Lemma 1: A Gödel circle γ with radius $r > r_c$ has total acceleration

$$\pi\,\mathrm{sh}\,2r(2\,\mathrm{sh}^2 r - 1)/(\mathrm{sh}^4 r - \mathrm{sh}^2 r)^{1/2}.$$

Proof: The unit tangent to the circle is $\xi^n = f(\partial/\partial\varphi)^n$, where $f = (\mathrm{sh}^4 r - \mathrm{sh}^2 r)^{-1/2}$. Clearly $\xi^n\nabla_n f = 0$. The acceleration vector α_n is given by

$$\alpha_n = f^2\left(\frac{\partial}{\partial\varphi}\right)^m\nabla_m\left(\frac{\partial}{\partial\varphi}\right)_n = -f^2\left(\frac{\partial}{\partial\varphi}\right)^m\nabla_n\left(\frac{\partial}{\partial\varphi}\right)_m = \frac{-f^2}{2}\nabla_n(\mathrm{sh}^4 r - \mathrm{sh}^2 r) = \frac{-f^2}{2}\mathrm{sh}\,2r(2\,\mathrm{sh}^2 r - 1)\nabla_n r.$$

(For the second equality we have used the fact that $(\partial/\partial\varphi)^n$ is a Killing field.) Hence

$$a = (-\alpha^n\alpha_n)^{1/2} = (f^2/2)\mathrm{sh}\,2r(2\,\mathrm{sh}^2 r - 1).$$

Therefore

$$\mathrm{TA}(\gamma) = \int_\gamma a\, ds = \int_0^{2\pi} a f^{-1}\, d\varphi = \frac{2\pi a}{f},$$

and our claim follows. (In the second equality we have used $d\phi/ds = \xi^n\nabla_n\phi = f$.) ■

Notice that $\mathrm{TA}(\gamma)$ blows up as $r\to\infty$ and $r\to r_c$. A minimal value for total acceleration is reached when r satisfies $\mathrm{sh}^2 r = (1+\sqrt{3})/2$. It comes out to $2\pi(9+6\sqrt{3})^{1/2}$, which is approximately 27.67.

III. AN INEQUALITY

We know from our proof of statement (4) above that the function E_φ (as determined relative to any cylindrical coordinate system) cannot increase along a timelike curve if the curve is a geodesic. A key idea in our proof is that when E_φ *does* increase, its *magnitude* of increase can be used to monitor the accumulation of total acceleration along the curve.

We start with a quite general inequality.[4]

Lemma 2: Let λ^m be a Killing field, not necessarily timelike, in a space-time (M, g_{mn}). Let γ be an arbitrary timelike curve in (M, g_{mn}) with tangent ξ^n, and let $E = \xi^m\lambda_m$. Then

$$|\xi^n\nabla_n E| \leqslant a[E^2 - \lambda^m\lambda_m]^{1/2}.$$

Proof: Direct computation shows

$$\xi^n\nabla_n E = \xi^n\xi^m\nabla_n\lambda_m + \xi^n\lambda_m\nabla_n\xi^m = \lambda_m\alpha^m = h_{mn}\lambda^m\alpha^n,$$

where $h_{mn} = g_{mn} - \xi_m\xi_n$ is the (negative semidefinite) "spatial metric" which projects g_{mn} orthogonal to ξ^m. Hence, by the Schwarz inequality (applied to $-h_{mn}$),

$$|\xi^n\nabla_n E| = |-h_{mn}\lambda^m\alpha^n| \leqslant (-h_{mn}\alpha^m\alpha^n)^{1/2}(-h_{mn}\lambda^m\lambda^n)^{1/2} = a[E^2 - \lambda^m\lambda_m]^{1/2}.$$ ■

We are interested in the case where (M, g_{mn}) is Gödel space-time, λ^n is $(\partial/\partial\varphi)^n$, and E is E_φ. So the inequality comes to

$$|\xi^n\nabla_n E_\varphi| \leqslant a[E_\varphi^2 - (\mathrm{sh}^4 r - \mathrm{sh}^2 r)]^{1/2}.$$

There are two subcases to consider. If $r > r_c$, then $(\mathrm{sh}^4 r - \mathrm{sh}^2 r) > 0$ and the square root term is dominated by $\sqrt{E_\varphi^2}$. If $0 \leqslant r \leqslant r_c$, then $(\mathrm{sh}^4 r - \mathrm{sh}^2 r) \leqslant 0$ and the term assumes a minimal (negative) value of $-\frac{1}{4}$ when $\mathrm{sh}\, r = 1/\sqrt{2}$. So in both cases we have

$$|\xi^n\nabla_n E_\varphi| \leqslant a[E_\varphi^2 + \tfrac{1}{4}]^{1/2}.$$

This is the inequality we shall exploit.

Lemma 3: Let γ be an arbitrary timelike curve in Gödel space-time passing through the point p. Let the rotational Killing field $(\partial/\partial\varphi)^n$ be centered at p, and let q be any point on γ. Then

$$\mathrm{TA}(\gamma) \geqslant \ln[2E_\varphi(q) + (4E_\varphi^2(q) + 1)^{1/2}].$$

Proof: Just integrate

$$\mathrm{TA}(\gamma) \geqslant \int_{\substack{\gamma \\ p \text{ to } q}} \frac{|\xi^n\nabla_n E_\varphi|}{[E_\varphi^2 + \frac{1}{4}]^{1/2}}\, ds \geqslant \int_{\substack{\gamma \\ p \text{ to } q}} \frac{dE_\varphi}{[E_\varphi^2 + \frac{1}{4}]^{1/2}} = \ln[2E_\varphi(q) + (4E_\varphi^2(q) + 1)^{1/2}].$$

IV. THE THEOREM

Now we concentrate attention on *closed* timelike curves in Gödel space-time. Given any one such γ, and any two points p and q on γ, there is a well-defined (radial) distance between p and q. Let the *diameter* of γ be the maximal value of this distance as p and q range over γ. The second key idea in our proof is the demonstration that this diameter cannot be arbitrarily small. Here we invoke statement (3) from Sec. II. We show that if the diameter were less than some minimal value, then the entire curve would have to fall within some critical cylinder; and that is impossible. The only slightly delicate point is that in computing that minimal value we cannot fall back on Euclidean plane geometry. The radial distance function is hyperbolic, not Euclidean.

Lemma 4: Let γ be a closed timelike curve in Gödel space-time with diameter D. Then $\operatorname{ch} D \geqslant (1+\sqrt{5})/2$.

Proof: We are really interested not so much in γ itself, but rather its (possibly self-intersecting) projection in some $t = \text{const}$, $y = \text{const}$ submanifold of Gödel space-time. Let γ' be this projection and let p and q be points on γ' which are maximally distant from one another. [So $d(p,q) = D$, where d is our distance function.] Further let s be the midpoint of the line segment connecting p and q. We show that if $\operatorname{ch} D < (1+\sqrt{5})/2$, then γ' is fully contained in the (open) disk of radius r_c centered at s. It will follow that γ itself is contained in the critical cylinder which has this disk as its projection, and we shall be done. (See Fig. 1.)

Let u be any point on γ'. Then $d(p,u) \leqslant D$ and $d(q,u) \leqslant D$. The angles $\measuredangle psu$ and $\measuredangle qsu$ cannot both be acute. Without loss of generality assume the former is not. By the counterpart to the "law of cosines" which holds in hyperbolic plane geometry[8] we have

$$\operatorname{ch}[d(p,u)] = \operatorname{ch}[d(p,s)]\operatorname{ch}[d(s,u)] - \operatorname{sh}[d(p,s)]\operatorname{sh}[d(s,u)]\cos\measuredangle psu.$$

Since $\cos\measuredangle psu \leqslant 0$ it follows that

$$\operatorname{ch}[d(s,u)] \leqslant \frac{\operatorname{ch}[d(p,u)]}{\operatorname{ch}[d(p,s)]} \leqslant \frac{\operatorname{ch} D}{\operatorname{ch} D/2} = \frac{\sqrt{2}\operatorname{ch} D}{[\operatorname{ch} D + 1]^{1/2}}.$$

But now if $\operatorname{ch} D < (1+\sqrt{5})/2$, then $\operatorname{ch}[d(s,u)] < \sqrt{2}$ and we may conclude that $d(s,u) < r_c$.[9] ■

Our proposition is a simple consequence of Lemmas 3 and 4. All we need is the fact that given any two timelike vectors λ^m, μ^m at a point (in any space-time), $\gamma^m \mu_m \geqslant (\lambda^m \lambda_m)^{1/2} (\mu^m \mu_m)^{1/2}$.

Proposition: Let γ be a closed timelike curve in Gödel space-time. Then

$$\mathrm{TA}(\gamma) \geqslant \ln(2+\sqrt{5}).$$

Proof: Let p and q be any two points on γ which are maximally distant from one another. Consider a cylindrical coordinate system adapted to p. By Lemma 4 the r coordinate of q satisfies $\operatorname{ch} r \geqslant (1+\sqrt{5})/2$. Hence $(\operatorname{sh}^4 r - \operatorname{sh}^2 r)^{1/2} \geqslant 1$, and q falls outside the critical cylinder centered at p. Since $(\partial/\partial\varphi)^n$ is timelike at q, it must be the case at that point that

$$E_\varphi = \xi^n \left(\frac{\partial}{\partial\varphi}\right)_n \geqslant \left[\left(\frac{\partial}{\partial\varphi}\right)^n \left(\frac{\partial}{\partial\varphi}\right)_n\right]^{1/2} = (\operatorname{sh}^4 r - \operatorname{sh}^2 r)^{1/2} \geqslant 1.$$

Our claim now follows from Lemma 3. ■

It is important that the point p in our proof need not be the one point on γ where there is a kink (if there is one at all).[2] Even if it is not, at least one of the connecting segments of γ between p and q must be smooth, and Lemma 3 can be applied to that one. If γ is smooth everywhere, then the argument can be applied *twice*, once on each segment, and the lower bound on $\mathrm{TA}(\gamma)$ can be raised by another factor of 2.

We can get some sense for magnitudes of total acceleration by considering another inequality[4] involving "fuel consumption." Suppose a point particle "rocket ship" traverses a timelike curve γ. Suppose its (rest) mass at any point is m. Then $\xi^n \nabla_n m \leqslant 0$ (since the rocket uses up fuel during the trip). Let J^n be the energy momentum of the rocket's exhaust. Assuming that the rocket is suitably isolated, J^n must balance precisely the rate at which the rocket itself loses energy momentum. So

$$J^n = -\xi^p \nabla_p (m \xi^n) = -[\xi^n (\xi^p \nabla_p m) + m\alpha^n].$$

Now J^n must be causal, i.e., $J^n J_n \geqslant 0$. Therefore,

$$(\xi^p \nabla_p m)^2 - m^2 a^2 \geqslant 0.$$

Since $(\xi^p \nabla_p m) \leqslant 0$ it follows that

$$a \leqslant -\xi^p \nabla_p (\ln m).$$

If m_i and m_f are, respectively, the initial and final mass of the rocket, then (by integration)

$$\mathrm{TA}(\gamma) \leqslant \ln(m_i/m_f).$$

This is the inequality we were looking for. It gives us a lower bound on that percent of the rocket's initial mass which must be in the form of fuel. Since $m_f - m_i$ is the fuel expended during the trip, we have

$$\text{Percent of initial mass as fuel} \geqslant \frac{m_i - m_f}{m_i} \geqslant 1 - \frac{1}{e^{\mathrm{TA}(\gamma)}}.$$

If $\mathrm{TA}(\gamma) = \ln(2+\sqrt{5})$ then the percent must already be greater than 76%. If $\mathrm{TA}(\gamma) = 2\pi(9+6\sqrt{3})^{1/2}$ (recall our calculation for Gödel circles), then the percent cannot differ from 100% by more than 2×10^{-12}.

We close by mentioning explicitly several questions which our discussion leaves open.

(a) Is there any closed timelike curve in Gödel space-time with total acceleration less than $2\pi(9+6\sqrt{3})^{1/2}$?

If the answer is yes, then we have the following questions.

(b) What is the greatest lower bound of $\mathrm{TA}(\gamma)$ as γ ranges over all closed timelike curves?

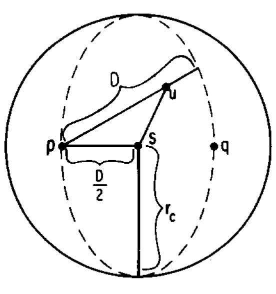

FIG. 1. Figure for Lemma 4.

(c) Is that lower bound realized?

(d) What do the curves look like which realize or approach the bound?

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[1]See, for example, K. Gödel, Rev. Mod. Phys. **21**, 447 (1949); W. Kundt, Z. Phys. **145**, 611 (1956); S. Chandrasekhar and J. P. Wright, Proc. Natl. Acad. Sci. **47**, 341 (1961); H. Stein, Philos. Sci. **37**, 589 (1970); J. Pfarr, Gen. Relativ. Gravit. **13**, 1073 (1981).

[2]In what follows, "timelike curves" will be taken to be smooth everywhere unless they are closed, in which case smoothness will be allowed to fail at initial (= terminal) points.

[3]By means of such a rescaling we can always make the maximal value of a along γ as small as we like. The point is that any such "saving" is exactly balanced by a corresponding increase in elapsed proper time along γ.

[4]S. Chakrabarti, R. Geroch, and G. Liang, J. Math. Phys. **24**, 597 (1983).

[5]Let $h_{mn} = g_{mn} - (\partial/\partial t)_m(\partial/\partial t)_n$ be the (negative semidefinite) metric which results from projecting g_{mn} orthogonal to $(\partial/\partial t)^m$. Since $(\partial/\partial t)_m = \nabla_m t + \sqrt{2}\,\mathrm{sh}^2 r \nabla_m \varphi$, we have

$$-h_{mn} = (\nabla_m r)(\nabla_n r) + \tfrac{1}{4}\mathrm{sh}^2 2r(\nabla_m \varphi)(\nabla_n \varphi).$$

Let S be any $t = \text{const}$, $y = \text{const}$ submanifold, and construe $-h_{mn}$ as a (positive definite) metric on S. It suffices for us to show that $(S, -h_{mn})$ is a complete Riemannian manifold of constant negative curvature. (The value of curvature is $-\frac{1}{4}$.) There are various ways to do this. One is the following. Consider new coordinates on S defined by

$$x_1 = \tfrac{1}{2}\mathrm{ch}\, 2r, \quad x_2 = \tfrac{1}{2}\mathrm{sh}\, 2r \cos\varphi, \quad x_3 = \tfrac{1}{2}\mathrm{sh}\, 2r \sin\varphi.$$

For all r and φ we have $x_1 > 0$ and $x_1^2 - x_2^2 - x_3^2 = \frac{1}{4}$. Furthermore, in these coordinates the metric assumes the form

$$-h_{mn} = -(\nabla_m x_1)(\nabla_n x_1) + (\nabla_m x_2)(\nabla_n x_2) + (\nabla_m x_3)(\nabla_n x_3).$$

Therefore, $(S, -h_{mn})$ is isometric to the upper half of a two-sheeted hyperboloid of radius $\frac{1}{2}$ in R^3, with respect to the metric induced on the latter by a background flat metric of Lorentz signature. It is a standard result that this hyperboloid (under the induced metric) is a complete Riemannian manifold of constant curvature $-\frac{1}{4}$. [See, for example, J. Wolf, *Spaces of Constant Curvature* (Publish or Perish, Boston, 1974), Chap. 2.]

[6]It must be shown that $\nabla_m t$ is timelike and future directed within a critical cylinder. That is easy. The inverse to g_{mn} is given by

$$g^{mn} = \frac{1}{(\mathrm{sh}^4 r + \mathrm{sh}^2 r)}\Big[-(\mathrm{sh}^4 r - \mathrm{sh}^2 r)\Big(\frac{\partial}{\partial t}\Big)^m\Big(\frac{\partial}{\partial t}\Big)^n - (\mathrm{sh}^4 r + \mathrm{sh}^2 r)\Big(\frac{\partial}{\partial r}\Big)^m\Big(\frac{\partial}{\partial r}\Big)^n - (\mathrm{sh}^4 r + \mathrm{sh}^2 r)\Big(\frac{\partial}{\partial y}\Big)^m\Big(\frac{\partial}{\partial y}\Big)^n - \Big(\frac{\partial}{\partial \varphi}\Big)^m\Big(\frac{\partial}{\partial \varphi}\Big)^n + 2\sqrt{2}\mathrm{sh}^2 r\Big(\frac{\partial}{\partial \varphi}\Big)^{(m}\Big(\frac{\partial}{\partial t}\Big)^{n)}\Big],$$

and hence

$$(\nabla_m t)(\nabla^m t) = (1 - \mathrm{sh}^2 r)/(1 + \mathrm{sh}^2 r).$$

So clearly $\nabla_n t$ is timelike if and only if $r < r_c$. Also, $\nabla_n t$ is future directed within the cylinder since $(\partial/\partial t)^n(\nabla_n t) = 1$.

[7]The angular coordinate ϕ is not defined at p, but that does not matter. The vector $(\partial/\partial\varphi)^n$ goes to the zero vector as p is approached, and E_φ goes to 0.

[8]See almost any book on non-Euclidean geometry; e.g., W. T. Fishback, *Projective and Euclidean Geometry* (Wiley, New York, 1969), p. 257.

[9]Of course the value $(1 + \sqrt{5})/2$ was obtained by working this computation backwards.