The class of continuous timelike curves determines the topology of spacetime*

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The title assertion is proven, and two corollaries are established. First, the topology of every past and future distinguishing spacetime is determined by its causal structure. Second, in *every* spacetime the path topology of Hawking, King, and McCarthy codes topological, differential, and conformal structure.

1. SUMMARY

Suppose one has two spacetimes (M,g) and (M',g') together with a bijection f: $M \to M'$, where both f and f^{-1} preserve continuous timelike curves; i.e., if γ: $I \to M$ is a continuous timelike curve in (M,g), then $f\circ\gamma$: $I \to M'$ is a continuous timelike curve in (M',g'); and symmetrically for f^{-1}. We show that f must be a homeomorphism. In this sense the class of continuous timelike curves in spacetime determines its topology.

The result is of interest because, at least in some sense, we directly experience whether events on our worldlines are "close" or not. That experience alone, it appears, allows a complete determination of topological structure. The result also has two consequences which are of independent interest.

It is well known that in all strongly causal spacetimes the Alexandroff topology is equal to the manifold topology.[1] Hence, at least in strongly causal spacetimes, if one knows of all points p and q whether it is possible that a particle travel from p to q, then one can recover the topology of spacetime. The question naturally arises whether the condition of strong causality is necessary for this recovery. We show that it is not. The weaker condition of past and future distinguishability suffices. One has the following result: If (M,g) and (M',g') are past and future distinguishing spacetimes and if f: $M \to M'$ is a causal isomorphism (i.e., a bijection where both f and f^{-1} preserve the causal connectibility relation $\ll$), then f must be a homeomorphism. But we also show that the assertion becomes false if the hypothesis of past *and* future distinguishability is relaxed to that of future distinguishability (or past distinguishability) alone.

A second consequence of our theorem is an improvement of a result of Hawking, King, and McCarthy.[2] They define a *path topology* on spacetimes and prove that, in the presence of strong causality, the path topology "codes" (standard) topological, differential, and conformal structure. We show that their hypothesis of strong causality is unnecessary. Indeed their result is true of *all* spacetimes.

2. STANDARD DEFINITIONS AND RESULTS

In what follows a spacetime (M,g) is taken to be a connected, four-dimensional smooth manifold without boundary M, together with a smooth pseudo-Riemannian metric of Lorentz signature g. Spacetimes are assumed to be temporally orientable and endowed with a particular temporal orientation.

Given subsets A and O of M with O open, $I^+(A,O)$ is the set of points q in O such that there exists a future directed smooth timelike curve γ: $I\to O$ (where $I \subseteq R$ is connected) and points $t_1, t_2 \in I$ such that $t_1 < t_2$, $\gamma(t_1)\in A$, and $\gamma(t_2)=q$. $I^+(A,O)$ is called the *chronological future of A relative to O. The causal future of A relative to O*, $J^+(A,O)$, is the union of $A\cap O$ with the set of points q in O such that there exists a future directed smooth causal curve (i.e., a smooth curve whose tangent vectors are everywhere nonvanishing, nonspacelike, and future directed) γ: $I\to O$ and points $t_1,t_2\in I$ such that $t_1<t_2$, $\gamma(t_1)\in A$, and $\gamma(t_2)=q$. Finally, the *horismos future of A relative to O*, $E^+(A,O)$, is the set $J^+(A,O) - I^+(A,O)$. These sets have duals $I^-(A,O)$, $J^-(A,O)$, and $E^-(A,O)$ which are defined analogously (substitute past directed curves for future directed curves). $I(A,O)$ is the union $I^+(A,O)\cup I^-(A,O)$. The sets $J(A,O)$ and $E(A,O)$ are defined similarly.

If $A=\{p\}$, we write $I^+(p,O)$ instead of $I^+(A,O)$ and $I^+(p)$ instead of $I^+(p,M)$. Similarly for the other I, J, E sets. The relations $q\in I^+(p,O)$, $q\in J^+(p,O)$, and $q\in E^+(p,O)$ will sometimes be written as $p\ll q(O)$, $p<q(O)$, $p\to q(O)$. Furthermore, $p\ll q(M)$, $p<q(M)$, and $p\to q(M)$ will sometimes be written as $p\ll q$, $p<q$, and $p\to q$.

The I, J, E sets have the following basic properties.[3] If $q\in I^+(p,O)$, then $p\in I^-(q,O)$ and conversely (similarly for the J and E sets). Both $I^+(p,O)$ and $I^-(p,O)$ are open. If $p\ll q(O)$ and $q<r(O)$, then $p\ll r(O)$. Similarly, if $p<q(O)$ and $q\ll r(O)$, then $p\ll r(O)$. If $p\to q(O)$, then, if γ: $[0,1]\to 0$ is a future directed smooth causal curve with $\gamma(0)=p$ and $\gamma(1)=q$, γ must be a null geodesic.

An open set O is *convex* iff given any two points p and q in O there is a geodesic γ: $[0,1]\to 0$ with $\gamma(0)=p$, $\gamma(1)=q$ and γ is unique (up to reparametrization). If O is an open convex set, then, for all points p in O, $J^+(p,O)=\mathrm{Cl}[I^+(p,O)]=$ the closure in O of $I^+(p,O)$; and $E^+(p,O)=\mathrm{Bnd}[I^+(p,O)]=$ the boundary of $I^+(p,O)$ in O. {These assertions are false in general if O is not convex. But $J^+(p,O)\subseteq \mathrm{Cl}[I^+(p,O)]$ and $E^+(p,O)\subseteq \mathrm{Bnd}[I^+(p,O)]$ are always true.} Dual assertions hold for J^- and E^-. The open convex sets form a basis for the manifold topology; i.e., given any point p and any open set U containing p, there is an open convex set O with $p\in O\subseteq U$.

A set A is *achronal* in O iff for all points p and q in $A\cap O$, it is not the case that $p\ll q(O)$.

A spacetime (M,g) is *chronological* iff it admits no closed, future-directed smooth timelike curves. (M,g) is *causal* iff it admits no closed, future-directed smooth causal curves.

A spacetime (M,g) is *future* (resp. *past*) *distinguishing* iff for all p and q: $I^+(p)=I^+(q) \Rightarrow p=q$ (resp. $I^-(p)=I^-(q) \Rightarrow p=q$). Equivalently, (M,g) is future (resp. past) distinguishing iff for all p in M and all open sets O containing p, there exists an open set O_1 with $p \in O_1 \subseteq O$ such that no future (resp. past) directed smooth timelike curve through p which leaves O_1 ever returns to it.

Finally, a spacetime is *strongly causal* iff, for all points p and all open sets O containing p, there exists an open set O_1 with $p \in O_1 \subseteq O$ such that no future directed smooth timelike curve which leaves O_1 (whether or not if passes through p) ever returns to O_1.

If (M,g) is a spacetime and $O \subseteq M$ is a connected open set, then we may think of $(O, g_{|O})$ as a spacetime in its own right. If O is convex, $(O, g_{|O})$ is necessarily strongly causal.

These "causality conditions" can be ordered in terms of (strictly) increasing strength:

strong causality
⇓
future *and* past distinguishability
⇓
future (or past) distinguishability
⇓
causality
⇓
chronology

The respective converse implications are all false.

If (M,g) is a spacetime, the *Alexandroff topology on* M, $\mathcal{T}_A$, is the coarsest topology on M in which all sets $I^+(p)$ and $I^-(q)$ are open. The collection of all sets of form $I^+(p) \cap I^-(q)$ form a basis for $\mathcal{T}_A$. If $\mathcal{T}$ is the (standard) manifold topology on M, then it is always true that $\mathcal{T}_A \subseteq \mathcal{T}$. But the condition $\mathcal{T}_A = \mathcal{T}$ is equivalent to strong causality. Suppose (M,g) is strongly causal. Then the condition that a set $A \subseteq M$ be open (in $\mathcal{T}$) is explicitly definable in terms of the relation $\ll$: A is open iff, for all points p in A, there exist points r and s in A such that $p \in I^+(r) \cap I^-(s) \subseteq A$.

Given two spacetimes (M,g) and (M',g'), a bijection f: $M \to M'$ is a *smooth isometry* iff f and f^{-1} are smooth, and $f_*(g) = g'$. f is a *smooth conformal isometry* iff f and f^{-1} are smooth, and there is a smooth nonvanishing map Ω: $M' \to R$ such that $f_*(g) = \Omega^2 g'$.

So far "causal structure" has been developed entirely in terms of smooth curves. For our purposes it is essential to work with the larger class of continuous curves. Suppose $\gamma: I \to M$ is a continuous curve. We say that γ is *future directed* and *timelike* iff, for all $t_0 \in I$ and all open convex sets O containing $f(t_0)$, there exists an open (i.e., open in the relative topology on I) subinterval $\bar{I} \subseteq I$ containing t_0 such that

$$\left.\begin{array}{l} t \in \bar{I} \text{ and } t < t_0 \Rightarrow \gamma(t) \ll \gamma(t_0) \quad (O), \\ t \in \bar{I} \text{ and } t_0 < t \Rightarrow \gamma(t_0) \ll \gamma(t) \quad (O). \end{array}\right\} \quad (*)$$

We say that γ is *future directed* and *causal* iff the above condition obtains but with $\ll$ replaced by $<$ in (*). Finally, we say that γ is a *future directed null geodesic* iff the above condition obtains but with (*) replaced by

$$t_1, t_2 \in \bar{I} \text{ and } t_1 < t_2 \Rightarrow \gamma(t_1) \to \gamma(t_2) \quad (O).$$

Note that every future directed continuous null geodesic can be reparametrized so as to become a (smooth) future directed null geodesic. (The corresponding assertions for continuous timelike and causal curves are false.) Dual definitions can be given for past directed continuous timelike (causal, null geodesic) curves.

The sets $I^+(A,O)$, $J^+(A,O)$, $E^+(A,O)$ could be redefined in terms of continuous curves, but doing so would not affect the resultant point sets. For example, $p \ll q(O)$ (according to our definition involving smooth timelike curves) iff there is a future directed continuous timelike curve $\gamma: I \to C$ and points $t_1, t_2 \in I$ with $t_1 < t_2$, $\gamma(t_1) = p$, and $\gamma(t_2) = q$.

When there is no chance of confusion we shall not distinguish between curves $\gamma: I \to M$ and their point set images $\gamma[I]$. Also, we shall sometimes refer, simply, to continuous (causal, null geodesic) curves and it should be understood that the curves are *either* future or past directed.

3. FROM TOPOLOGICAL STRUCTURE TO DIFFERENTIAL AND CONFORMAL STRUCTURE

We shall prove that the class of future directed continuous timelike curves determines the topology of spacetime. Having done so, it will follow automatically that this class of curves also determines the differential and conformal structure of spacetime. This is all that one can hope for since all conformally equivalent Lorentz metrics on a manifold induce the same continuous timelike curves.

That differential and conformal structure will follow on the heels of topological structure is a consequence of:

Hawking's theorem[4]: Suppose (M,g) and (M',g') are spacetimes and f: $M \to M'$ is a homeomorphism where both f and f^{-1} preserve future directed continuous null geodesics. Then f is a smooth conformal isometry.

To avail ourselves of this result, we need a simple lemma.

Lemma 1: Suppose (M,g) and (M',g') are spacetimes and f: $M \to M'$ is a homeomorphism where both f and f^{-1} preserve future directed continuous timelike curves. Then both f and f^{-1} preserve future directed continuous null geodesics.

Proof: It suffices to observe that the future directed continuous null geodesics of a spacetime (M,g) can be characterized in terms of its future directed continuous timelike curves and its topology.

First, given any open set U and points p,q in U, we have that $q \in \text{Bnd}[I^+(p,U)]$ iff for all future directed continuous timelike curves σ: $(0,1) \to U$, if $\sigma(t_0) = q$ for some t_0 where $0 < t_0 < 1$, then there exist t_1, t_2 where $0 < t_1 < t_0 < t_2 < 1$ such that $\sigma(t_1) \notin I^+(p,U)$, but $\sigma(t_2) \in I^+(p,U)$.

Next, note that if $\gamma: I \to M$ is a continuous curve, then γ is a future directed null geodesic iff for all $t_0 \in I$ and all open sets O containing $\gamma(t_0)$, there exists an open set $U \subseteq O$ containing $\gamma(t_0)$ such that for all $t_1, t_2 \in I$ with $t_1 < t_2$, if $\gamma(t_1)$, $\gamma(t_2) \in U$ then $\gamma(t_2) \in \mathrm{Bnd}[I^+(\gamma(t_1), U)]$./

4. THE PRINCIPAL RESULT AND ITS CONSEQUENCES

Theorem 1: Suppose (M,g) and (M',g') are spacetimes and $f: M \to M'$ is a bijection where both f and f^{-1} preserve future directed continuous timelike curves. Then f is a homeomorphism. (By Hawking's theorem f must also be a smooth conformal isometry.)

A proof of the theorem is given in the next section.

As it is stated, the hypothesis of the theorem is slightly stronger than necessary. It suffices that f and f^{-1} take (past or future directed) continuous timelike curves to (past or future directed) continuous timelike curves.[5] This follows immediately from the following lemma.

Lemma 2: Suppose (M,g) and (M',g') are spacetimes and $f: M \to M'$ is a bijection. Suppose further that both f and f^{-1} preserve continuous timelike curves. Then either: (a) Both f and f^{-1} preserve future directed continuous timelike curves, or (b) both f and f^{-1} take future directed continuous timelike curves to past directed continuous timelike curves.

Proof: Let p be any point in M. Suppose there are future directed continuous timelike curves γ and σ through p such that $f\circ\gamma$, but not $f\circ\sigma$, is future directed in (M',g'). Let γ^- be the "lower segment" of γ with future end point p. Let σ^+ be the "upper segment" of σ with past end point p. Then the continuous timelike curve which results from "linking" γ^- with σ^+ is one whose image under f is not a continuous timelike curve at all. This is impossible. So at least as restricted to continuous timelike curves through some particular point in M, f either systematically preserves or systematically reverses orientation.

Let A (resp. B) be the set of points in M at which f preserves (resp. reverses) orientation. We show A is open. Suppose p is in A and $p \ll q$ for some point q. Then there is an open set O with $p \in O \subseteq I^-(q)$. Let γ be a future directed continuous timelike curve with initial point p and terminal point q. Suppose now there is a point $r \in O \cap B$. Let σ be any future directed continuous timelike curve with initial point r and terminal point q. Then the result of linking γ with σ is not a continuous timelike curve, but its image under f *is* a continuous timelike curve. This is impossible since f^{-1} preserves continuous timelike curves. Therefore, $O \subseteq A$ and so A is open as claimed. A symmetric argument establishes that B is open.

It thus follows that f either systematically preserves or systematically reverses the orientation of continuous timelike curves. The same argument applies to f^{-1} and, of course, f preserves orientation iff f^{-1} does too./

We consider now the question whether the topological structure of spacetime can be recovered from its causal structure. Rather than thinking of the topological, differential, and conformal structure of spacetime as given and abstracting a causal connectibility relation $\ll$, we ask if the construction can be turned "on its head" with the relation $\ll$ construed as primitive. It turns out that it can be if the spacetime in question is sufficiently well behaved in its causal structure. "Sufficiently well behaved" means "at least past and future distinguishing."

If (M,g) and (M',g') are spacetimes, a map $f: M \to M'$ is a *causal isomorphism* iff f is a bijection and for all points p and q in M: $p \ll q \Longleftrightarrow f(p) \ll f(q)$. Our result follows from the following lemma.

Lemma 3: Suppose (M,g) and (M',g') are past and future distinguishing spacetimes and that $f: M \to M'$ is a causal isomorphism. Then f and f^{-1} preserve future directed continuous timelike curves.

Proof: Suppose $\gamma: I \to M$ is an arbitrary future directed continuous timelike curve in (M,g). Suppose $p = \gamma(t_0)$ with $t_0 \in I$, and suppose O' is an arbitrary open convex set containing $f(p)$. We must show that there exists an open subinterval $\bar{I} \subseteq I$ with $t_0 \in \bar{I}$ such that

$$t \in \bar{I} \text{ and } t < t_0 \Rightarrow (f\circ\gamma)(t) \ll f(p) \quad (O'),$$

$$t \in \bar{I} \text{ and } t_0 < t \Rightarrow f(p) \ll (f\circ\gamma)(t) \quad (O'). \qquad (*)$$

Since (M',g') is future distinguishing, there is an open set U' with $f(p) \in U' \subseteq O'$ such that no future directed timelike curve from $f(p)$ which leaves U' ever reenters. Let $f(q)$ be any point in $I^+(f(p), U')$. Since $f(p) \ll f(q)$, we must have $p \ll q$. So there must exist an open convex set O with $p \in O \subseteq I^-(q)$. Since γ is a future directed continuous timelike curve, there must exist an open subinterval $\bar{I}_1 \subseteq I$ with $t_0 \in \bar{I}_1$ such that

$$t \in \bar{I}_1 \text{ and } t_0 < t \Rightarrow p \ll \gamma(t) \quad (O).$$

We claim now that

$$p \ll \gamma(t) \quad (O) \Rightarrow f(p) \ll (f\circ\gamma)(t) \quad (O').$$

For, if $p \ll \gamma(t)(O)$, we have $p \ll \gamma(t) \ll q$. Hence $f(p) \ll (f\circ\gamma)(t) \ll f(q)$. So there exists a future directed smooth timelike curve through $f(p)$, $(f\circ\gamma)(t)$, and $f(q)$ in sequence. We know that this curve cannot leave U' between $f(p)$ and $f(q)$. So we must have $(f\circ\gamma)(t) \in I^+(f(p), U') \subseteq I^+(f(p), O')$.

A parallel argument using past distinguishability of (M',g') establishes that there is an open subinterval $\bar{I}_2 \subseteq I$ with $t_0 \in \bar{I}_2$ such that:

$$t \in \bar{I}_2 \text{ and } t < t_0 \Rightarrow (f\circ\gamma)(t) \ll f(p) \quad (O').$$

Hence the set $\bar{I} = \{t \in \bar{I}_1 / t \geq t_0\} \cup \{t \in \bar{I}_2 / t \leq t_0\}$ is an open subinterval of I with $t_0 \in \bar{I}$ which satisfies (*)./

Thus we have

Theorem 2: Suppose (M,g) and (M',g') are past and future distinguishing spacetimes and $f: M \to M'$ is a causal isomorphism. Then f is a homeomorphism. (By Hawking's theorem f must also be a smooth conformal isometry.)

As was the case with Theorem 1, Theorem 2 can be recast so as to be completely "time symmetric" in formulation.[5] Let τ be the symmetric causal connectibility relation on spacetime points defined by $p\tau q \Longleftrightarrow p \ll q$ or $q \ll p$. Given two spacetimes (M,g) and (M',g') a map $f: M \to M'$ is a *symmetric causal isomorphism* iff f is a bijection and for all points p and q in M: $p\tau q \Longleftrightarrow f(p)\tau f(q)$. To recast Theorem 2 in symmetric form, it suffices to prove the following lemma and invoke Lemma 2.

Lemma 4: Suppose (M,g) and (M',g') are past and future distinguishing spacetimes and that $f: M \to M'$ is a symmetric causal isomorphism. Then f and f^{-1} preserve continuous timelike curves.

One proves the lemma by compounding the constructions of Lemmas 2 and 3. We skip the argument as it is somewhat tedious and involves no new ideas.

The following example shows that the hypothesis of past and future distinguishability in Theorem 2 (and hence Lemma 3) cannot be relaxed to either future distinguishability or past distinguishability alone. We give the example in a two-dimensional version to simplify matters.

Start with the two-dimensional plane carrying a metric:

$$ds^2 = (\cosh t - 1)^2(dt^2 - dx^2) + dt\,dx$$

with respect to global Cartesian coordinates t, x. Next form a vertical cylinder by identifying the point $(t,0)$ with all points $(t,2n)$ for all n. Finally excise two closed half-lines: $\{(t,x): x=0 \text{ and } t \geq 0\}$ and $\{(t,x): x=1 \text{ and } t \geq 0\}$ (see Fig. 1.) Along the "equator" $t=0$ the metric reduces to the form $ds^2 = dt\,dx$ and its associated null cones are horizontal, pointing in the direction of increasing x. But as $|t| \to \infty$, the cones "tip to the left" and asymptotically approach the upright position they have in Minkowski spacetime. Because of the excisions the spacetime is future distinguishing. But it is not past distinguishing. Every point on the $t=0$ equator has for its chronological past the entire region of the spacetime falling below the equator.

Now let f be a bijection of the spacetime onto itself defined by

$$f: (t,x) \to \begin{matrix} (t,x) & \text{if } t<0, \\ (t,x+1) & \text{if } t \geq 0. \end{matrix}$$

f leaves the "lower open half" of the spacetime fixed, but reverses the position of the two upper slabs. f is surely discontinuous along the $t=0$ equator; it "cuts" continuous timelike curves which cross the equator.

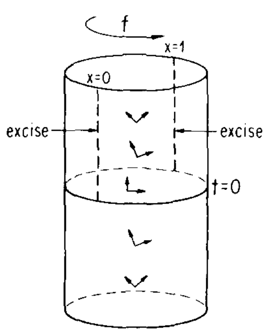

FIG. 1.

But f is a causal isomorphism. The important thing to notice here is that every point below the $t=0$ equator has all points in both upper slabs in its chronological future.

This establishes that the condition in the hypothesis of Theorem 2 cannot be relaxed to future distinguishability. A symmetric example (with excisions below the $t=0$ equator) shows that it cannot be relaxed to past distinguishability either.

Finally, we use Theorem 1 to generalize a result of Hawking, King, and McCarthy.[2] They define the *path topology* on a spacetime to be the finest topology which induces on all continuous timelike curves the same topology induced on them by the standard manifold topology. Equivalently, if (M,g) is a spacetime with $A \subseteq M$, A is open in the path topology on M iff given any continuous timelike curve $\gamma: I \to M$ there exists a (standard) open set O such that $\gamma[I] \cap A = \gamma[I] \cap O$. Their interest in the new topology is motivated in part by the belief that, in some sense, we "experience" continuity along future directed continuous timelike curves. The standard topology, they claim, has no comparable physical significance.

Hawking, King, and McCarthy prove that given any strongly causal spacetime (M,g), if $f: M \to M$ is a homeomorphism with respect to the path topology, then f must be a smooth conformal isometry. But along the way they prove the following:

Lemma 5[2]: If (M,g) is a spacetime and $f: M \to M$ is a homeomorphism with respect to the path topology, then both f and f^{-1} preserve continuous timelike curves.

Thus it follows immediately that we have

Theorem 3: If (M,g) is an arbitrary spacetime and $f: M \to M$ is a homeomorphism with respect to the path topology, then f is a smooth conformal isometry.

One can easily reformulate the theorem so as to be parallel in form to Theorems 1 and 2. One simply takes $f: M \to M'$ to be a path topology homeomorphism between arbitrary spacetimes (M,g) and (M',g'). The conclusion is affected not at all.

5. PROOF OF THEOREM 1

If it were assumed that f preserves *all* continuous curves, it would follow immediately that f is continuous. Given any sequence $\{p_i\}$ converging to p, one could find a continuous curve "threading" all the p_i in sequence and then p. Its image would have to be a continuous curve threading all the $f(p_i)$ in sequence and then $f(p)$. Hence $\{f(p)\}$ would have to converge to $f(p)$. Under our hypotheses, however, this construction can only cope with sequences $\{p_i\}$ which converge chronologically to p. The problem is with those sequences $\{p_i\}$ which converge to p but are locally spacelike related to p.

Our proof is rather long and so is divided into a sequence of lemmas. The crucial idea is this: To show that f is continuous at p, one proves that one may as well assume that f is continuous over a nice-looking region near p (Lemma E). Then one uses continuous null geodesic segments in that "safe region" to characterize the convergence of points to p. This does the trick because (by Lemma 1 above) continuous null geodesics in the safe region are necessarily preserved by f.

In what follows $\mathcal{D}$ (resp. $\mathcal{D}'$) is taken to be the set of points at which f (resp. f^{-1}) is discontinuous.

Lemma A: If O is an open set in M, O' is an open convex set in M', and $f[O] \subseteq O'$, then $O \subseteq M - \mathcal{D}$.

Proof: Let p be any point in O. To show f is continuous at p, it suffices to show that given any open set U' containing $f(p)$, $f^{-1}[O' \cap U']$ is open in M. Since O' is convex, the spacetime $(O', g_{|O'})$ is strongly causal. So the Alexandroff topology on O' is equal to the relative manifold topology induced in O'. Thus $U' \cap O'$ is open in the Alexandroff topology on O'. But $f_{|O}: O \to O'$ is certainly continuous with respect to the Alexandroff topologies on O and O'. So $f^{-1}[U' \cap O']$ must be open in the Alexandroff topology on O. *A fortiori* $f^{-1}[U' \cap O']$ is open in (the manifold topology on) M./

Lemma B: Given p in M, there is an open set O in M containing p such that $I(p, O) \subseteq M - \mathcal{D}$. (So f is at least continuous over "local futures and pasts.")

Proof: Let O' be an open convex set containing $f(p)$. We show first that there is an open set O containing p such that $f[I^+(p, O)] \subseteq O'$.

Suppose there is no such O. Then given any open O_1 containing p there must be a point p_1 in O_1 such that $p_1 \in I^+(p, O_1)$ but $f(p_1) \notin O'$. Since $I^-(p_1, O_1)$ is open, we can find an open set $O_2 \subseteq O_1$ containing p such that $O_2 \subseteq I^-(p_1, O_1)$. There must exist a p_2 in O_2 such that $p_2 \in I^+(p, O_2) \subseteq I^+(p, O_1)$ but $f(p_2) \notin O'$. Clearly $p_2 \ll p_1 (O_1)$. Continuing in this way, we can generate a nested sequence of open sets $O_1 \supseteq O_2 \supseteq O_3 \cdots$ all containing p, and a sequence of points $\{p_i\}$ where, for all i, $p_i \in O_i$, $p_{i+1} \ll p_i(O_i)$, $p \ll p_i(O_i)$, but $f(p_i) \notin O'$ (see Fig. 2). Furthermore, we may choose the $\{O_i\}$ so that they converge to p (i.e., so that their intersection is $\{p\}$). Now we can certainly join p_{i+1} to p_i with a continuous future directed timelike curve segment γ_i contained in O_i. Linking these segments together and adjoining the point p, we obtain a future directed continuous timelike curve γ through p which "threads" all the p_i. By our construction no initial segment of $f \circ \gamma$ can intersect O'. But this is impossible since $f \circ \gamma$ is a continuous timelike curve through $f(p)$.

Therefore, as claimed, there is an open set containing p—call it O_1—such that $f[I^+(p, O_1)] \subseteq O'$. Similarly, there is an open set O_2 such that $f[I^-(p, O_2)] \subseteq O'$. Let $O = O_1 \cap O_2$. Then clearly, $f[I(p, O)] \subseteq O'$. It now follows by Lemma A that $I(p, O) \subseteq M - \mathcal{D}$./

Lemma C: f and f^{-1} preserve continuous causal curves.

Proof: Let $\gamma: I \to M$ be a future directed continuous causal curve in M with $\gamma(t_0) = p$ for some $t_0 \in I$. Let O' be any open convex set containing $f(p)$. We must show that there exists an open subinterval $\bar{I} \subseteq I$ containing t_0 such that:

$$t \in \bar{I} \text{ and } t < t_0 \Rightarrow (f \circ \gamma)(t) < f(p) \quad (O'),$$

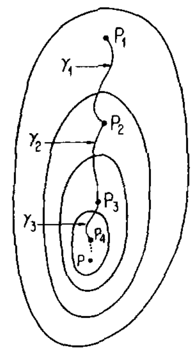

FIG. 2.

$$t \in \bar{I} \text{ and } t_0 < t \Rightarrow f(p) < (f \circ \gamma)(t) \quad (O'). \qquad (*)$$

Just as in the proof of Lemma B we can show that there must exist an open set O in M containing p such that $f[I(p, O)] \subseteq O'$. By moving to a subset we may take O to be convex. We choose $\bar{I} \subseteq I$ containing t_0 so that:

$$t \in \bar{I} \text{ and } t < t_0 \Rightarrow \gamma(t) < p \quad (O),$$

$$t \in \bar{I} \text{ and } t_0 < t \Rightarrow p < \gamma(t) \quad (O).$$

Now if $\gamma(t) < p$, then every continuous timelike curve segment through $\gamma(t)$ intersects $I^-(p, O)$. Hence every continuous timelike curve segment through $(f \circ \gamma)(t)$ intersects $I^-(f(p), O')$. Thus $(f \circ \gamma)(t) \in \mathrm{Cl}[I^-(f(p), O')]$ and therefore, since O' is convex, $(f \circ \gamma)(t) \in J^-(f(p), O')$. Thus the first half of (*) is established. The second half is symmetric. Hence $f \circ \gamma$ is a future directed continuous causal curve. (The argument for f^{-1} is, of course, symmetric.)/

Lemma D: (i) $\mathcal{D}$ is closed in M; $\mathcal{D}'$ is closed in M'.

(ii) For all $p \in M$, $p \in \mathcal{D}$ iff $f(p) \in \mathcal{D}'$.

(iii) If $p \in \mathcal{D}$, then there is an inextendible future directed continuous causal curve through p fully contained in $\mathcal{D}$.

Proof: Suppose f is continuous at p. Let O' be any open convex set containing $f(p)$. Let O be an open set with $p \in O \subseteq f^{-1}[O']$. Then, applying Lemma A, we have that $O \subseteq M - \mathcal{D}$. Thus $M - \mathcal{D}$ is open. Similarly $M' - \mathcal{D}'$ is open. So (i).

Suppose p is in $\mathcal{D}$. Then there exists a sequence $\{p_i\}$ which converges to p and an open convex set O' in M' which contains $f(p)$ but none of the $f(p_i)$. We can find sequences $\{r_i\}$ and $\{s_i\}$ converging chronologically to p from below and above respectively such that for each i there is a *local* future directed continuous timelike curve γ_i through p_i with initial point r_i and terminal point s_i. The only accumulation point of the γ_i is p.

Now $\{f(r_i)\}$ and $\{f(s_i)\}$ must converge to $f(p)$. So (passing to a subsequence if necessary) we may assume that all $f \circ \gamma_i$ begin and end in O'. But since $f(p_i) \notin O'$, each of these curves $f \circ \gamma_i$ must leave O' as well. There will be a future directed inextendible continuous causal curve Δ through $f(p)$ every point of which is an accumulation point of the $f \circ \gamma_i$.[6] Since the only accumulation point of the γ_i is p, it must be the case that $\Delta - \{f(p)\} \subseteq \mathcal{D}'$. Since $\mathcal{D}'$ is closed, it follows that $\Delta \subseteq \mathcal{D}'$. Thus $p \in \mathcal{D} \Rightarrow f(p) \in \mathcal{D}'$. The converse is symmetric. So we have (ii). For (iii) we need only repeat this past argument with respect to $f(p)$ and f^{-1}./

Lemma E: If $\mathcal{D} \neq \emptyset$, then there exists an open convex set O with $\mathcal{D} \cap O \neq \emptyset$ such that:

(i) $\mathcal{D}$ is achronal in O.

(ii) Through each point p in $\mathcal{D} \cap O$ there passes a unique continuous null geodesic Γ_p such that $\Gamma_p \cap O \subseteq \mathcal{D}$.

(iii) Given any continuous null geodesic Γ which intersects $\mathcal{D} \cap O$, either $\Gamma \cap O \subseteq \mathcal{D}$ or $\Gamma \cap O \cap \mathcal{D}$ is a singleton.

Proof: First note that (ii) and (iii) follows from (i) in view of Lemma D. For (i) suppose $\mathcal{D} \neq \emptyset$ but no O exists satisfying the required conditions. Let O_1 be any open convex set meeting $\mathcal{D}$ with compact closure. By our assumption we can find points r_1 and s_2 in $O_1 \cap \mathcal{D}$ such that $r_1 \ll s_1(O_1)$. Now let O_2 be any open convex set where $r_1 \in O_2 \subseteq I^-(s_1, O_1)$. Repeating the argument with respect to O_2, we can find points r_2 and s_2 in $O_2 \cap \mathcal{D}$ such that $r_2 \ll s_2(O_2)$. Certainly $s_2 \ll s_1(O_1)$. Continuing in this fashion, we generate a sequence $\{s_i\}$ in $O_1 \cap \mathcal{D}$ with $s_{i+1} \ll s_i(O_1)$ for all i. This sequence must have an accumulation point s. But now if we apply Lemma B to s, we find that there must exist an open set O containing s such that $I^+(s, O) \subseteq M - \mathcal{D}$. This leads to a contradiction since eventually all the s_i must enter $I^+(s, O_1)$./

Proof of the Theorem: Suppose $\mathcal{D} \neq \emptyset$ and O is as in Lemma E. Let p be any point in $\mathcal{D} \cap O$ with corresponding Γ_p. Clearly $I(\Gamma_p \cap O, O) \subseteq M - \mathcal{D}$. There must exist a sequence $\{p_i\}$ converging to p and an open convex set O' containing $f(p)$ but none of the $f(p_i)$.

Let Ω be *any* future directed continuous null geodesic segment through p distinct from Γ_p which is sufficiently "short" that $f \circ \Omega$ is fully contained in O'. There exist continuous null geodesic segments Ω_i within O, passing through p_i respectively, which converge to Ω in the sense that every open set which intersects Ω intersects eventually all Ω_i. We may choose $\{\Omega_i\}$ so that it has no convergence points off Ω. Eventually all Ω_i enter $I(\Gamma_p \cap O, O)$ and hence $M - \mathcal{D}$. It follows from Lemma E (iii) that, for eventually all i, $\Omega_i \cap \mathcal{D}$ is either empty or a singleton. The intersection point of Ω_i with $\mathcal{D}$ (if there is one) comes either "before p_i," at p_i itself, or "after p_i." Without loss of generality we may assume that there is an infinite subset of $\{\Omega_i\}$ in each member of which the intersection point with $\mathcal{D}$ (if there is one) does not come before p_i. Now let Ω_i^- be the "lower-half" of Ω_i with future end point p_i included. By moving to a subsequence we can thus find a sequence of continuous null geodesic segments $\{\Omega_i^-\}$ in O with the following properties (see Fig. 3):

(i) $\{\Omega_i^-\}$ converges to the lower half Ω^- of Ω, but has no convergence points off Ω^-.

(ii) For each i, $\Omega_i^- \cap \mathcal{D} \subseteq \{p_i\}$.

From (ii), Lemma C, and Lemma 1, it follows that each image curve $f \circ \Omega_i^-$ is a continuous null geodesic segment in M'. From (i) and the fact that $\Omega^- - \{p\} \subseteq M - \mathcal{D}$,

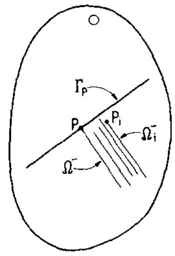

FIG. 3.

it follows that these segments converge to $f \circ \Omega^-$.

Now recall that no point $f(p_i)$ lies within O'. So, though the $f \circ \Omega_i^-$ converge to $f \circ \Omega^-$, they must all leave O' before reaching their respective $f(p_i)$. Let $f(q)$ be any point of the null geodesic extension of $f \circ \Omega^-$. We claim $f(q) \in \mathcal{D}'$. For suppose to the contrary that $f(q) \in M - \mathcal{D}'$. Then, since $f(q)$ is a convergence point of $\{f \circ \Omega_i^-\}$, q must be a convergence point of $\{\Omega_i^-\}$. This is impossible since $q \notin \Omega^-$.

In our construction we assumed that Ω satisfied the "not before p_i," clause for an infinite subset of Ω_i. Dropping that assumption, we have the following conclusion. If Ω^- and Ω^+ are the respective lower and upper segments of Ω, then *either* the future null geodesic extension of $f \circ \Omega^-$ or the past null geodesic extension of $f \circ \Omega^+$ is a future directed continuous causal curve segment through $f(p)$ lying within $\mathcal{D}'$. But this is true of *all* future directed continuous null geodesic segments; Ω was chosen arbitrarily. Thus, since f is a bijection, it follows that there exist *distinct* future directed continuous causal curves through $f(p)$ lying within $\mathcal{D}'$. Their preimages under f^{-1} must be distinct future directed continuous causal curves through p lying within $\mathcal{D}$. But this contradicts our assumption that $\mathcal{D}$ is achronal in O.

Thus, $\mathcal{D}$ is empty, and, hence, $\mathcal{D}'$ is empty as well./

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[1]E.H. Kronheimer and R. Penrose, Proc. Camb. Phil. Soc. 63, 481 (1967).

[2]S.W. Hawking, A.R. King, and P.J. McCarthy, J. Math. Phys. 17, 174 (1976).

[3]Proofs of these and subsequent claims can be found in S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge U.P., Cambridge, 1973); R. Penrose, *Techniques of Differential Topology in Relativity* (SIAM, Philadelphia, 1972).

[4]A proof is given in Hawking, King, and McCarthy (Ref. 2). The theorem is not formulated in exactly this from, but the argument carries over intact.

[5]This version of the theorem is applicable to all temporally orientable spacetimes whether or not a particular temporal orientation is distinguished.

[6]Hawking and Ellis (Ref. 3) prove this in detail in their Lemma 6.2.1.