A tool for teaching General Relativity

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GRTensorJ - Books is an active interface to a small part of the computer algebra systems GRTensorII (for Maple) and GRTensorM (for Mathematica) with the specific intent of providing students of General Relativity with an advanced programmable calculator-style tool. All standard functions associated with a classical tensor approach to the subject are available thus reducing these to "elementary functions". This is not a traditional database. The database entries are spacetimes and calculations are done in real time. All spacetimes are referenced directly by equation number in ten current (and classic) texts in notation as close as possible to the original text. The tool is now available free of charge from grtensor.org/teaching.

I. MOTIVATION

Recently Hartle [1] has emphasized the increasing importance of General Relativity on the frontiers of both the very largest and the very smallest distance scales, a fact reflected by the increasing number of undergraduate physics majors being introduced to the subject. Of course, one can only learn physics by doing it, but many of the standard functions of General Relativity, when taught through classical tensor methods, involve only partial differentiation and summation. I believe that little is to be gained by repetitive routine calculations of this nature. For this reason, I would like to introduce here a tool that will be of use for all students starting their study of General Relativity. The purpose of the tool is to turn all standard (tensor based) functions into "elementary functions" - those immediately available with the click of a button - so that the inner beauty of General Relativity shines through.

II. BACKGROUND

The tool described here involves and update [2] to the Java [3] interface GRTensorJ [4]. The workings of the interface have been described in detail previously [5]. The version of GRTensorJ described here cannot be run over the internet but rather is a local application. The interface currently covers every spacetime (referenced by equation number and in notation as close to the original as possible) in the following texts:

- Carroll, Sean. Spacetime and Geometry: An Introduction to General Relativity, Addison Wesley (San Francisco), 2004. ISBN 0-8053-8732-3
- d'Inverno, Ray. Introducing Einstein's Relativity, Clarendon Press (Oxford), 1992. ISBN 0-19-859686-3
- Hartle, James. *Gravity: An Introduction to Einstein's General Relativity*, Addison Wesley (San Francisco), 2003. ISBN 0-8053-8662-9
- Hawking, Stephen, and G.F.R. Ellis. *The Large Scale Structure of Spacetime*, Cambridge University Press (Cambridge), 1973. ISBN 0-521-09906-4
- Misner, Charles, Kip Thorne and John Wheeler. *Gravitation*, W.H. Freeman and Co. (San Francisco), 1973. ISBN 0-7167-0344-0
- Poisson, Eric. A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics, Cambridge University Press (Cambridge), 2004. ISBN 0-521-83091-5
- Rindler, Wolfgang. *Relativity: Special, General and Cosmological*, Oxford University Press (Oxford), 2001. ISBN 0-19-850836-0
- Schutz, Bernard. A First Course in General Relativity, Cambridge University Press (Cambridge), 1990. ISBN 0-521-27703-5
- Wald, Robert. General Relativity, University of Chicago Press (Chicago), 1984. ISBN 0-226-87033-2
- Weinberg, Steven. Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, John Wiley and Sons (New York), 1972. ISBN 0-471-92567-5

GRTensorJ menu item GRTensorJ sub-menu command Object(s)

Metric	Metric	g_{ab}
	Signature	Sig
	Line Element	ds^2
	Constraints	
	Determinant of Metric	det(q)
	Inverse Metric	a^{ab}
	Partial Derivative of Metric	gab c
Christoffel Symbols	Chr(dn.dn.dn)	Γ_{abc}
	Chr(dn.dn.up)	Γ^{c}_{cb}
Geodesic	····(····,···,·· · ·	Writes out the geodesic equations
Riemann	B(dn,dn,dn,dn)	Rahad
	B(up dn dn dn)	B^a_{bad}
	B(up, up, dn, dn)	B^{ab} ,
	$\mathbf{R}(\mathbf{u}\mathbf{p},\mathbf{u}\mathbf{p},\mathbf{u}\mathbf{n},\mathbf{u}\mathbf{n})$	Pabed a state of the state of t
	Krotschmann	D D^{abcd}
Word	C(dn dn dn dn)	$n_{abcd}n$
weyi	$\mathcal{O}(an,an,an,an) \dots$	C_{abcd}
D: :	Dual $CStar(dn, dn, dn, dn)$	$C_{abcd} \equiv \frac{1}{2} \epsilon_{abef} C^{cf}_{cd}$
Ricci	$R(an, an) \dots$	$R_{ab} \equiv R^{-}_{acb}$
Trace-free Ricci	$S(dn,dn) \dots$	$S_{ab} \equiv R_{ab} - \frac{1}{N}g_{ab}R$
Einstein	G(dn,dn)	$G_{ab} \equiv R_{ab} - \frac{1}{2}g_{ab}R$
Invariants	Invariants-Ricci	$R \equiv R^a_{\ a}, R1 \equiv \frac{1}{4}S^a_{\ b}S^o_{\ a}, R2 \equiv \frac{-1}{8}S^a_{\ b}S^o_{\ c}S^c_{\ a}$
		$R3 \equiv \frac{1}{16} S^a{}_b S^b{}_c S^c{}_d S^d{}_a$
	Invariants-Weyl	$W1R \equiv \frac{1}{8}C_{abcd}C^{abcd}, W1I \equiv \frac{1}{8}C^*{}_{abcd}C^{abcd}$
		$W2R \equiv \frac{-1}{16} C_{ab} ^{cd}C_{cd} ^{ef}C_{ef} ^{ab}$
		$W2I \equiv \frac{-1}{16} C^*_{ab} C^*_{cd} C_{cd} C_{ef}^{ab}$
	Invariants-Mixed	$M1R \equiv \frac{1}{2}S^{ab}S^{cd}C_{abcd}, M1I \equiv \frac{1}{2}S^{ab}S^{cd}C^*_{abcd}$
		$M2R \equiv \frac{1}{16} S^{cd} S_{ef} (C_{acdb} C^{aefb} - C^*_{acdb} C^{aefb}_*)$
		$M2I \equiv \frac{1}{8}S^{bc}S_{ef}(C^*_{abcd}C^{aefd})$
		$M3 \equiv \frac{1}{16} S^{cd} S_{ef} \left(C_{acdb} C^{aefb} + C^*_{acdb} C^{aefb}_* \right)$
		$M4 \equiv \frac{-1}{32} S^{cg} S^{ef} S^{c} {}_{d} (C_{ac} {}^{db} C_{befg} + C^{*}_{ac} {}^{db} C^{*}_{befg})$
		$M5R \equiv \frac{1}{22} S^{cd} S^{ef} C^{aghb} (C_{acdb} C_{aefh} + C^*_{acdb} C^*_{aefh})$
		$M5I \equiv \frac{3^2}{22} S^{cd} S^{ef} C_*^{aghb} (C_{acdb} C_{gefh} + C^*_{acdb} C^*_{gefh})$
		$M6R \equiv \frac{32}{22} S_a{}^e S_e{}^c S_b{}^f S_f{}^d C^{ab}{}_{cd}$
		$M6I \equiv \frac{1}{20} S_a^{\ e} S_e^{\ c} S_b^{\ f} S_f^{\ d} C^{*ab}_{\ cd}$
Differential Invariants	diRicci	$R_{ab:c}R^{ab;c}$
	diRiem	$R_{abcd:e}^{abcd;e}$
	diS	$S_{aba}S^{ab;c}$
	diWeyl	$C_{ab,c\sim}$ $C^{abcd;e}$
Bel-Robinson	T(dn dn dn dn)	$T_{abca;e} \subset T_{abca;e} \subset T_{$
Weyl-Schouton	Weyl-Schouten	$WS_{I} = R_{I} - R_{I} - (a \cdot R^{e} - a \cdot R^{f} \cdot 1)/4$
Poob	P(dn, dn)	$m \mathcal{D}_{abc} = n_{ab;c} - n_{ac;b} - (g_{ab}n_{e;c} - g_{ac}n_{f;b})/4$ $D = -O + \frac{1}{2}b^{b}dO$
Dach	$D(an,an) \dots$	$D_{ac} = C_{abcd} + \frac{1}{2} \kappa C_{abcd}$

TABLE I: Initial set of preprogrammed objects available for coordinate calculations (condensed). Note: "..." means that all index combinations are immediately available as with Riemann.

III. PREPROGRAMMED CALCULATIONS

In addition to a complete help system, the initial set of preprogrammed objects available for coordinate calculations is shown in TABLE 1. Note that the table is condensed as, for example, all index configurations are immediately available. Note also that context - sensitive mathematical information is embedded as explained below. The introduction of further objects is also described below.

IV. LOADING A SPACETIME AND WORKING WITH IT

FIG. 1 shows a screen shot of the interface running Maple 10 [6] under Windows XP [7] and samples the coverage associated with the classic text by Misner, Thorne and Wheeler. Assuming you have the text, you simply click on

the equation number to load the spacetime.

FIG. 2 shows loading of the Kruskal - Szekeres covering of the Schwarzschild vacuum as discussed by Poisson. We use this example here because the coordinates are implicitly defined and therefore involve constraints which can be seen in the figure. Without these constraints the spacetime is not defined. FIG. 3 shows the result of applying constraints to the calculation of the Kretschmann scalar (see TABLE 1).

As mentioned above, the interface contains context - sensitive embedded mathematical information. An example is shown in FIG. 4 where one notices that there are only four Ricci invariants available for calculation. The reason for this is given in the explanation as shown in FIG. 5 [8].

V. INTERFACE EXPANSION

At the time of writing, virtually any calculation one encounters in classical tensor analysis is already programmed into the interface. Further routines can be easily added through the GRTensorII grdef and GRTensorM grdefine facilities. This includes tetrad calculations which we do not discuss here. To see how to program the interface, let us look at the construction of the Weyl-Schouten tensor (equivalently, the Cotton-York tensor which is also available). This is a three index tensor which in three dimensions is zero if and only if the space is conformally flat. As can be see from TABLE 1, it is defined by

$$WS_{abc} \equiv R_{ab;c} - R_{ac;b} - (g_{ab}R^{e}_{e;c} - g_{ac}R^{f}_{f;b})/4,$$

where the terms are defined in the table. This relation is programmed into the interface by including a plain text file (Weyl-Scouten.ts) the entire contents of which are:

grdef('WS{a [b c] }:=R{a b ; c}-R{a c ; b}-(g{a b}*R{^e e;c}-g{a c}*R{^f f ; b})/4');

grcalc(WS(dn,dn,dn));

*grdisplay(_);

VI. SYSTEM REQUIREMENTS

To run GRTensorJ - Books you need Java [3] installed on your computer and a version of Maple or Mathematica. (You do not need to have GRTensorII or GRTensorM already installed.) The Maple version (tested on Maple 7, 8, 9, 9.5.2, 10 and student 10) for Windows (tested on 2000 and XP) is available now from grtensor.org/teaching. Versions for other platforms, and updates, will be announced at the web site as they are available.

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^[1] James B. Hartle, "General Relativity in the Undergraduate Physics Curriculum" arXiv:gr-qc/0506075 .

^[2] The update to the interface was written by Kamen Angelov.

^[3] Java is ©Sun Microsystems.

^[4] Mustapha Ishak, Peter Musgrave, John Mourra, Jonathan Stern and Kayll Lake, "GRLite and GRTensorJ: Graphical user interfaces to the computer algebra system GRTensorII", AIP Conference Proceedings 493, Edited by C. P. Burgess and R. C. Myers arXiv:gr-qc/9911012.

- [5] Mustapha Ishak and Kayll Lake, Class. Quant. Grav. 19 505-514 (2002) <code>arXiv:gr-qc/0111008</code> .
- [6] Maple is ©Maplesoft.
- [7] Windows is ©Microsoft.
- [8] Kevin Santosuosso, Denis Pollney, Nicos Pelavas, Peter Musgrave and Kayll Lake, Comput. Phys. Commun. 115 381-394 (1998) arXiv:gr-qc/9809012 .
- [9] Mathematica is ©Wolfram Research.

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FIG. 1: A look at the middle section of the classic text by Misner, Thorne and Wheeler.

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FIG. 2: Loading of the Kruskal - Szekeres covering of the Schwarzschild vacuum as discussed by Poisson.



FIG. 3: Applying constraints to the calculation of the Kretschmann scalar (see Table 1).

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FIG. 4: Calculation of Ricci invariants.



FIG. 5: Explanation for the choice of Ricci invariants.