Initial-value problem of general relativity. III. Coupled fields and the scalar-tensor theory *

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The initial-value problem of general relativity is treated in the case where the external sources are electromagnetic or neutrino fields. Taking into account the initial conditions that must be satisfied by these fields, we show that the resulting equations in each case form a quasilinear elliptic system of a type that has been treated extensively in previous work. We also treat the initial-value problem of the scalar-tensor theory of gravitation. Throughout this work we use first-order "canonical" gravitational variables. The principal mathematical tools are conformal transformations and a covariant decomposition of symmetric tensors.

I INTRODUCTION

In previous work[1] we have discussed the construction of solutions of the initial-value problem of general relativity. The gravitational field on a spacelike hypersurface is found by prescribing freely certain of the data (unbarred variables in the succeeding analysis) in a conformally covariant manner. It was shown how to construct a particular conformal transformation of the given independent quantities in such a way that the transformed data (barred variables) satisfy the constraints. The transformations of the various parts of the gravitational field and its sources, if any, lead to four coupled quasilinear elliptic equations which almost always have unique solutions.[2] Definitions, conventions, and relevant results from papers I and II are given in the Appendix.

In this paper we shall discuss in more detail the initial-value problem in the presence of pure electromagnetic and neutrino fields, as well as the initial-value problem of the scalar-tensor theory of gravitation. We focus attention here on these particular examples because of their practical interest. Elsewhere,[3] the initial-value analysis for gravity, including gravity with torsion, coupled to general boson and fermion fields will be dealt with thoroughly using conformal techniques of the type developed in recent work[1] and used in the present study.

External sources of the gravitational field are characterized by a stress-energy tensor $\overline{T}^{\mu\nu}$ with energy density $\overline{\mu}=\overline{T}^{\mu\nu}n_\mu n_\nu$ and current density $\overline{\nu}^i=\perp^i_\mu \overline{T}^{\mu\nu}n_\nu$, where n^μ is the unit timelike normal of the initial hypersurface and $\perp^i_\mu$ is an operator of projection into this hypersurface. The Einstein equations constrain the initial data to satisfy

$$-2\overline{\nabla}_j\overline{p}^{ij}=16\pi\overline{\nu}^i, \tag{1.1}$$

$$\overline{R}-(\overline{p}_{ij}p^{ij}-\tfrac{1}{2}\overline{p}^2)=16\pi\overline{\mu}. \tag{1.2}$$

The gravitational data are the spacelike three-metric $\overline{g}_{ij}$ of the initial slice (with covariant derivative $\overline{\nabla}_i$ and scalar curvature $\overline{R}$) and a "momentum" tensor $\overline{p}^{ij}=\overline{K}\overline{g}^{ij}-\overline{K}^{ij}$, where $\overline{K}_{ij}$ is the second fundamental tensor of the slice and $\overline{K}=\overline{g}^{ij}\overline{K}_{ij}$. In the following these quantities will be dealt with just as in previous papers. (See the Appendix for a summary.) In particular, the dependent data for which (1.1) and (1.2) are solved are a scalar conformal factor $\phi(x)>0$ and a three-vector $W^i(x)$. The scalar defines a conformal mapping $\overline{g}_{ij}=\phi^4 g_{ij}$ and the vector defines the "longitudinal" or constrained part of $\overline{p}^{ij}$ in (1.2). On asymptotically flat manifolds, we choose the boundary conditions $\phi\to 1$ and $W^i\to 0$ at spatial infinity. On closed manifolds there are no boundary conditions.

The two levels at which $\overline{\mu}$ and $\overline{\nu}^i$ may be dealt with are (1) as scalar and three-vector point functions $\overline{\mu}(x)$, $\overline{\nu}^i(x)$ on the initial manifold, and (2) as scalar and three-vector functions constructed from a basic underlying field, e.g., in the case of electromagnetism one has $\overline{\mu}=(8\pi)^{-1}(\overline{E}^i\overline{E}^j+\overline{B}^i\overline{B}^j)\overline{g}_{ij}$. We shall first consider $\overline{\mu}$ and $\overline{\nu}^i$ as point functions and ask how they might change under a conformal transformation of the other initial data. We may assume for simplicity that $\overline{\mu}=\overline{\mu}[\phi,\mu]$, $\overline{\nu}^i=\overline{\nu}^i[\phi,\nu^j]$, where μ and ν^i are the "trial" energy and current densities which must be transformed, i.e., conformally deformed, into $\overline{\mu}$ and $\overline{\nu}^i$ along with the other data in such a way that (1.1) and (1.2) are satisfied. Elementary physical considerations guide us in constructing these transformations. To simplify the analysis and guarantee physically meaningful results, we may require that (a) $\mu(x)\geq 0$ implies

$\bar{\mu}(x) \geq 0$, and that (b) $\mu^{-2}\nu_i\nu^i \leq 1$ implies $\bar{\mu}^{-2}\bar{\nu}_i\bar{\nu}^i \leq 1$. Condition (a) guarantees the local positiveness of energy density and (b) guarantees the dominance-of-energy requirement, which rules our sources with spacelike local energy-momentum four-vectors. For pragmatic reasons, we can require both (a) and (b) to hold for all $\phi(x) > 0$ because we cannot know the value of ϕ before the complete initial-value problem is solved. It follows that from (a) and (b) $\bar{\mu}^{-2}\bar{\nu}_i\bar{\nu}^i$ is independent of ϕ. A final natural condition to impose is (c) the transformations reduce to the identity when $\phi = 1$. From (a), (b), and (c), we conclude that an acceptable transformation is

$$\bar{\mu} = \phi^{\alpha}\mu, \quad \bar{\nu}^i = \phi^{(\alpha-2)}\nu^i, \tag{1.3}$$

where α is a real number. We wish to point out that transformations of the type (1.3) are not actually *necessary* in order to construct physically significant solutions for arbitrary types of matter-field sources.[4] However, they have the virtue of simplicity and are precisely of the form required in the problems being treated in the present work, as we shall see.

A more fundamental point of view is possible whenever the energy and current densities are constructed from an underlying field. This source field may have an initial-value problem of its own that must be solved along with the gravitational constraints. The relations between the "trial" quantities μ and ν^i and the physically relevant final values $\bar{\mu}$ and $\bar{\nu}^i$ in this case are found by requiring that the initial-value constraints on the source fields should be solved as far as possible without dependence on the unknowns ϕ and W^i of the gravitational problem. That is, the gravity constraints and the source constraints are to be posed in such a way that their mutual coupling is minimal. This program is carried out for electromagnetic and neutrino fields in Secs. II and III. One finds in these cases the *unique* relations $\bar{\mu} = \mu\phi^{-8}$, $\bar{\nu}^i = \nu^i\phi^{-10}$, which are in accord with (1.3) in the case $\alpha = -8$. This result then turns out to imply that the gravitational constraints have precisely the form for which existence and uniqueness[2] and stability of solutions of the initial-value equations[5] have been established.

The scalar-tensor theory[6] has an initial-value problem that can be posed in terms of a scalar field and the same tensor variables $\bar{g}_{ij}$ and $\bar{p}^{ij}$ that are used in conventional general relativity.[7] In Sec. IV this problem is treated and the constraints are written as elliptic equations that have the same properties as they do in general relativity. The final section contains a brief discussion of the results and how they may be easily generalized to encompass a number of other physically important examples.

It is interesting to note that the conformal analysis of the initial-value problem of gravitation with sources, as developed in this paper and in previous papers, can be used as a basis for formulating Wheeler's version of Mach's principle. This issue is discussed elsewhere.[8]

II. INITIAL-VALUE PROBLEM WITH ELECTROMAGNETIC SOURCES

The coupled initial-value problem for the Einstein-Maxwell field consists of the gravitational initial-value equations (1.1) and (1.2), and the electromagnetic constraints

$$\bar{\nabla}_i\bar{E}^i = \partial_i\bar{E}^i + \bar{E}^j\bar{\Gamma}^i_{ji} = 0\ , \tag{2.1}$$

$$\bar{\nabla}_i\bar{B}^i = \partial_i\bar{B}^i + \bar{B}^j\bar{\Gamma}^i_{ji} = 0\ , \tag{2.2}$$

which must also be satisfied on the initial spacelike hypersurface. In (1.1) and (1.2) we have the electromagnetic field current density and energy density as sources of gravity,

$$\bar{\nu}^i = (4\pi)^{-1}\bar{g}^{1/2}\bar{g}^{im}\epsilon_{mjk}\bar{E}^j\bar{B}^k\ , \tag{2.3}$$

$$\bar{\mu} = (8\pi)^{-1}\bar{g}_{ij}(\bar{E}^i\bar{E}^j + \bar{B}^i\bar{B}^j)\ , \tag{2.4}$$

where ϵ_{mjk} is the covariant unit alternating tensor density of weight -1 with values $+1, -1, 0$. We observe that

$$\bar{g}_{ij} = \phi^4 g_{ij} \Rightarrow \bar{\Gamma}^i_{ji} = \Gamma^i_{ji} + 6\phi^{-1}\partial_j\phi\ . \tag{2.5}$$

Hence, if we define

$$\bar{E}^i = \phi^{-6}E^i, \quad \bar{B}^i = \phi^{-6}B^i, \tag{2.6}$$

we find[9]

$$\bar{\nabla}_i\bar{E}^i = \phi^{-6}\nabla_i E^i, \quad \bar{\nabla}_i\bar{B}^i = \phi^{-6}\nabla_i B^i\ . \tag{2.7}$$

In the gravitational initial-value problem, the "base" or "trial" metric g_{ij} is prescribed freely. Then we can construct in a well-known manner[10] two covariantly transverse vector fields E^i_T and B^i_T, i.e., solutions of the equations

$$\nabla_i E^i_T = \partial_i E^i_T + E^j_T\Gamma^i_{ji} = 0\ , \tag{2.8}$$

$$\nabla_i B^i_T = \partial_i B^i_T + B^j_T\,\Gamma^i_{ji} = 0\ . \tag{2.9}$$

Once E^i_T and B^i_T have been chosen in this way, (2.6) and (2.7) ensure that $\bar{\nabla}_i\bar{E}^i_T = 0$, $\bar{\nabla}_i\bar{B}^i_T = 0$ for any ϕ, and thus in particular for that ϕ which satisfies the gravitational constraints. The transformation (2.6) is unique in this respect. The sources (2.3) and (2.4) become, using (2.6),

$$\bar{\nu}^i = \phi^{-10}\nu^i = \phi^{-10}(4\pi)^{-1}g^{1/2}g^{im}\epsilon_{mjk}E^j_T B^k_T\ , \tag{2.10}$$

$$\bar{\mu} = \phi^{-8}\mu = \phi^{-8}(8\pi)^{-1}g_{ij}(E^i_T E^j_T + B^i_T B^j_T). \tag{2.11}$$

Therefore, $\alpha = -8$ in the transformations (1.3). The gravitational constraints, which constitute four of the ten Einstein equations, have been extensively developed and discussed in terms of the scalar and vector potentials ϕ and W^i in previous articles.[1,2] These equations are summarized in the Appendix. Referring to the equations for gravity (A5), (A12), and (A13) in the Appendix, we see that the complete gravitational-electromagnetic initial-value equations are (2.8), (2.9), and

$$\nabla_j p^{ij}_{TT} = 0 \ , \tag{2.12}$$

$$\nabla_j (LW)^{ij} + 6(LW)^{ij}\nabla_j \ln\phi + \tfrac{1}{2}\nabla^i\tau + 2\phi^{-6} g^{1/2} g^{im}\epsilon_{mjk} E^j_T B^k_T = 0, \tag{2.13}$$

$$-8\nabla^2\phi = -R\phi + M_{TT}\phi^{-7} + 2M_{TL}\phi^{-1} + (M_L - \tfrac{3}{8}\tau^2)\phi^5 + 2g_{ij}(E^i_T E^j_T + B^i_T B^j_T)\phi^{-3} \ , \tag{2.14}$$

where

$$\begin{aligned} &\tau = \tfrac{2}{3}p = \tfrac{2}{3}\bar{p} \ , \\ &M_{TT} = g_{ij}g_{kl}p^{ik}_{TT}p^{jl}_{TT}, \\ &M_{TL} = g_{ij}g_{kl}p^{ik}_{TT}(LW)^{jl}, \\ &M_L = g_{ij}g_{kl}(LW)^{ik}(LW)^{jl}. \end{aligned} \tag{2.15}$$

To satisfy this system of equations for a given g_{ij}, one first constructs in a straightforward manner[11] the solutions E^i_T, B^i_T, and p^{ij}_{TT} of (2.8), (2.9), and (2.12). These quantities, together with g_{ij} and a freely specified scalar τ, are the input data of (2.13) and (2.14), which are then solved for W^i and ϕ. Now we have a complete set of initial data $\bar{g}_{ij}$, $\bar{p}^{ij}$, $\bar{E}^i$, $\bar{B}^i$ that satisfy the Einstein-Maxwell constraints (1.1), (1.2), (2.1), and (2.2) simultaneously. In terms of the unbarred variables, we have the solution

$$\begin{aligned} &\bar{g}_{ij} = \phi^4 g_{ij} \ , \\ &\bar{p}^{ij} = \phi^{-10}p^{ij}_{TT} + \phi^{-4}[\,(LW)^{ij} + \tfrac{1}{2}\tau g^{ij}] \ , \\ &\bar{E}^i = \phi^{-6}E^i_T \ , \\ &\bar{B}^i = \phi^{-6}B^i_T \ . \end{aligned} \tag{2.16}$$

III. INITIAL-VALUE PROBLEM WITH NEUTRINO SOURCES

The relativistic theory of neutrinos requires the introduction of spinors. In order to facilitate extension of the results below to the coupled gravity-electron theory (Sec. V), we shall use here the language of Dirac four-spinors. We shall denote by D_μ the four-dimensional covariant derivative. (In the preceding sections, only the covariant derivative ∇_i induced on three-dimensional spacelike hypersurfaces has been used.) The stress-energy tensor for the neutrino field in Einstein's equations is given by[12]

$$\bar{T}_{\mu\nu} = \tfrac{1}{4}\,[\bar{\psi}^*\bar{\gamma}_\mu\bar{D}_\nu\bar{\psi} - (\bar{D}_\nu\bar{\psi}^*)\bar{\gamma}_\mu\bar{\psi} + \bar{\psi}^*\bar{\gamma}_\nu\bar{D}_\mu\bar{\psi} - (\bar{D}_\mu\bar{\psi}^*)\bar{\gamma}_\nu\bar{\psi}] \ , \tag{3.1}$$

where the four-spinor $\bar{\psi}$ satisfies the massless Dirac equation

$$\bar{\gamma}^\mu\bar{D}_\mu\bar{\psi} = 0 \ . \tag{3.2}$$

Here, let us recall our notational convention that a bar over a quantity indicates that it satisfies the constraints, as opposed to the unbarred "trial" quantities. For a spinor $\bar{\psi}$, we denote its adjoint by $\bar{\psi}^*$ = (Hermitian conjugate of $\bar{\psi}$) × ("Hermitizing matrix").[12] The Dirac matrices satisfy

$$\bar{\gamma}_\mu\bar{\gamma}_\nu + \bar{\gamma}_\nu\bar{\gamma}_\mu = 2\,{}^{(4)}\bar{g}_{\mu\nu} I \ , \tag{3.3}$$

and the covariant derivative $\bar{D}_\mu$ acts on spinors according to the rule

$$\bar{D}_\mu\bar{\psi} = \partial_\mu\bar{\psi} - \bar{\Gamma}_\mu\bar{\psi} \ , \tag{3.4}$$

where $\bar{\Gamma}_\mu$ is the spinor connection determined up to a multiple of the identity matrix by[12]

$$\bar{D}_\mu\bar{\gamma}_\nu = \partial_\mu\bar{\gamma}_\nu - \bar{\Gamma}^{\alpha}_{\mu\nu}\bar{\gamma}_\alpha - \bar{\Gamma}_\mu\bar{\gamma}_\nu + \bar{\gamma}_\nu\bar{\Gamma}_\mu = 0 \ . \tag{3.5}$$

The neutrino field must also satisfy

$$(I - i\gamma_5)\bar{\psi} = 0 \ , \tag{3.6}$$

where

$$\gamma_5 = [4!\,(-{}^{(4)}\bar{g})^{1/2}]^{-1}\epsilon^{\alpha\beta\gamma\delta}\bar{\gamma}_\alpha\bar{\gamma}_\beta\bar{\gamma}_\gamma\bar{\gamma}_\delta \ . \tag{3.7}$$

Equation (3.6) incorporates the experimentally established fact that neutrinos have only one helicity.

The initial-value problem for the gravity-neutrino theory can be posed in terms of the gravitational variables $\bar{g}_{ij}$, $\bar{p}^{kl}$ together with the spinors $\bar{\psi}$ and $\bar{\psi}^*$. To show this, following Dirac,[13] we must first pick a "hypersurface-compatible" orthonormal tetrad $\bar{\lambda}^\mu_{(\alpha)}$ that satisfies

$$\begin{aligned} &\bar{g}_{ij} = \bar{\lambda}^{(k)}_i\bar{\lambda}^{(l)}_j\delta_{(k)(l)} , \\ &n^\mu = \bar{\lambda}^\mu_{(0)} , \end{aligned} \tag{3.8}$$

where n^μ is the timelike unit normal of the hypersurface. One can now express all the needed quantities in a form suitable for the initial-value analysis. By introducing the operator that projects quantities onto the hypersurface,

$$\perp^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu = \bar{\lambda}^\mu_{(i)}\bar{\lambda}^{(i)}_\nu ,$$

we may write the neutrino equations (3.2) in the form

$$n_\nu\bar{\gamma}^\nu n^\mu\bar{D}_\mu\bar{\psi} = \perp^j_i\bar{\gamma}^i\bar{D}_j\bar{\psi} \ , \tag{3.9}$$

which expresses the time derivative of $\bar{\psi}$ in terms of quantities that have been projected onto the hypersurface and that are, therefore, expressed in terms of initial data. Equation (3.9) will be useful below. In contrast to the dynamical equations (3.2) or (3.9), the helicity condition (3.6) constitutes a constraint on the initial value of $\overline{\psi}$.[14] If this constraint is satisfied initially, it continues to hold at later times by virtue of the equations of motion.

The four gravitational constraints are given by (1.1) and (1.2) with $\overline{\mu}$ and $\overline{\nu}^i$ determined by contracting $\overline{T}_{\mu\nu}$ in (3.1) with $\overline{\lambda}^{\mu}_{(0)}=n^{\mu}$ and $\overline{\lambda}^{\mu}_{(i)}$. Using (3.9), one finds

$$\overline{\mu}=\tfrac{1}{2}[\overline{\psi}^*(\perp_i^j\overline{\gamma}^i\overline{D}_j\overline{\psi})-(\perp_i^j\overline{\gamma}^i\overline{D}_j\overline{\psi}^*)\overline{\psi}], \tag{3.10}$$

$$\overline{\nu}^i=-\tfrac{1}{4}[\overline{\psi}^*\overline{\gamma}_n\perp^i_j\overline{D}^j\overline{\psi}-(\perp^i_j\overline{D}^j\overline{\psi}^*)\overline{\gamma}_n\overline{\psi} +\overline{\psi}^*\overline{\gamma}^i\overline{\alpha}^k\perp^j_k\overline{D}_j\overline{\psi}-(\perp^j_k\overline{\alpha}_k\overline{D}_j\overline{\psi}^*)\overline{\gamma}^i\overline{\psi}], \tag{3.11}$$

where $\overline{\alpha}^i=\overline{\gamma}_n\overline{\gamma}^i$ and $\overline{\gamma}_n=n_\sigma\overline{\gamma}^\sigma$. Note that, as required, $\overline{\mu}$ and $\overline{\nu}^i$ contain no reference to the lapse and shift functions $(-{}^{(4)}g^{00})^{-1/2}$ and ${}^{(4)}g_{0i}$. The only role of the lapse and shift functions is to describe how the spacetime coordinate system is to be continued away from the initial hypersurface. Clearly, then, the initial-value equations are independent of these quantities. In fact, the only part of the gravitational initial data $\{\overline{g}_{ij},\overline{p}^{kl}\}$ on which $\overline{\mu}$ and $\overline{\nu}^i$ depend is, as is the case with the electromagnetic field, the three-metric $\overline{g}_{ij}$ (and the $\overline{\lambda}^k_{(i)}$'s) and its spatial derivatives. This point is more easily seen by examining Dirac's equivalent expressions, (35) and (36) in Ref. 13, or those in Ref. 3, both of which involve a more complete (3 + 1)-dimensional breakup than we have presented here. However, (3.9), and (3.10) are in very convenient form for the remaining part of our analysis.

The complete initial-value problem is now the gravity constraints (1.1) and (1.2), with (3.10) and (3.11) incorporated, plus (3.6). We want to choose, as in the electromagnetic case, appropriate conformal mappings that uncouple the constraints as much as possible and result in the simplest possible equations. Since the constraint (3.6) is algebraic, it is invariant with respect to any transformation of the type $\overline{\psi}=\psi\phi^{\alpha}$($\gamma_5$ is clearly scale-invariant). Thus the choice of α can be determined by requiring that the resulting transformations $\overline{\mu}\to\mu$ and $\overline{\nu}^i\to\nu^i$ are homogeneous in ϕ and involve no terms such as $\nabla_i\phi$, etc. To carry out the conformal transformation of (3.10) and (3.11), it is convenient to work in terms of a four-dimensional conformal mapping ${}^{(4)}\overline{g}_{\mu\nu}={}^{(4)}g_{\mu\nu}\phi^4$. Of course, only the three-dimensional part $\overline{g}_{ij}=\phi^4 g_{ij}$ [and $\overline{\lambda}^{(k)}_i=\phi^2\lambda^{(k)}_i$ from (3.8)] is actually involved in (3.10) and (3.11) since these expressions are independent of ${}^{(4)}g_{00}$ and ${}^{(4)}g_{0i}$. The transformed spinor connection is given by

$$\overline{\Gamma}_\lambda=\Gamma_\lambda+\tfrac{1}{2}\,{}^{(4)}g_{\lambda\nu}(\partial_\mu\ln\phi)(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu). \tag{3.12}$$

One finds from the above that if $\overline{\psi}=\psi\phi^{-3}$ (and $\overline{\psi}^*=\psi^*\phi^{-3}$) then $\overline{\mu}=\mu\phi^{-8}$ and $\overline{\nu}^i=\nu^i\phi^{-10}$ just as for the case of an electromagnetic field. Inasmuch as the product (Hermitian conjugate of ψ)$\times$(ψ) is the probability density (per unit proper three-volume), one sees that the transformation $\overline{\psi}=\psi\phi^{-3}$ is entirely natural.

Our analysis can now be summarized as follows: On an initial spacelike hypersurface pick a three-metric g_{ij} (and a related hypersurface-compatible set of λ vectors), a transverse-tracefree tensor p^{ij}_{TT}, and a scalar τ. We also choose a single-helicity spinor ψ that satisfies the energy conditions $\mu\geq 0$, $\mu^{-2}\nu_i\nu^i\leq 1$. Using the conformal transformations we arrive at a set of equations for ϕ and W^i of precisely the form (A.12) and (A.13). Then, for the ϕ and W^i that satisfy these equations, a complete set of initial data satisfying the constraints is given by

$$\begin{aligned}&\overline{g}_{ij}=\phi^4 g_{ij},\\ &\overline{p}^{ij}=\phi^{-10}p^{ij}_{TT}+\phi^{-4}[(LW)^{ij}+\tfrac{1}{2}\tau g^{ij}],\\ &\overline{\psi}=\phi^{-3}\psi.\end{aligned} \tag{3.13}$$

IV. SCALAR-TENSOR THEORY

Although originally expressed in a form with a variable gravitational constant, the scalar-tensor theory of gravitation was put by Dicke in an alternative form in which the scalar field acts along with other sources in the Einstein equations to produce a spacetime metric.[15] The scalar field, here denoted by λ, obeys field equations in addition to the Einstein equations, but introduces no additional constraints on the choice of initial data. This fact makes it quite straightforward to adapt our techniques to this case, as only the initial-value equations (1.1) and (1.2), suitably modified in terms of source structure, need to be considered.

A first-order formalism for the above version of scalar-tensor theory was developed by Toton.[7] He used the "canonical" variables g_{ij} and $\pi^{ij}=g^{1/2}p^{ij}$ to describe the tensor field and introduced a canonical pair which we denote by λ and α to characterize the scalar field, where $\alpha=g^{1/2}a$ is the canonical momentum density. In terms of these variables, the initial-value equations become

$$-2\overline{\nabla}_j\overline{p}^{ij}=\overline{a}\overline{\nabla}^i\overline{\lambda}+16\pi\overline{\nu}^i, \tag{4.1}$$

$$\overline{R}-(\overline{p}_{ij}\overline{p}^{ij}-\tfrac{1}{2}\overline{p}^2)=(6+4\omega)\overline{\lambda}^{-2}(\overline{\nabla}_k\overline{\lambda})(\overline{\nabla}^k\overline{\lambda}) +(6+4\omega)^{-1}\overline{\lambda}^2\overline{a}^2+16\pi\overline{\mu}. \tag{4.2}$$

The dimensionless coupling constant for the scalar field is denoted by ω and, as before, $\overline{\mu}$ and $\overline{\nu}^i$ are the energy density and current density of other fields (not including the scalar and tensor fields). We shall assume here that $\bar{\mu} = \mu\phi^{-8}$ and $\bar{\nu}^i = \nu^i\phi^{-10}$ as in the preceding sections.

Examination of the structure of (4.1) and (4.2) indicates that the conformally transformed scalar-field initial data should be defined by

$$\bar{\lambda} = \lambda, \quad \bar{a} = \phi^{-6}a, \tag{4.3}$$

in order to maintain consistency with the transformation properties in the tensor and matter variables. Note that (4.3) implies $\bar{\alpha} = \alpha$ so that the commutator (Poisson bracket) of $\bar{\lambda}$ and $\bar{\alpha}$ is preserved under this mapping. Also, (4.3) ensures that no new terms involving $\nabla_i\phi$ appear to complicate the analysis. Taking (4.3) and the previous results embodied in (A12) and (A13) into account we find

$$\nabla_j(LW)^{ij} + 6(LW)^{ij}\nabla_j \ln\phi + \tfrac{1}{2}\nabla^i\tau + \phi^{-6}(8\pi\nu^i + \tfrac{1}{2}a\nabla^i\lambda) = 0 \,, \tag{4.4}$$

$$\begin{aligned} -8\nabla^2\phi = &-[R - (6+4\omega)\lambda^{-2}(\nabla_k\lambda)(\nabla^k\lambda)]\phi \\ &+[M_{TT} + (6+4\omega)^{-1}\lambda^2 a^2]\phi^{-7} \\ &+2M_{TL}\phi^{-1} + (M_L - \tfrac{3}{8}\tau^2)\phi^5 + 16\pi\mu\phi^{-3}. \end{aligned} \tag{4.5}$$

It is of some interest to compare the behavior of the scalar field as a source of the tensor field to the behavior of the other sources we have treated. In particular, note that the current density of the scalar field $\bar{a}\bar{\nabla}^i\bar{\lambda}$ transforms according to $\bar{a}\bar{\nabla}^i\bar{\lambda} = \phi^{-10}a\nabla^i\lambda$, just as for the other sources, where we found $\bar{\nu}^i = \phi^{-10}\nu^i$. On the other hand, the energy density has two parts that transform differently[16]: (1) a "density of potential energy" $\bar{\lambda}^{-2}\bar{g}^{ij}(\bar{\nabla}_i\bar{\lambda})(\bar{\nabla}_j\bar{\lambda}) = \phi^{-4}\lambda^{-2}g^{ij}(\nabla_i\lambda)(\nabla_j\lambda)$, which can be compared with the analogous expression for the tensor field $\bar{R} = \phi^{-4}R - 8\phi^{-5}\nabla^2\phi$, and (2) a "density of kinetic energy" $\bar{\lambda}^2\bar{a}^2 = \phi^{-12}\lambda^2 a^2$, which is analogous to the kinetic term $\bar{g}_{ij}\bar{g}_{kl}\bar{p}^{ik}_{TT}\bar{p}^{jl}_{TT} = \phi^{-12}g_{ij}g_{kl}p^{ik}_{TT}p^{jl}_{TT}$ of the tensor field. Of course, it is the inhomogeneous term in the scalar curvature $\bar{R}$ involving $\nabla^2\phi$ that enables the gravitational initial-value equations to be solved by the present method. Since this term arises from the tensor gravitational field, the presence of an additional scalar field makes no essential difference in the character of the resulting equations.

In previous papers,[1,2] the existence and uniqueness of solutions for ϕ and W^i were established for the exact equations when τ = constant and $\nu^i = 0$. That analysis does not require modification when a scalar field is present whose current density vanishes initially. We also treated by perturbation methods the more general case when $\tau = \text{constant} + \delta\tau(x)$, $\nu^i = \delta\nu^i(x)$.[1] Likewise, here the same affirmative results are obtained if the scalar field current density $a\nabla^i\lambda$ is assumed to be small. The proofs of these assertions rest entirely upon the facts that (1) the term added to M_{TT} in (4.5) is positive-definite, and (2) the sign of the term added to R is irrelevant, as shown in Ref. 2. Thus, the analysis of existence and uniqueness of solutions given in Ref. 2 carries directly over to the present case and we shall not repeat it here. Our conclusion is, therefore, that the character of initial-value equations for the scalar-tensor theory does not differ in any important way from that of standard general relativity.

V. DISCUSSION AND GENERALIZATIONS

In this paper we have illustrated the analysis of initial-value problems for fundamental fields coupled to the gravitational field in general relativity. The results we have given can be readily generalized in several directions.

The electromagnetic field can possess charges and currents as sources. The only change in the initial-value equations is that the constraint on the electric field becomes

$$\bar{\nabla}_i \bar{E}^i = 4\pi\bar{\rho} \,, \tag{5.1}$$

where $\bar{\rho}$ is the charge density (per unit proper three-volume). However, it is easy to see that this equation is conformally invariant for all $\phi > 0$ if we set

$$\bar{E}^i = \phi^{-6}E^i, \quad \bar{\rho} = \phi^{-6}\rho. \tag{5.2}$$

Therefore, the gravitational initial-value equations have the same form as when $\bar{\rho} = 0$.

The massless Dirac equation treated in Sec. III can be modified by the addition of the usual mass term. This removes the helicity constraint. The only important change in the gravity equations comes through the appearance of an additional term in the source energy density. If the Dirac mass, say $\bar{m}$, is to be incorporated in a manner that is *independent* of the solution ϕ that sets the scale, i.e., if $\bar{m}$ is not scaled, then the additional term transforms as $\bar{\mu}' = \mu'\phi^{-6}$, and the scale equation (A13) is modified only to the extent of having an additional term $16\pi\mu'\phi^{-1}$ on the right-hand side. It is important to note that this extra term does not affect the theorems on existence and uniqueness of solutions for ϕ proved in Ref. 2.[17]

There are many other cases that can be treated using these methods. Some of them are treated in Ref. 3.

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APPENDIX

Latin indices run through 1, 2, 3, and Greek indices denote 0, 1, 2, 3 with $x^0 = t =$ time. The signature of the spacetime metric is $(-+++)$. The Einstein equations have the form ${}^{(4)}R_{\mu\nu} - \frac{1}{2}{}^{(4)}g_{\mu\nu}{}^{(4)}R = 8\pi T_{\mu\nu}$, with the curvature tensor satisfying $(D_\mu D_\nu - D_\nu D_\mu)A_\sigma = A_\rho^{(4)}R^\rho{}_{\sigma\nu\mu}$ and the Ricci tensor defined by ${}^{(4)}R_{\mu\nu} = {}^{(4)}R^\rho{}_{\mu\rho\nu}$. The scalar curvature of a spacelike three-sphere is positive with these conventions.

The metric of a spacelike three-manifold is denoted by g_{ij}. The second fundamental tensor of this embedded manifold is denoted by K_{ij}. We define $p^{ij} = Kg^{ij} - K^{ij}$ and decompose this three-tensor orthogonally with respect to g_{ij} by the prescription[11]

$$p^{ij} = p^{ij}_{TT} + (LW)^{ij} + \tfrac{1}{3}pg^{ij}, \tag{A1}$$

where

$$g_{ij}p^{ij}_{TT} = g_{ij}(LW)^{ij} = 0, \tag{A2}$$

$$(LW)^{ij} = \nabla^i W^j + \nabla^j W^i - \tfrac{2}{3}g^{ij}\nabla_k W^k, \tag{A3}$$

$$p = g_{ij}p^{ij}, \tag{A4}$$

$$\nabla_j p^{ij}_{TT} = 0 \Longleftrightarrow \nabla_j(LW)^{ij} = \nabla_j(p^{ij} - \tfrac{1}{3}pg^{ij}). \tag{A5}$$

A conformal transformal transformation of g_{ij} is defined by $\bar{g}_{ij} = \phi^4 g_{ij}$, $\phi(x) > 0$. This leads to

$$\bar{g}^{ij} = \phi^{-4}g^{ij}, \tag{A6}$$

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + 2\phi^{-1}(\delta^i_j\partial_k\phi + \delta^i_k\partial_j\phi - g_{jk}g^{im}\partial_m\phi), \tag{A7}$$

and

$$\bar{R} = R\phi^{-4} - 8\phi^{-5}\nabla^2\phi,\ \nabla^2\phi = g^{ij}\nabla_i\nabla_j\phi\ . \tag{A8}$$

The conformal mappings of the parts p^{ij} are carried out as follows. We set

$$\bar{p}^{ij}_{TT} = \phi^{-10}p^{ij}_{TT} \tag{A9}$$

as this mapping uniquely preserves the defining properties for all $\phi(x) > 0$. We do not transform W^i. Then (A7) shows that

$$(\bar{L}W)^{ij} = \phi^{-4}(LW)^{ij}\ . \tag{A10}$$

The scalar trace is part of the freely specified data (i.e., independent of ϕ), therefore we define $\bar{p} = p = \frac{3}{2}\tau$, which leads to

$$\begin{aligned}\tfrac{1}{3}\bar{p}\bar{g}^{ij} &= \phi^{-4}(\tfrac{1}{3}pg^{ij})\\ &= \tfrac{1}{2}\phi^{-4}g^{ij}\tau.\end{aligned} \tag{A11}$$

Combining these results and applying them to the constraints (1.1) and (1.2) give[1]

$$\nabla_j(LW)^{ij} + 6(LW)^{ij}\nabla_j\ln\phi + \tfrac{1}{2}\nabla^i\tau + 8\pi\nu^i\phi^{-6} = 0, \tag{A12}$$

$$\begin{aligned}-8\nabla^2\phi = {}& -R\phi + M_{TT}\phi^{-7} + 2M_{TL}\phi^{-1}\\ &+ (M_L - \tfrac{3}{8}\tau^2)\phi^5 + 16\pi\mu\phi^{-3},\end{aligned} \tag{A13}$$

where τ and the M's are defined in (2.15) and $\bar{\mu} = \mu\phi^{-8}$, $\bar{\nu}^i = \nu^i\phi^{-10}$, as described in Sec. I.

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[1]N. Ó Murchadha and James W. York, Jr., Phys. Rev. D 10, 428 (1974); 10, 437 (1974). These are parts I and II of the present series. See also James W. York, Jr., Phys. Rev. Lett. 26, 1656 (1971); 28, 1082 (1972); J. Math. Phys. 14, 456 (1973).

[2]N. Ó Murchadha and James W. York, Jr., J. Math. Phys. 14, 1551 (1973).

[3]J. Isenberg and J. Nestor, Bull. Amer. Phys. Soc. 21, 37 (1976).

[4]An example in which more complicated relations between μ, ν^i and $\bar{\mu}$, $\bar{\nu}^i$ are found is the case of gravity with both *torsion* fields and a massive vector field. This case is analyzed in Ref. 3.

[5]Ref. 1, paper II.

[6]R. H. Dicke, Nature 192, 440 (1961); P. Jordan, Z. Phys. 157, 112 (1959).

[7]E. Toton, J. Math. Phys. 11, 1713 (1970).

[8]J. Isenberg (unpublished).

[9]Clearly, one could equally well work in terms of the vector density $\mathcal{E}^i = g^{1/2}E^i$ in which case $\nabla_i\mathcal{E}^i = \partial_i\mathcal{E}^i$ and $\bar{\mathcal{E}}^i = \mathcal{E}^i$. The magnetic vector can be dealt with in the same way.

[10]One can use, for example, the covariant orthogonal vector decomposition $V^i = V^i_T + \nabla^i\theta$, for an arbitrary V^i.

[11]For the construction of p^{ij}_{TT}, see J. W. York, Jr., J. Math. Phys. 14, 456 (1973); Ann. Inst. Henri Poincaré 21, 319 (1974).

[12]For a discussion of the coupled gravitational and neutrino fields, see D. Brill and J. A. Wheeler, Rev. Mod. Phys. 29, 465 (1957).

[13]P. A. M. Dirac, in *Recent Developments in General Relativity* (Pergamon, New York, 1962).

[14]Note that in the two-component spinor form of the neutrino equations, there is no constraint equation analogous to (3.6).

[15]R. H. Dicke, Phys. Rev. 125, 2163 (1962).

[16]The different transformations for the scalar field's "potential" and "kinetic" energies can also be traced from the fact that these terms contain different powers of the gravitation constant G, which we have given the fixed value unity throughout this paper.

[17]It is not difficult to show that the hypotheses of theorems I, II, and III of Ref. 2 are unaffected by the presence of an extra term of the form $\mu'\phi^{-1}$, with $\mu' \geq 0$.