

Visiting the Gödel Universe

Frank Grave and Michael Buser

Abstract— Visualization of general relativity illustrates aspects of Einstein's insights into the curved nature of space and time to the expert as well as the layperson. One of the most interesting models which came up with Einstein's theory was developed by Kurt Gödel in 1949. The Gödel universe is a valid solution of Einstein's field equations, making it a possible physical description of our universe. It offers remarkable features like the existence of an optical horizon beyond which time travel is possible. Although we know that our universe is not a Gödel universe, it is interesting to visualize physical aspects of a world model resulting from a theory which is highly confirmed in scientific history.

Standard techniques to adopt an egocentric point of view in a relativistic world model have shortcomings with respect to the time needed to render an image as well as difficulties in applying a direct illumination model. In this paper we want to face both issues to reduce the gap between common visualization standards and relativistic visualization. We will introduce two techniques to speed up recalculation of images by means of preprocessing and lookup tables and to increase image quality through a special optimization applicable to the Gödel universe. The first technique allows the physicist to understand the different effects of general relativity faster and better by generating images from existing datasets interactively. By using the intrinsic symmetries of Gödel's spacetime which are expressed by the Killing vector field, we are able to reduce the necessary calculations to simple cases using the second technique. This even makes it feasible to account for a direct illumination model during the rendering process. Although the presented methods are applied to Gödel's universe, they can also be extended to other manifolds, for example light propagation in moving dielectric media. Therefore, other areas of research can benefit from these generic improvements.

Index Terms—General relativity, Gödel universe, nonlinear ray tracing, time travel.

1 INTRODUCTION

The theory of relativity is difficult to understand intuitively, because relativistic effects only appear under extreme conditions. Traveling near concentrated masses like black holes or at velocities close to the speed of light has severe effects on the evolution of time or the structure of space. The mathematical description of general relativity is provided by Einstein's field equations. On the one hand the three spatial dimensions and the time dimension have to be described as a continuum and not as separate quantities. This spacetime then is "curved" by gravitation resulting from mass distributions like galaxies. Although these features can be described mathematically, it is hard to develop an imagination just by considering these equations. Visualization of general relativity can help to gain a deeper insight into the theory for an expert as well as the layperson. We visualize what a physical observer is able to see when he is situated in a general relativistic environment, focussing on a particularly interesting solution and introducing two new techniques for the visualization process.

In 1949 Kurt Gödel found a special solution of Einstein's field equations with many remarkable features. This Gödel universe represents a rotating universe which has a crucial impact on the way light propagates. For example, light rays are always restricted to a limited region which therefore represents an optical horizon. Furthermore, Gödel's universe was the first solution of Einstein's field equations which allowed time traveling. Exactly this optical horizon divides the spacetime into two parts. In the inner part time travel is not possible and causality is conserved. When traveling beyond the optical horizon it is possible to travel into the past of a person located in the causal inner region. Although it is proven that our universe is not rotating and therefore *not* a Gödel universe, this solution indicates that time travel is *in principle* possible, if we only take general relativity into account.

For visualizing general relativity standard raytracing methods have to be generalized, because light rays are curved lines and the velocity of light is not infinite. These light rays are called null geodesics to stress the geometrical origin of the underlying spacetime. The numerical calculation of such geodesics is very time-consuming and therefore this task is usually performed on large CPU-clusters. In our paper we want to present two new techniques of visualizing general relativity.

First, we will introduce a specific way to store image data to enable us to fine-tune the scene parameters in a post rendering process. Image data is usually stored as RGB data, which does not contain any information on e.g. the spectrum of radiation emitted by an object. If a spectrum is transformed into RGB data, information will be lost, so shifting a spectrum stored as plain RGB data will lead to false results. In such a case it is useful to store more than the actual color values of each pixel. It can also be necessary to visualize just a certain aspect of general relativity, because the large amount of very different visual effects can be confusing and misleading. With our flexible approach, the task of visualizing a specific scene is split into two consecutive sub-tasks. We only have to render a certain scene once. After that, we are able to change the look of this scene or the effects displayed for the currently given arrangement of objects and observer interactively. Then it is possible to display an interesting scene with an optimized set of parameters in reasonable time.

Second, we can make use of the specific symmetries of the given world model to achieve further improvements. Exploiting symmetries is a fundamental principle to simplify and solve physical and mathematical problems. Therefore it is worth to make use of symmetries also in visualization. Light rays in curved spacetimes can be very complicated and it is not always possible to find the proper geodesics. Shadow rays, for example, have to be constructed between two given points in spacetime which is hardly feasible to accomplish with standard numerical techniques. By considering the symmetries, however, an appropriate transformation can be found to map each geodesic onto a geodesic with simple initial conditions. We used this approach to calculate shadow rays and render illuminated objects in the Gödel universe.

Because Gödel's universe is an outstanding example for the consequences of general relativity and has not been visualized yet, we used it to demonstrate our techniques to give a first insight into this universe. The application of those techniques is not limited to a particular spacetime.

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2 RELATED WORK

Visualization of astronomical data or astrophysical models has gained importance as well as quality over the last years, in which a variety of methodical approaches for different target audiences have been developed and improved.

Magnor et al.[17] introduced an interactive visualization tool for realistic rendering of arbitrary dust distributions which are also interesting for a general audience in a planetarium show, for example. Strengert et al.[24] have extended GPU-based ray casting to spectral volume rendering to demonstrate effects of selective absorption and dispersion in certain materials. A comparable pre-processing technique is deferred shading[3], where a large framebuffer is used to store surface properties of each pixel before calculating the final pixel color. Open source software like *Celestia*[1] let the user explore the universe on a common desktop computer. In this way we can become aware of the vast empty regions in our universe. Li et al. developed[16] the scalable world-in-miniature (WIM) map to display the astrophysical universe using a logarithmic scale to simplify way finding and navigation. It is also able to depict the uncertainty of the underlying astrophysical data due to observational errors[15].

Weiskopf[26] provided a systematic approach to visualize many fields of the theory of relativity. His work was in part based on an easily extendible ray tracing system written by Gröne[7]. Müller[19] later improved the visualization of special and general relativity regarding more complex world models. Due to the rapidly increasing computing power of modern GPUs, there exist several interactive simulations of special relativistic content like the work of Savage et al.[23]. A very nice round-up over visualization in special and general relativity as well as in astrophysics can be found in [27]. Focussing on a general audience of laypersons, Kraus et al.[11] provide a large number of media files for educational purposes.

Leonhardt et al. showed in [13, 14] that there is a link between the optics in a moving media and relativity. A black hole or other relativistic effects can be emulated by propagating light in such media.

In 1949 Gödel published a spacetime which represents an exact solution of Einstein's field equations[8]. In his paper he also went into the most important properties of his model. A few years later Kundt[12] integrated the geodesic equation for Gödel's spacetime, followed by Chandrasekhar[2] and Novello[21]. A more detailed description of the inner structure of the Gödel universe can be found in [9].

3 GÖDEL'S UNIVERSE

3.1 General Relativity

In this section, only a very short introduction to the concepts of general relativity can be given. A very detailed presentation and the following equations can be found in [22] for example.

3.1.1 Einstein's Insights

In 1905, Einstein found the theory of special relativity[4], where the correct transformation between coordinate systems in relative motion especially at a high relative velocity is described correctly. Special relativity has profoundly changed our view of space and time. Lengths, elapsed times or even the mass of an object become dependent on the relative speed between observer and object. For example, a clock of a fast moving observer runs slower than the clock of a resting observer when both times are measured by the resting observer. But the moving observer will measure no time dilation effect regarding to his own watch.

General relativity, a geometric theory of gravitation, was discovered in 1915[5]. The theory extends special relativity regarding gravitation. It is shown that even the pure presence of mass has effects on space and time similar to the effects in special relativity. Like a slower ticking clock, a person living near a black hole will be aging slower than a person far away from the black hole. Moreover, the whole spacetime becomes "curved", making for example the circumference of a circle $o \neq 2\pi r$ or the volume of a sphere $V \neq 4/3\pi r^3$, where r is a radial coordinate.

3.1.2 Manifolds and Metrics

A general relativistic model is mathematically described by a line element ds which specifies distances on a four-dimensional pseudo-riemannian manifold, the so-called spacetime. The squared infinitesimal distance ds^2 between given points \mathbf{x} and $\mathbf{x} + d\mathbf{x}$ on the manifold is given by the line element

$$ds^2 = \sum_{\mu, \nu=0}^3 g_{\mu\nu}(\mathbf{x}) dx^\mu dx^\nu,$$

where $g_{\mu\nu}(\mathbf{x})$ is an entry of a 4×4 matrix, the metric tensor at the point \mathbf{x} . dx^μ is an infinitesimal distance in the μ -direction of the coordinate system in which the spacetime is represented. Properties like the curvature of the spacetime or the path of a moving object can be obtained from this line element. In a flat spacetime without masses the spacetime in cylindrical coordinates is expressed with the simple Minkowski line element

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2, \quad (1)$$

with the speed of light c . Obviously, the "space" part of the line element $-dr^2 - r^2 d\phi^2 - dz^2$ is Euclidian.

3.1.3 Geodesic Equation

Paths of light rays or massive objects under the influence of gravitational forces resemble basically "curved" paths in a given four-dimensional coordinate system. These paths contain one time component and three spatial components and can be obtained by solving the geodesic equation

$$\frac{d^2 x^\sigma}{d\lambda^2} + \sum_{\mu, \nu=0}^3 \left(\Gamma_{\mu\nu}^\sigma \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right) = 0, \quad (2)$$

where $\Gamma_{\mu\nu}^\sigma$ are the Christoffel symbols for the corresponding metric and the index $\sigma \in \{0; 1; 2; 3\}$. If the spacetime is not highly symmetric, this equation is almost always impossible to solve analytically in order to obtain $\mathbf{x}(\lambda)$, where λ is an affine parameter.

3.2 Gödel's Metric

3.2.1 Einstein's Birthday

As a good friend of Albert Einstein, Kurt Gödel published an article entitled "A remark about the relationship between relativity theory and idealistic philosophy"[6] in 1949 to pay tribute to Einstein on the occasion of his 70th birthday. One of Gödel's thoughts, which he discussed in this article, was the influence of the general relativity on the nature of time. He discussed the possibility of a reversed time ordering and as consequence the breakdown of causality. Indeed, in the same year Gödel published a mathematical model of a universe, which adheres strictly to Einstein's equations of general relativity, and allows an observer to go on a mind-boggling journey into his own past. This Gödel universe was the first model of a spacetime which showed the severe consequences of Einstein's theory that clearly.

The Gödel universe is an example of a rotating, not isotropic but homogeneous universe. In mathematical terms this particular spacetime is given by its line element[10]

$$ds^2 = c^2 dt^2 - \frac{dr^2}{1 + \left(\frac{r}{2a}\right)^2} - r^2 \left(1 - \left(\frac{r}{2a}\right)^2 \right) d\phi^2 - dz^2 + \frac{2r^2 c}{\sqrt{2a}} dr d\phi.$$

The time is given by t , whereas r , ϕ and z denote the spatial position and are given in cylindrical coordinates. It is remarkable that in the limit $a \rightarrow \infty$ the Gödel metric results in a flat spacetime metric, given in cylindrical coordinates by equation (1). A closer look at the metric reveals that along the z -coordinate the metric has the structure of flat space. Beside the time t , it is therefore often sufficient to contemplate only the subspace which is identified by the r and ϕ coordinates. We will call this subspace the $r - \phi$ subspace.

Even though in the following sections all the considerations are done with respect to the origin, this is not really a constriction. All statements can be generalized due to the homogeneity of the Gödel universe. The homogeneity allows us to regard each spacetime point as the center of the coordinate system.

3.2.2 Time Traveling

In the Gödel metric one decisive parameter appears. It is the Gödel parameter a , which determines the rotation as well as certain geometric properties. From that parameter we can define $2a$ as the critical Gödel radius, which separates the spacetime into two parts. The inner part consists of all points with $r \leq 2a$. In geometrical terms this means that the inner part is enclosed by an infinite extended cylinder with $2a$ as radius. Hence, all spacetime points beyond the Gödel radius belong to the outer part.

It is intriguing in Gödel's universe that the causality breaks down at the transition between both parts. While two consecutive events at the origin never change their order for an observer located in the inner part, an observer in the outer part may perceive a reversed order of both events. Although the proper time of such an observer evolves as usual, in respect of the origin it has the opposite direction. This effect can be used to go on an abstruse journey into one's own past. For example an observer who is located at the origin at the beginning travels through the outer part to move back in time. This enables him to return to the origin at an earlier time than he had started.

3.2.3 Optical Horizon

Beside the causality breakdown between the inner and the outer part, we have also a remarkable optical feature. The view of an observer located at the origin is always limited to the inner part. A look at the structure of the geodesics reveals this behavior. Although a particle or a photon moves along the z -direction linearly like in flat space, in the r - ϕ subspace they always return to their source. Moreover, the geodesics are even closed and of elliptical shape. Fig. 1 shows some of such geodesics in the r - ϕ subspace. Whatever geodesic is being considered, as long as it passes the origin the whole geodesic is completely within the inner part. As a result light emitted at the origin never exceeds the Gödel radius and thus does not reach the outer part. Because the Gödel universe is homogeneous we can also argue the other way around. A photon emitted beyond the Gödel radius never reaches the origin. This restriction is the reason, why an observer at the origin can never see or communicate with a time traveling person.

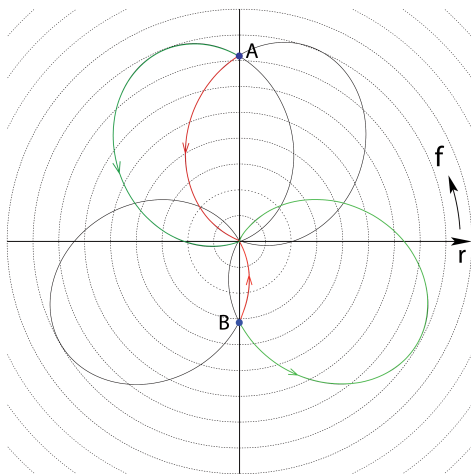


Fig. 1. Geodesics in the r - ϕ subspace. The light rays evolve clockwise into the past and counter-clockwise into the future respectively. An object at position A could be seen twice by an observer situated at the origin (see section 6.1).

4 NONLINEAR 4D RAY TRACING

4.1 4D Ray Tracing

Classical 4D ray tracing is an extension to 3D ray tracing, where the speed of light is regarded as infinite. In four dimensions, the speed of light is limited to $c = 3 \times 10^8$ m/s or $c = 1$ in geometrical units. If the so-called Lorentz-transformation between moving coordinate systems and all resulting influences are implemented, one is able to visualize effects of special relativity ([11], [26] and [23]) within the Minkowski metric of flat spacetime. The more challenging task of obtaining correct visualizations of characteristics in general relativity will be explained below.

4.2 Ray Tracing in Curved Spacetimes

4.2.1 Defining a scene

Objects in a general relativistic scene can be implemented in two different ways. Properties like size or center of a coordinate object are described in pseudo-cartesian coordinates on the manifold of the corresponding spacetime. Large objects can be put in a scene, when coordinate objects are used. This kind of object-definition does not take the curvature of the spacetime into account, so we are restricted to relatively flat regions of the spacetime when using these objects. This is used to define, for example, a background texture to the surroundings of a black hole. When objects in a highly curved area of spacetime are needed, we can use local objects. In general relativity it is shown that there exists a locally flat spacetime in the surrounding area of any point on a manifold, where local objects are characterized with respect to Euclidian coordinates ([20], implemented in [19]). Points and directions on the manifold resemble positions and velocities of local objects. Hereby it is possible to visualize the correct appearance of a geodesically moving small object regarding for example the finiteness of light. The camera is also specified with respect to a locally flat spacetime.

4.2.2 Solving the Geodesic Equation

The geodesic equation (2) consists of four second order ordinary differential equations. To solve them numerically we have to define initial conditions concerning position and direction. The initial position has to be a valid point on the manifold; the initial direction is subject to a normalization condition which assures that the object has a valid velocity. Photons travel on *lightlike* rays at the speed of light c and massive particles on *timelike* curves with a velocity less than c .

4.2.3 Calculating intersections

The numerical integration yields a curved ray, represented by usually about 1000 straight segments in a four-dimensional spacetime. Each segment then has to be tested for intersections with the objects of the scene. If the object is moving, the corresponding timelike curve contains the information about the object reaching a certain point at a certain time. An intersection is found, when a light ray intersects the object regarding *both* space and time.

5 EXTENSIONS AND IMPROVEMENTS

In this section, we will illustrate our new approaches in visualizing general relativity exemplified with the Gödel universe. The discussion on new physical insights as well as the advantages of these new methods can be found in section 6.

Visualization is a powerful tool to make complicated relations accessible to human beings. To understand a relativistic world model like the Gödel universe intuitively, auxiliary visualization might be decisive. For example if one is interested in the optical distortion of an object he might try to find an appropriate texture to display this effect. Also, the Doppler effect or the time order of visible objects could be in the focus of interest. It is desirable to be able to generate visualizations interactively. Unfortunately, it is hardly possible to accomplish this task in general. For example, the geodesics in Gödel's universe are too complicated to utilize them for interactive visualizations. Either we choose an approach which is directly applicable for

many spacetimes or we try to find an analytic solution of the geodesic equation for a specific spacetime.

Sections 5.1 and 5.2 address the first approach. To be able to interactively visualize a given scene it is necessary to preprocess the information non-interactively. After having calculated the necessary data, it is possible to visualize the scene interactively in a postrendering step. In this step lookup tables will be generated from the preprocessed data. Section 5.3 offers a solution to the second approach. Here, the intrinsic symmetries will be used to visualize the Gödel universe faster. To achieve more realism the visualization is also done in consideration of direct illumination.

5.1 Preprocessing

In a first step, the desired images are precalculated using a custom file format. Two files per image are being generated, one containing the RGB data of the calculated image (RGB-file) for the currently used scene, the other file holding detailed information about each intersection of each ray with the scene (data-file). This data is used to generate several lookup tables for this image concerning a pixel mask to stencil empty pixels, frequency-shift as well as time-shift data, details regarding the intersection of the curved ray with an object and properties of the ray itself to be able to calculate polarization effects in the future.

The calculation of this data is the most expensive part of image generation for two reasons: First, the numerical integration of the geodesic equation is very time-consuming. To approximate the curved ray with line segments usually 1000 to 5000 integration steps have to be carried out. Each step takes about 10 μ s to finish. Second, to intersect a curved ray with the scene each segment has to be intersected with the complete scene in the worst case, which takes about 1 μ s per object. These calculations carried out on a CPU can be accelerated very simply by parallelization on a cluster, because all pixels are independent from each other ([19], [26]).

5.2 Post Rendering and Interactivity

In the post rendering process, the data-file is being processed to generate lookup tables which are then used as textures in the corresponding fragment shader as shown in Fig. 2. With these shaders and the user input concerning the parameters of the desired effect it is possible to visualize the precalculated data in various ways interactively. Once the user has found interesting parameters for certain effects it is possible to script the application in order to generate several new image sequences very quickly.

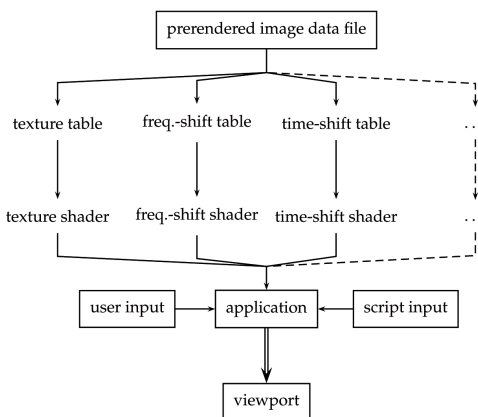


Fig. 2. Post rendering process.

Three shaders will be explained in this work, starting with the re-texturing of an image.

5.2.1 Re-texturing

It is often helpful to replace just the texture of an object in order to understand its appearance better. A checkerboard texture can be useful to understand the visual distortions on a certain region of an object. A

real image will help to see if an object appears rotated or not, because it is possible to distinguish front and back sides of it. If one is only interested in the visual size of this object, a bright single color will be sufficient.

To be able to switch between textures two things are necessary: First, an object lookup table is used to decide which pixel belongs to which object. After that, the uv lookup table is queried to determine the texture coordinate of this pixel. Rendering can be further sped up by using the pixel mask for stenciling empty regions of an image. This is used for all other shaders as well. Now the user can select new textures for all visible objects and can also alter uv -coordinates by means of offsets and scaling factors. With this method it is for example possible to let a sphere rotate only by appropriately changing the texture coordinates and without having to precalculate the image again. Fig. 6 illustrates the re-textured reference image in Fig. 5.

5.2.2 Frequency-shift

A frequency-shift occurs when electromagnetic radiation that is emitted from an object is shifted to lower frequencies (“red-shift”) or to higher frequencies (“blue-shift”) of the electromagnetic spectrum. In classical physics as well as special relativity, frequency-shifting happens when light source and observer move with a nonzero relative velocity. This is the so-called (relativistic) Doppler effect. In general relativity there is an additional gravitational frequency-shift effect. Simply speaking, when a photon moves away from a strong gravitational source it loses energy. Therefore, its wavelength is shifted towards the lower energetic red end of the spectrum. Vice versa, a photon is blue-shifted when moving towards a gravitational source.

Mathematically speaking, the photon is parallel-transported from the emission event to the observer along the geodesic and the frequencies ν of the corresponding photon and the total frequency-shift are then connected via

$$z_{\text{total}} = \frac{v_{\text{observer}}}{v_{\text{emission}}}.$$

Light is red-shifted, if $z_{\text{total}} \in (0; 1)$. For z_{total} being larger than 1, the photon is blue-shifted.

The temperature of an object is associated to a certain spectrum. This spectrum is then shifted according to the equation above. After transforming the shifted spectrum to chromaticity coordinates XYZ through folding the spectrum with the CIE color matching functions, the RGB values for this temperature can be calculated in a given color system. We have developed a shader implementation of [25]. If we assume black body radiation, the spectrum of the emitted radiation is a Planck spectrum. Shifting such a spectrum is easy, because a shifted Planck spectrum is again a Planck spectrum. So a one-dimensional lookup texture, associating a spectrum to the visible color, could be used to obtain the shifted colors. We have chosen the general and extensible approach of [25], although we consider black body radiation at the moment.

The frequency-shift data is stored in a lookup table for the frequency-shift shader. The user can alter the temperature of an object and see the change of this operation immediately. It could be interesting to visualize an object with a surface temperature of the sun in order to see how it would appear in the Gödel universe. In Fig. 7 we see the reference data visualized with a surface temperature of 5000K, 1000K less than the surface temperature of the sun.

5.2.3 Time-shift

A very elementary property even in classical physics is the finiteness of the speed of light. If we ask for the visual impression an observer has, we need all photons which the observer *receives at the same time*. In general, these photons will not be *emitted at the same time*. From astronomy one learns that observing a galaxy far away is like looking into the past of this galaxy. The visual appearance of the Large Magellanic Cloud *now and here* results from light emitted over 150,000 years ago, the appearance of the Andromeda Galaxy *now and here* consists of light which was emitted over 2,500,000 years ago. By storing the time a light ray needs from object to observer we are able to visualize such time-distance effects.

In a first approach each object is multitextured with two textures such as earth by day and a checkerboard pattern. By defining an offset t_0 and a time difference Δt we are able to depict which photons in the current picture have been emitted in a time span $t \in [t_0; t_0 + \Delta t]$ as illustrated in Fig. 8. The time-order of visible images of an object can thus be determined. Next, we are able to simulate time-dependent textures. Imagine the sphere on which simultaneously¹ blinking flashlights are placed. These blinking spheres will not seem simultaneous to the observer due to the time the photons need to travel.

5.3 Isometries and Direct Light

Each point P of a Riemannian manifold is accompanied by its own vector space called the tangent vector space T_P . In order to connect the tangent spaces of two adjacent points the affine connection $\Gamma_{\mu\nu}^\sigma$ which are also designated as Christoffel symbols are needed as they appear in the geodesic equation (2). For ordinary flat space all $\Gamma_{\mu\nu}^\sigma$ vanish since each point has the identic vector space as its tangent space. Therefore the geodesic equation gets cut down into its simplest version with straight lines as solution.

For an arbitrary space time, however, such a solution is difficult to obtain. Usually the geodesic needs to be approximated numerically. Unfortunately, the lack of the whole analytical solution results in restrictions of the visualization process. Standard numerical algorithms like Runge-Kutta for example only solve an initial value problem. Therefore the initial point and the initial direction have to be determined first, in order to compute the geodesic. For ray casting, where the light rays are just traced back from the camera into the scenery, a proper set of initial conditions can be obtained easily. But for direct illumination also shadow rays, i.e. geodesics between the intersection point and the light source have to be taken into account. However, the necessary initial conditions of such geodesics are unknown and thus shadow normally cannot be considered during the rendering process in a curved space time. Nevertheless, we found a method to construct such geodesics in a simple way. In physics it is a common way to exploit symmetries to reduce the complexity of a problem. A similar approach we use here.

As symmetry we want to designate a continuous transformation $\mathbf{x}' = \mathbf{x}'(\mathbf{x}, \eta)$ determined by a parameter η which has no influence on the physical behavior. In terms of the geodesics such a transformation means that a solution of the geodesic equation is mapped on another solution. The Gödel universe includes five such symmetries. Three of them are trivial and can be revealed by having a look at the metric. It turns out that the metric remains unchanged by adding an arbitrary constant to the t , ϕ or z -coordinate. Therefore a geodesic can be shifted in time, rotated or moved along the z -axis without being changed. In order to obtain the remaining two symmetries we need to apply the concept of the Killing vectors. A Killing vector $\xi(\mathbf{x})$ is defined at each point in space time and thus constitutes a field. They designate the direction at each particular point in which an infinitesimal displacement given by the transformation

$$x'^\alpha = x^\alpha + \varepsilon \xi^\alpha(x^\beta) \tag{3}$$

with small $\varepsilon > 0$ leaves the metric unchanged. The requirement of a non changing metric after such an infinitesimal transformation can also be constituted by the expression

$$g'_{\mu\nu}(x^\alpha) = g_{\mu\nu}(x^\alpha).$$

The necessary equation, such a Killing vector has to fulfill to meet the above requirements, is the so-called Killing equation given as

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0. \tag{4}$$

¹In general relativity the concept of simultaneity is rather difficult. Whether an observer measures or perceives two events as simultaneous or not depends on his motion. This is not only an optical effect but also a deeply physical effect. Moreover, clocks or blinking spheres quickly become desynchronized only due to their position within a gravitational field. At the moment, we are regarding the coordinate time, where the latter effect plays no role. For more information about time measurement in the theory of relativity, see [22].

The subscripts α and β indicate the four different coordinates t , r , ϕ and z . $\xi_\mu = g_{\mu\nu} \xi^\nu$ are the covariant components of the Killing vector ξ and the semicolon denotes the covariant derivative. For details concerning the Killing equation see [18].

To extend the infinitesimal transformation of equation (3) to a continuous transformation $\mathbf{x}'(\mathbf{x}, \eta)$ with parameter η we have to conduct the displacement given by the Killing vectors successively. Therefore the transformation we are looking for is the solution of the differential equation

$$\frac{dx'^\mu(\eta)}{d\eta} = \xi^\mu(x'^\nu) \tag{5}$$

in which the Killing vector field can be interpreted as flux through which a trajectory of a particle has to be found. The construction of the transformation by making use of the Killing vector field ensures that the metric and thus the geometry remains unchanged while applying the transformation. Hence in mathematical terms these transformations are isometric and for this reason we call this kind of transformations isometries. The isometric property also is responsible for the fact that a the solution of the geodesic equation $\mathbf{x}(\lambda)$ can be mapped onto another solution $\mathbf{x}'(\mathbf{x}(\lambda), \eta)$.

It turns out that five different Killing vector fields are inherent to Gödel's spacetime. Three of them are related to the symmetries already mentioned and due to eq. (5) they are of a very simple structure. The remaining two Killing vector fields, however, are more complex. The exact solution of the Killing equation (4) can be found in [10]. In

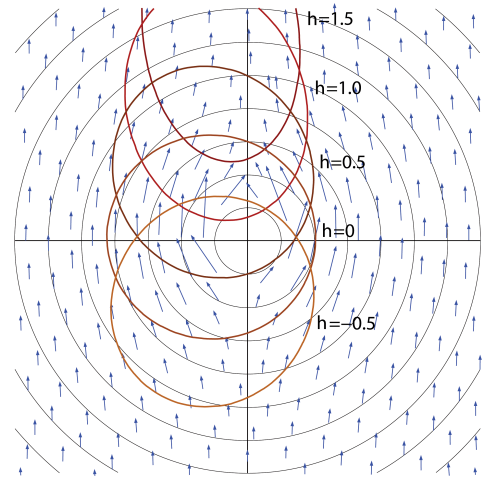


Fig. 3. Geodesic in the $r - \phi$ subspace under isometric transformation. The arrows depict the particular Killing vector field whereby the geodesic is moved along this vector field.

Fig. 3 one of those more complex Killing vector fields of the Gödel spacetime is shown. The elliptical curves are geodesics which emerge from the same isometric transformation but each for a different parameter η . The isometric transformation itself is induced by the presented Killing vector field by equation (5). Contrary to the trivial transformation of t , ϕ and z as presented above, the isometry given in Fig. 3 also has influence on the radial coordinate r . Thus a proper concatenation of the available isometries delivers a transformation with which two arbitrary spacetime points can be mapped onto each other.

In this way a geodesic starting at an arbitrary point in spacetime can be obtained very easily. All what has to be done is to apply such a transformation to an already existing geodesic, for example a geodesic through the origin O that means $r = 0$ and $z = 0$ as initial point. Fortunately, for this sort of geodesics the exact solution of the geodesic equation (2) is available and thus speeds up the computation process tremendously. Furthermore, with the exact solution it is also possible to determine a geodesic originating from a given point P and passing O . Such a geodesic does not necessarily exist, because the geodesics are enclosed in an limited area determined by the Gödel

horizon around P . Only when O is close enough to P it is possible that such a geodesic exists which can be regarded as shadow ray. If the shadow ray is not intersected by another object light from a light source located at the origin O can reach the considered point P and thus this point is illuminated by the light source.

The presented method enables us to render images which account for shadows. However, the illumination is limited to a light source in O . In order to overcome this restriction we merely need to deploy a proper isometric transformation to move the given light source situated at Q into the origin O . The considered intersection points and all objects existing in the scene have to be transformed as well. In the transformed system the shadow rays can be computed as already described. An example for a directly illuminated sphere is given in Fig. 4.

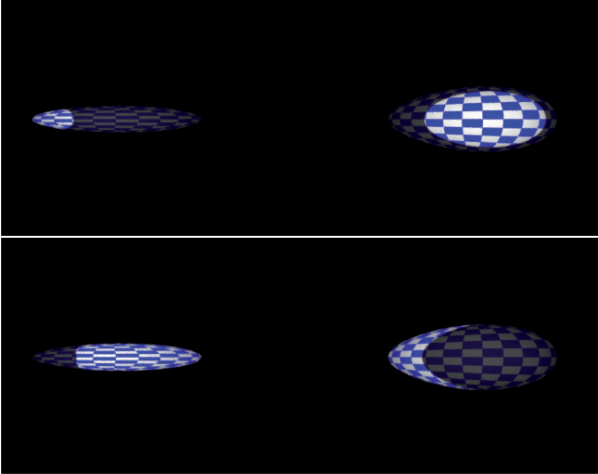


Fig. 4. Both pictures show a sphere at rest in the Gödel universe. In the upper picture the observer, the sphere and the light source is aligned, whereby the light source is located beyond the optical horizon. In the lower picture the sphere is illuminated laterally. In both cases the observer is located at the origin.

6 DISCUSSION

At first, we want to discuss the new physical content of our work before evaluating the advantages of our approach. The physical effects will be shown in more detail in the accompanying video.

In Fig. 5 we can see an earth sphere, which passes the observer's position at 70% of the speed of light and is moving in the $r - \phi$ plane. It is freely falling, meaning the only force on the sphere results from the rotating matter which defines the Gödel universe. The picture itself was calculated using a panorama camera with a field of view of $220^\circ \times 25^\circ$. It is used as a reference image for the following images.

6.1 New Insights

In Fig. 4 we can see the sphere appear exactly twice. This can be expected when taking a look at Fig. 1, where the geodesics in the $r - \phi$ subspace are depicted. A small static object at position "A" can be seen twice, because light reaches the observer at the origin along the two paths. The observer can see the front side of the object along the red path and the back side along the green path. Equally, an object at position "B" can be seen twice as well. But this is only valid for a static object. The geodesic on which the object is moving on is of the same shape as a null geodesic but returns at a smaller radius $r < 2a$. Furthermore the object emits light all the time and at every position. A fraction of this light always returns to the observer, because the geodesics are closed in the $r - \phi$ subspace. So the image of the sphere at an arbitrary position can be seen at a certain time from the observer's position. If we assume that the sphere has been moving on this geodesic all the time, the observer can see many images of this sphere at every observation time.

In our images, we restricted the time the photon was traced back into the past so that the number of visible earth images is restricted to three. The accompanying video reveals that over 20 representations appear if this tracing back is increased by one order of magnitude. Although possible with our framework, we did not use complex objects like meshes or CSG-objects. This made it possible to extract the essential physical features while conserving clarity of all images.

If we take a look at Fig. 6, we can clearly tell that the middle and right representation of the sphere show the same side. Using a checkerboard texture with an additional color gradient it is obvious that the right representation is mirrored when compared to the central sphere. This is hardly visible in the original image.

The image resulting from the frequency-shift shader shows another interesting effect. It can be shown that the relativistic Doppler effect due to the movement of an object is the only contribution to the overall frequency-shift. Therefore blue-shifted objects move towards the observer and red-shifted objects fly away. Clearly, the left representation of the sphere in Fig. 7 appears to be at rest, because the visible color equals the unshifted reference color for an object at rest in a flat spacetime. The other two representations are moving away from the observer at a high velocity.

Fig. 8 displays effects of the speed of light. We can find the time order of all visible objects when using the multitextured sphere and the time-shift shader. The upper subpicture shows a checkerboard ring corresponding to photons which were emitted a short time ago. If we adjust this time-span we find that this representation of the sphere is the closest image concerning time. In the lower subpicture we can see the right representation appear with the checkerboard texture. These photons have been emitted a longer time ago. Consequently the left representation is an image of the sphere from even longer ago.

From this discussion we obtain a variety of information concerning the distance of objects, their orientation and relative speed just by investigating a single picture. The effects visualized in Fig. 6, 7 and 8 can also be seen in the first half of the accompanying video.

It is quite astonishing that in Gödel's universe an object at rest can be simultaneously observed from two different sides as long as it emits light by itself. Fig. 4, however, shows the incident light on a sphere emitted by a light source. In the upper picture of Fig. 4 the sphere and the light source are aligned with the observer who is located in the origin. Albeit in the presented scenario the light source is beyond the optical horizon and therefore not discernible directly, its light is scattered on the surface of the sphere and thus reaches the observer. Moreover, the part of the sphere which is averted from the light source is also partially illuminated. This we can ascribe to the peculiar shape of the geodesics in Gödel's spacetime. The attached movie shows a freely moving sphere, illuminated laterally. The light source itself is depicted as a small yellow sphere in the movie. No time shift effects of the propagating light are taken into account. We therefore can observe each object only twice.

6.2 Evaluation

Nearly all images in this work have been created using only one pre-calculated dataset. In the last section we discussed this dataset concerning several different physical effects. Clearly, it is very helpful to be able to create these images and the first part of our additional video from one existing dataset. The checkerboard texture clearly shows the orientation of the visible spheres. The frequency-shift image helps to understand in which direction the spheres move without having to look at the corresponding movie. Therefore, it is not necessary to render a complete sequence in order to gain a first insight into the relative motion of the spheres. It is, however, necessary to apply an adequate temperature to the object in order to see the direction in which the object is moving. Assuming a too high temperature would let all the three visible spheres appear blue and the differences in the frequency-shifts would hardly be visible. In such a case the direction of flight could not be identified from one single image. Without the possibility of applying the post-rendering technique, the complete dataset would have to be calculated again and again until an adequate temperature had been found.

The path of the freely moving earth sphere was restricted to the causal region of the Gödel universe. Even here, we can see at least three representations of this sphere at every moment, if the object is freely moving and the propagation of light is fully taken into account. Further representations originate from light which was emitted even longer ago. It is very helpful to be able to find the time-ordering of these representations to conserve clarity and to understand what is visible in a certain image region. Again, all this data could be generated without our techniques with very time-consuming operations which do not guarantee that the best output is generated in the first pass. Examination of the data is adjourned with long idle times, when the data is generated. With the introduced post rendering technique this amount of time can be reduced immensely. The image sequence introduced in the first part of the accompanying video consists of 250 different images, which took about five days to render on a 50 node dual core CPU cluster. After the preprocessed data of an image is read, the parameters like temperature or texture coordinates can be changed interactively.

We have rendered a reference Schwarzschild scenario (512x512 pixels, black hole surrounded by 16 objects) on an ordinary AMD 64X2 4600+ desktop PC with a GeForce 8800 GTS. Directly rendering the image without the application of our techniques took about one hour. The generation of the precalculated data file takes about 10 minutes longer due to the numerous file operations. These could be optimized when rendering small resolution data sets but are indispensable when rendering high resolution images. The uncompressed data file is 79 MB large while the directly generated image usually takes less than one MB disc space.

The postrendering speed critically depends on the applied shader. The retexturing and timeshift algorithms reach framerates of about 2450, the frequency shift only yields 8.4 fps. The latter framerate could be increased immensely if we restricted the algorithm to utilize a Planck spectrum (see section 5.2.2), but a general approach results in such low framerates. Precalculation speed depends approximately in a linear manner on the image resolution and the number of objects visualized. The postrendering framerate linearly depends on the effective resolution, i.e. the number of non-black pixels.

In the second part of the paper isometric transformations were employed to demonstrate how direct illumination can be taken into account for the rendering process. But this is not the only achievement of the presented method. It is also worth mentioning how an often used technique in standard ray tracing gets accessible to a curved spacetime like the Gödel universe. In standard ray tracing it is common to use different coordinate systems during the rendering process. Since flat space is subject to standard ray tracing, a switch between different coordinate systems can be done with ease whereas in curved space this task comes along with difficulties. In the Gödel universe this problem was solved by using the intrinsic symmetries. Proper isometric transformations were applied to transform particular points into the origin. The other way around one could argue that not the particular spacetime point itself is transformed but a new coordinate system whose origin is located at that specific point describes the new situation. Thus, an effective method to switch between different frames of reference is also available for the Gödel universe.

7 CONCLUSION AND FUTURE WORK

All methods concerning preprocessing (5.1) and postrendering (5.2) are not restricted to the Gödel universe. The procedure to create a precalculated dataset can be applied to any spacetime. The introduced post rendering technique can be used as long as there is an algebraic expression for the metric. Furthermore, it can be used for non-relativistic ray tracing as well.

Finding the time order of visible objects is crucial when visualizing the Gödel universe. Even the restriction to the causal region leads to multiple images. In a next step, the sphere will be pushed across the Gödel horizon where time travel is possible. This will lead to even more representations of the object, because it can reappear before it even disappeared behind this optical and causal horizon. Despite the philosophical problems of this effect, it will become very hard to find visualizations that can be understood. With the possibility to find the

time order of these representations for a given dataset interactively we are expecting to overcome these visualization problems.

For taking direct illumination in (5.3) into account we needed to consider shadow rays which were only computable as long as the origin was involved. With isometric transformations, however, the scope on which this technique can be applied was extended to the whole spacetime. Beside direct illumination we also want to consider the attenuation of propagating light to obtain more realistic pictures. In combination with isometric transformations it is therefore sufficient to find an applicable description of the attenuation of light which is emitted at the origin.

Although the presented method was adapted only to Gödel's spacetime, isometries can be exploited in other spacetimes as well depending on the given problem. Fortunately, many problems are accompanied by symmetries, which is a decisive principle in nature.

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Fig. 5. Reference Illustration: An image of a freely moving earth sphere in the Gödel universe which is emitting light at every position. Three representations of the sphere are visible because the finiteness of the velocity of light and the motion of the sphere are both taken into account.

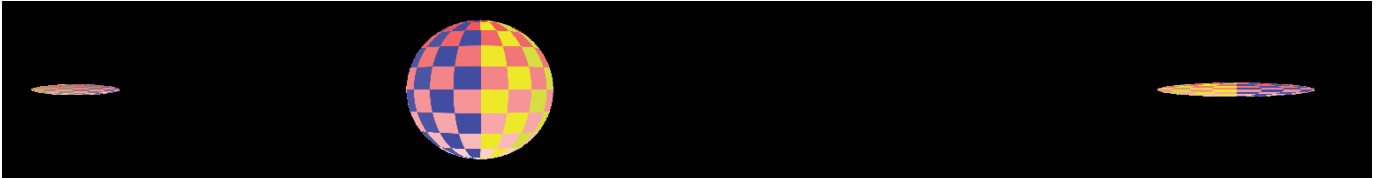


Fig. 6. Replacing the texture from the reference image with a checkerboard texture. Now it is visible that the right representation of the sphere shows the same side when compared to the center representation, but it appears mirrored. The left representation shows a color gradient from yellow to blue and is therefore showing the other side of the sphere.

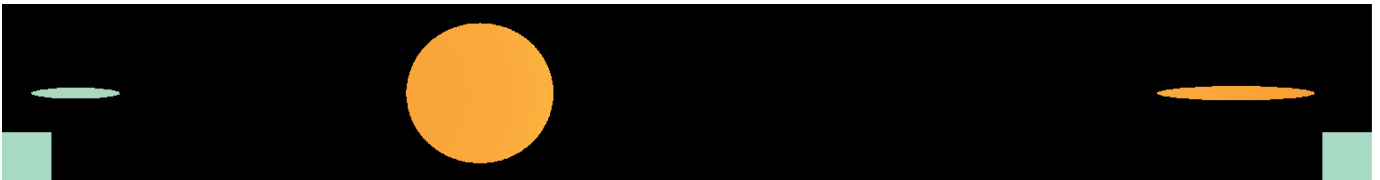


Fig. 7. Frequency-shift effects due to the relative velocity between sphere and observer. The sphere has a surface temperature of 5000K and the two squares at the bottom show the unshifted color of a resting reference object. The left representation seems to be at rest because it is colored with the reference color. The other two spheres are moving away from the observer's position at a high velocity, because they appear red-shifted.

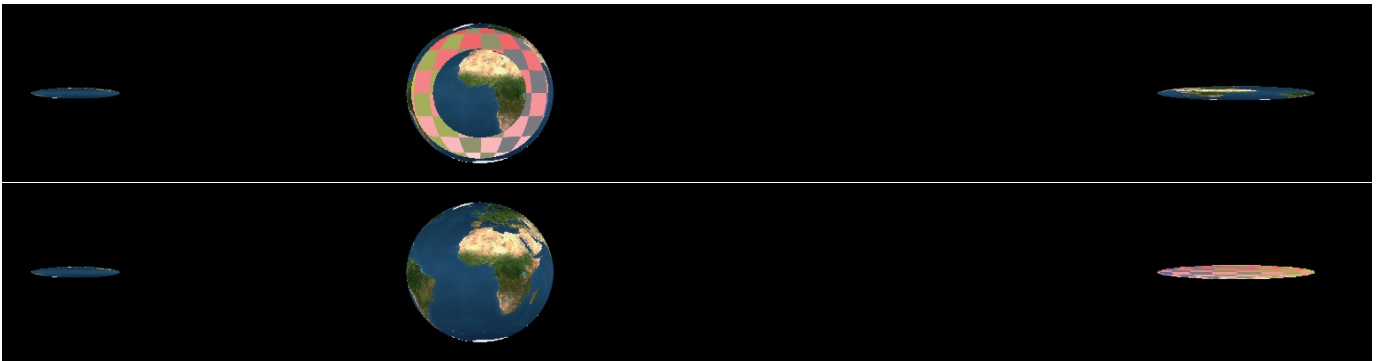


Fig. 8. The sphere is multitextured with a daylight earth texture and the checkerboard texture. The latter texture only appears when the time of emission is in a certain changeable interval. The center sphere is the closest representation, followed by the right and then the left sphere.

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