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Positive Energy in General Relativity

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An argument is given in favor of the conjecture that an isolated system in general relativity must have nonnegative total energy.

There is an old conjecture in general relativity to the effect that, for any object constructed of matter with nonnegative local mass density, the total mass-energy, measured in terms of the asymptotic gravitational field, must also be nonnegative. The conjecture, a bit more precisely, asserts that the total energy must be nonnegative for any space-time which is asymptotically flat (in order that "total energy" make sense), which is not too badly singular (in order to rule out such space-times as the $m < 0$ Schwarzschild), and whose stress-energy satisfies the local energy condition. This conjecture, despite the fact that it seems obvious physically that it should be true, remains open. The essential difficulty is that, because of the nonlinear character of Einstein's equation, the asymptotic behavior of the gravitational field is related only in a rather complicated way to the details of the matter inside the space-time.

Most attempts to resolve this conjecture have been along the following lines. One introduces a suitable three-dimensional surface in the space-time, and then, from the fields and their equations in the space-time, induces on this surface certain fields satisfying certain differential equations. One next characterizes the total energy in terms of the asymptotic behavior of these fields, and finally tries to relate this total energy to the local properties of the fields via their differential equations. In fact, there are two versions of this conjecture, depending on whether one chooses one's surface to be asymptotically a "null cone", or asymptotically a "spacelike three-plane." For the former, one imposes asymptotic flatness at null infinity [1], and the total energy becomes the Bondi energy [2]. Essentially all that is known [3] in this case is that a number of promising-looking potential counterexamples fail. For the latter, one imposes asymptotic flatness at spatial infinity [4, 5], and the total energy becomes the Arnowitt–Deser–Misner energy [5]. In this case, the conjecture is known to hold under a variety of simplifying assumptions, e.g., that the space-time is spherically symmetric [6], that the surface is area-extremal [7], or that the induced geometry on the surface is flat [6].

We here give a new argument in favor of this positive-energy conjecture. Its advantages over earlier results are that it is rather geometrical, that its structure is transparent physically, and that it is extremely simple. Its disadvantage is that it

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appears to be very difficult to formulate this argument as a precise theorem which captures the full strength of the conjecture.

The argument is the following. Let M, g_{ab} be a space-time representing an "isolated object", and let S be a Cauchy surface [8] for this space-time. Fix a timelike curve γ which remains within the isolated object, and whose length (both to the future and to the past from S) is infinite. Let τ be a positive number, and let p be the point of γ a distance τ into the future from S along γ, and p' the point a distance τ into the past. Since p and p' are within the domain of dependence of the Cauchy surface S, there exists [8] a maximal timelike geodesic μ from p' to p. Characterize this geodesic by the point at which it meets S and its tangent direction at that point. We thus obtain a family of points of S, one for each positive number τ.

Now suppose that our space-time satisfies the strong energy and generic conditions: We show as a consequence that this family of points of S cannot accumulate. Suppose, for contradiction, that there were an accumulation point q in S, and let ξ^a be a corresponding accumulation tangent direction. Consider the (timelike or null) geodesic through q, with tangent direction ξ^a. This geodesic must be complete, since γ has infinite length; and must be extremal between any two of its points, since it is an accumulation of extremal geodesics. But the existence of such a geodesic is incompatible [8] with the energy and generic conditions. We conclude, then, that the family of points of S obtained as the intersections of our maximal geodesics and the surface S cannot accumulate. In particular, this family cannot be contained in any compact subset of S.

Now suppose further that there is an asymptotic region of the space-time which includes all of S except for a compact subset. Then, from the above, there exists a timelike geodesic μ which begins on γ, meets S in the asymptotic region, and then returns to γ. But, since γ is to be "within the isolated object", this means physically that a free particle could be thrown from the object, reach a large distance from the object by "time" S, and then fall back to the object. But this behavior for a free particle corresponds physically to positive total mass-energy of the object.

This argument makes use of a number of conditions on the space-time. First, we require that the space-time satisfy the energy condition, i.e., that the local mass density of the matter be nonnegative. We also require the generic condition, which, since an argument of this type must conclude strictly positive total energy, is necessary to exclude examples with zero energy, such as Minkowski space-time. We impose asymptotic flatness and that the space-time be "not too badly singular" by requiring a Cauchy surface S such that all of S except for some compact subset is in an asymptotic region. This last condition may be formulated more precisely as follows. We require that one can attach to S a single point at spatial infinity such that the usual asymptotic conditions are satisfied, and such that the resulting three-dimensional manifold is compact. [Note that we do not require that the topology of S be $\mathbb{R}^3$, i.e., the argument allows nontrivial topologies.]

Certain aspects of the argument, however, are more difficult to formulate precisely. The first is the condition that the curve γ "remain within the isolated object". It would not suffice, for example, to require merely that γ not reach null infinity. We must ensure that γ remain "closer to the object" than this—and indeed enough so that the final timelike geodesic μ from γ to S and back to γ is in fact suggestive of positive total energy.[1] We require not only the existence of such a timelike curve γ, but furthermore one of infinite length. This is essentially a further restriction on the singular character of the space-time: The incompleteness cannot be so rampant that all timelike curves which choose to remain in the immediate vicinity of the object are forced to have finite length. The existence of such a curve may turn out to be a consequence of a suitable version of cosmic censorship: If all singular behavior were within a horizon, then perhaps one could choose the timelike curve γ to be just outside that horizon, and thus to enjoy infinite length while remaining sufficiently near the (presumably, now collapsed) object to make the argument work. A second difficulty involves the relationship between the behavior of the final timelike geodesic μ and the total energy as normally defined. The Arnowitt–Deser–Misner energy, for example, refers to the asymptotic behavior of the initial data on S, and not directly to the "infalling" of timelike geodesics. Although one of course expects these two to be related to each other, it seems to be difficult to express this relationship precisely.

These two difficulties are of course closely related to each other, for the initial timelike curve γ must be chosen to remain close enough to the object that the behavior of the final timelike geodesic μ implies positive Arnowitt–Deser–Misner or Bondi energy. It may be possible to make these issues precise, along the following lines. One would introduce a notion of a space-time "uniformly asymptotically flat" at, say, spatial infinity. The definition would require, roughly speaking, the existence of a radial function r on the space-time such that, for some r-value r_0, the monopole gravitational field dominates all other moments for $r > r_0$ and for all time. The conditions would be so formulated that a consequence of the existence of a timelike geodesic which begins at a small r-value, reaches $r > r_0$, and returns to a small r-value would be that the Arnowitt–Deser–Misner energy is positive. Having formulated such a definition, one would demand uniform asymptotic flatness for the argument above, would choose one's timelike curve γ to remain in the region of small r-values, and would choose the compact subset of S to include all points with $r \leqslant r_0$.

Is uniform asymptotic flatness, defined along the lines above, a physically reasonable condition on space-times? It appears that it is not. Let an otherwise isolated object emit a small, massive particle which escapes to infinity. Then, since the particle would at sufficiently late times reach $r > r_0$, and since the gravitational effects near the massive particle would certainly dominate the monopole moment of the object itself, uniform asymptotic flatness would seem to fail. Indeed, such an example can apparently be used to show that the argument above does not itself carry the full force of the positive-energy conjecture. Imagine that it were possible to construct an isolated object of matter with nonnegative local mass density, but such that the

[1] Indeed, consider Minkowski space-time, and let S be the plane $t = 0$. Then there exists a timelike curve γ which does not reach null infinity, but such that the maximal geodesics constructed in the argument intersect S in a family of points which leave every compact subset of S.

total energy of the system, measured asymptotically, were in fact negative. That is, suppose that one had a counterexample to the positive-energy conjecture. Suppose further that this example had the feature that a small massive particle escaped to infinity. Under these circumstances, one could have a timelike geodesic μ which leaves the central object, reaches a great distance from that object, and then "swings around" in the gravitational field of the small particle and returns to the object. Thus, the type of timelike geodesic demanded by the present argument would exist, although the system itself would have negative total energy.

There appear to be at least two possibilities for strengthening the present argument. For the first, suppose one had a space-time satisfying all of the properties required for the argument except uniform asymptotic flatness. One might try to modify the space-time, e.g., by changing the equation of state, such that the total energy is not changed and such that the other properties remain applicable—but such that uniform asymptotic flatness is now obtained. That is, one would attempt to "build rubber bands into the system to pull emitted particle back." One would then apply the argument to the modified space-time, to conclude that the total energy of the original space-time must have been positive.

The second possibility begins with the observation that it seems unlikely that the timelike geodesic μ which "swings around" the particle in the example above is actually maximal. But the final geodesic demanded by the present argument must of course be maximal, and so the argument may indeed rule out such negative-energy space-times. Thus, one would try to show that the existence of a timelike geodesic which reaches the asymptotic region and returns to the object—provided that geodesic is maximal—already implies positive total energy. The plan, in other words, would be to show that the argument as it stands, even without uniform asymptotic flatness, can be made to carry the full force of the conjecture.

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