Simple Derivation of the Weyl Conformal Tensor

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The Riemann-Christoffel tensor of differential geometry is the usual starting point for the unfolding of Einstein's theory of general relativity. However, there is another tensor that in some ways is more fundamental. It is called the Weyl conformal tensor, and it is responsible for a type of gravitational distortion that is quite different than that described by the Ricci terms in the RC tensor. Here I will derive the Weyl tensor for the n-dimensional case.

By way of summary, here is the RC tensor:

$$R^{\lambda}_{\ \nu\alpha\beta} = \begin{Bmatrix} \lambda \\ \nu\alpha \end{Bmatrix}_{|\beta} - \begin{Bmatrix} \lambda \\ \nu\beta \end{Bmatrix}_{|\alpha} + \begin{Bmatrix} \lambda \\ \sigma\beta \end{Bmatrix} \begin{Bmatrix} \sigma \\ \nu\alpha \end{Bmatrix} - \begin{Bmatrix} \lambda \\ \sigma\alpha \end{Bmatrix} \begin{Bmatrix} \sigma \\ \nu\beta \end{Bmatrix}$$

where the subscript notation $|\beta$ means partial differentiation with respect to x^{β}. The quantities in brackets are the Christoffel symbols of the second kind,

$$\begin{Bmatrix} \lambda \\ \nu\alpha \end{Bmatrix} = \frac{1}{2} g^{\lambda\sigma} \left[g_{\sigma\nu|\alpha} + g_{\sigma\alpha|\nu} - g_{\nu\alpha|\sigma} \right]$$

The RC tensor has a number of interesting symmetry and contraction properties. For one, it can be contracted to give the symmetric Ricci tensor

$$R^{\lambda}_{\ \nu\lambda\beta} = R_{\nu\beta}$$

It can also be converted to its lower-index form via

$$g_{\lambda\mu} R^{\lambda}_{\ \nu\alpha\beta} = R_{\mu\nu\alpha\beta}$$

and it exhibits two symmetry properties that we will employ to derive the Weyl tensor:

$$\begin{aligned} R_{\mu\nu\alpha\beta} &= -R_{\mu\nu\beta\alpha} \\ R_{\mu\nu\alpha\beta} &= -R_{\nu\mu\alpha\beta} \end{aligned}$$

The German mathematical physicist Hermann Weyl (1885–1955) made many fundamental and important contributions to physics, but perhaps he is most famous for his 1929 discovery of quantum-mechanical phase invariance. Phase invariance, known more properly (but misleadingly) as gauge invariance, is a symmetry that underlies all modern quantum theories. It basically is a statement that action Lagrangians are invariant with respect to the replacement

$$\Psi(x) \to e^{i\pi(x)} \Psi(x)$$

where Ψ is a wave function and π is an arbitrary function of space and time. Weyl's gauge theory sprang from an even earlier (1918) theory in which Weyl demanded that Einstein's theory of general relativity should be invariant with respect to the similar replacement

$$g_{\mu\nu}(x) \to e^{\pi(x)} g_{\mu\nu}(x)$$

which we shall call a *metric gauge transformation*. It is remarkable that the Weyl tensor can be deduced by simply demanding that it be invariant with respect to this transformation.

Using this gauge principle, Weyl was able to derive all of electrodynamics from a generalized Einstein-Maxwell Lagrangian. Sadly, this theory failed, but it gave birth to Weyl's 1929 discovery, which today is considered one of the most profound tenets of modern quantum physics.

Around the time of his 1918 theory, Weyl became interested in the cosmological aspects of general relativity, particularly the curvature properties of the universe. Perhaps still infatuated with metric gauge transformations, Weyl wanted to know if there was a tensor, similar to the RC tensor, that was invariant with respect to the above transformation. He found it, only to learn much later that his tensor, now called the *Weyl conformal tensor*, was of fundamental importance in the modern understanding of the two types of gravitational effects of matter: compression and tidal deformation.

In order to derive the Weyl tensor, we first note that the terms *conformal invariance* and *metric gauge invariance* are synonymous and involve the transformation

$$\overline{g}_{\mu\nu}(x) = e^{\pi(x)} g_{\mu\nu}(x) \tag{1}$$

For simplicity, let us consider only infinitesimal transformations:

$$\overline{g}_{\mu\nu}(x) = e^{\epsilon\pi(x)} g_{\mu\nu} \simeq (1+\epsilon\pi) g_{\mu\nu} \tag{2}$$

where ϵ is a small constant such that all quantities involving ϵ^2 and higher terms can be neglected. From (2) we have the variation

$$\begin{aligned} \delta g_{\mu\nu} &= \overline{g}_{\mu\nu} - g_{\mu\nu} \\ &= \epsilon\pi g_{\mu\nu} \end{aligned}$$

Similarly, we have

$$\begin{aligned} \delta g^{\mu\nu} &= \overline{g}^{\mu\nu} - g^{\mu\nu} \\ &= -\epsilon\pi g^{\mu\nu} \end{aligned}$$

Weyl decided that his tensor should consist solely of the RC tensor and its two possible contractions. First is the symmetric Ricci tensor,

$$R_{\beta\nu} = R_{\nu\beta} = R^{\lambda}_{\ \nu\lambda\beta}$$

while the other is the Ricci scalar

$$R = g^{\beta\nu} R_{\beta\nu}$$

Weyl called his tensor $C^{\lambda}_{\ \nu\alpha\beta}$; for simplicity, he wanted this tensor to have the same symmetry properties as $R^{\lambda}_{\ \nu\alpha\beta}$ but with the conformal condition

$$\delta C^{\lambda}_{\ \nu\alpha\beta} = 0$$

I'm not sure how Weyl proceeded at this point, but I do know that it is more convenient to deal with the lower-index form, which is given by

$$C_{\mu\nu\alpha\beta} = g_{\mu\lambda} C^{\lambda}_{\ \nu\alpha\beta}$$

from which we have the variation

$$\delta C_{\mu\nu\alpha\beta} = \epsilon\pi C_{\mu\nu\alpha\beta}$$

We now assume that the lower-index Weyl tensor is composed of the lower-index RC tensor along with all permutations of the Ricci tensor and Ricci scalar:

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + A_{\mu\alpha} R_{\nu\beta} + B_{\mu\beta} R_{\nu\alpha} + C_{\nu\alpha} R_{\mu\beta} + D_{\nu\beta} R_{\mu\alpha} + E_{\mu\nu} R_{\alpha\beta} + F_{\alpha\beta} R_{\mu\nu} + W_{\mu\nu\alpha\beta} R \tag{3}$$

where the various coefficients are tensors to be determined. Given the stated symmetry properties of $R_{\mu\nu\alpha\beta}$ and $R_{\mu\nu}$, we see immediately that the coefficients $E_{\mu\nu}$ and $F_{\alpha\beta}$ must be zero.

To calculate $\delta C_{\mu\nu\alpha\beta}$, we must take the variation of the right-hand side of (3). To save time, I will simply right down the required variations (which you can verify on some rainy night):

$$\begin{aligned} \delta R_{\mu\nu\alpha\beta} &= \epsilon\pi R_{\mu\nu\alpha\beta} + \frac{1}{2}\epsilon g_{\mu\alpha} \pi_{|\beta||\nu} - \frac{1}{2}\epsilon g_{\mu\beta} \pi_{|\alpha||\nu} + \frac{1}{2}\epsilon g_{\beta\nu} \pi_{|\mu||\alpha} - \frac{1}{2}\epsilon g_{\alpha\nu} \pi_{|\mu||\beta} \\ \delta R_{\mu\alpha} &= \frac{1}{2}\epsilon g_{\mu\alpha} g^{\rho\sigma} \pi_{|\rho||\sigma} + \frac{1}{2}(n-2)\,\epsilon\pi_{|\mu||\alpha} \\ \delta R &= (n-1)\epsilon g^{\rho\sigma} \pi_{|\rho||\sigma} - \epsilon\pi R \end{aligned}$$

where $\pi_{|\mu||\alpha} (= \pi_{|\alpha||\mu})$ is the double covariant derivative of the scalar π:

$$\pi_{|\mu||\alpha} = \pi_{|\mu|\alpha} - \pi_\lambda \begin{Bmatrix} \lambda \\ \mu\alpha \end{Bmatrix}$$

Plugging these identities into (3), we have the rather messy formula

$$\begin{aligned} \delta C_{\mu\nu\alpha\beta} \quad = \quad & \epsilon\pi R_{\mu\nu\alpha\beta} + \frac{1}{2}\epsilon g_{\mu\alpha}\pi_{|\beta||\nu} - \frac{1}{2}\epsilon g_{\mu\beta}\pi_{|\alpha||\nu} + \frac{1}{2}\epsilon g_{\beta\nu}\pi_{|\mu||\alpha} - \frac{1}{2}\epsilon g_{\alpha\nu}\pi_{|\mu||\beta} \\ & + R_{\nu\beta}\,\delta A_{\mu\alpha} + A_{\mu\alpha}\left[\frac{1}{2}\epsilon g_{\nu\beta}g^{\rho\sigma}\pi_{|\rho||\sigma} + \frac{1}{2}(n-2)\epsilon\pi_{|\nu||\beta}\right] \\ & + R_{\nu\alpha}\,\delta B_{\mu\beta} + B_{\mu\beta}\left[\frac{1}{2}\epsilon g_{\nu\alpha}g^{\rho\sigma}\pi_{|\rho||\sigma} + \frac{1}{2}(n-2)\epsilon\pi_{|\nu||\alpha}\right] \\ & + R_{\mu\beta}\,\delta C_{\nu\alpha} + C_{\nu\alpha}\left[\frac{1}{2}\epsilon g_{\mu\beta}g^{\rho\sigma}\pi_{|\rho||\sigma} + \frac{1}{2}(n-2)\epsilon\pi_{|\mu||\beta}\right] \\ & + R_{\mu\alpha}\,\delta D_{\nu\beta} + D_{\nu\beta}\left[\frac{1}{2}\epsilon g_{\mu\alpha}g^{\rho\sigma}\pi_{|\rho||\sigma} + \frac{1}{2}(n-2)\epsilon\pi_{|\mu||\alpha}\right] \\ & + R\,\delta W_{\mu\nu\alpha\beta} + W_{\mu\nu\alpha\beta}\left[(n-1)\epsilon g^{\rho\sigma}\pi_{|\rho||\sigma} - \epsilon\pi R\right] \end{aligned} \qquad (4)$$

We now require that all terms involving $\pi_{|\mu||\alpha}$ and their permutations cancel each other. For example, the term $\frac{1}{2}(n-2)\epsilon A_{\mu\alpha}\pi_{|\nu||\beta}$ must be set equal to $-\frac{1}{2}\,\epsilon g_{\mu\alpha}\pi_{|\nu||\beta}$, from which we get the identifications

$$A_{\mu\alpha} = -\frac{1}{n-2}g_{\mu\alpha}\,, \quad \delta A_{\mu\alpha} = -\frac{1}{n-2}\,\epsilon\pi g_{\mu\alpha}$$

Similarly, we have the identities

$$\begin{aligned} B_{\mu\beta} &= \frac{1}{n-2}g_{\mu\beta}\,, \quad \delta B_{\mu\beta} = \frac{1}{n-2}\,\epsilon\pi g_{\mu\beta} \\ C_{\nu\alpha} &= \frac{1}{n-2}g_{\nu\alpha}\,, \quad \delta C_{\nu\alpha} = \frac{1}{n-2}\,\epsilon\pi g_{\nu\alpha} \\ D_{\nu\beta} &= -\frac{1}{n-2}g_{\nu\beta}\,, \quad \delta D_{\nu\beta} = -\frac{1}{n-2}\,\epsilon\pi g_{\nu\beta} \end{aligned}$$

This leaves the terms involving $g^{\rho\sigma}\pi_{|\rho||\sigma}$, which we can similarly eliminate by setting

$$W_{\mu\nu\alpha\beta} = \frac{1}{(n-1)(n-2)}\left[g_{\mu\alpha}\,g_{\beta\nu} - g_{\mu\beta}\,g_{\alpha\nu}\right]\,, \quad \delta W_{\mu\nu\alpha\beta} = 2\epsilon\pi W_{\mu\nu\alpha\beta}$$

We have now identified all the coefficients, and it is easy to see that (4) reduces to

$$\delta C_{\mu\nu\alpha\beta} = \epsilon\pi C_{\mu\nu\alpha\beta}$$

as required, where

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \frac{1}{n-2}\left[g_{\mu\beta}\,R_{\nu\alpha} - g_{\mu\alpha}\,R_{\nu\beta} + g_{\nu\alpha}\,R_{\mu\beta} - g_{\nu\beta}\,R_{\mu\alpha}\right] + \frac{1}{(n-1)(n-2)}\left[g_{\mu\alpha}\,g_{\beta\nu} - g_{\mu\beta}\,g_{\alpha\nu}\right]R$$

The conformal, upper-index form of the tensor is obtained by raising the first index:

$$C^\lambda{}_{\nu\alpha\beta} = R^\lambda{}_{\nu\alpha\beta} + \frac{1}{n-2}\left[\delta^\lambda_\beta\,R_{\nu\alpha} - \delta^\lambda_\alpha\,R_{\nu\beta} + g_{\nu\alpha}\,R^\lambda{}_\beta - g_{\nu\beta}\,R^\lambda{}_\alpha\right] + \frac{1}{(n-1)(n-2)}\left[\delta^\lambda_\alpha\,g_{\beta\nu} - \delta^\lambda_\beta\,g_{\alpha\nu}\right]R$$

and this is the form that is usually encountered in the textbooks. It is obvious from this expression that the Weyl tensor must be zero for $n \leqslant 2$; what is perhaps not so obvious is that it vanishes for $n = 3$ as well. Incidentally, note that *any* contraction of the Weyl tensor (such as $C_{\nu\beta} = C^\lambda{}_{\nu\lambda\beta}$) is identically zero, even in a curved space where $R_{\nu\beta} \neq 0$.

The Weyl conformal tensor is essentially the RC tensor with the Ricci terms subtracted out. The Ricci terms themselves are tied to the matter tensor $T_{\mu\nu}$ via Einstein's gravitational field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu}$$

where G is the gravitational constant (I've left out the cosmological constant for brevity). Because gravity tends to compact matter, the Ricci terms are associated with gravitational compaction and collapse, a process that involves the reduction in the initial volume of a chunk of matter or a gas cloud. By comparison, the Weyl tensor is associated with a curvature phenomenon known as *tidal deformation*, which preserves the volume but distorts its shape (gravity waves passing through a planet, the tidal "scrunching" effect just outside the event horizon of a black hole, etc.). In the absence of matter, the matter tensor and Ricci terms all go to zero; however, the Weyl tensor does not vanish provided there is a source of gravity (i.e., mass-energy) somewhere in the universe. In this sense, the Weyl tensor is more fundamental than the RC tensor.

My intent here was simply to derive the Weyl tensor, not to elaborate on its properties. For a much more detailed description of the Weyl tensor and its importance in the evolution of the universe (especially the *Weyl curvature hypothesis*), entropy, the arrow of time, and tidal distortion, see the following books by the noted British mathematical physicist Roger Penrose:

1. R. Penrose, *The Road to Reality: A Complete Guide to the Laws of the Universe*. Knopf, 2004.
2. R. Penrose, *The Emperor's New Mind*. Penguin, 1989.