Generating Electromagnetic Waves from Gravity Waves in Cosmology

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Abstract

Examples of test electromagnetic waves on a Friedmann–Lemaître–Robertson–Walker (FLRW) background are constructed from explicit perturbations of the FLRW space–times describing gravitational waves propagating in the isotropic universes. A possible physical mechanism for the production of the test electromagnetic waves is shown to be the coupling of the gravitational waves with a test magnetic field, confirming the observation of Marklund, Dunsby and Brodin [Phys. Rev. D**62**, 101501(R) (2000)].

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1 Introduction

Marklund, Dunsby and Brodin [1] have made the important observation that gravitational wave perturbations of Friedmann–Lemaître–Robertson–Walker (FLRW) cosmological models coupled to weak magnetic test fields can generate electromagnetic waves. The magnitude of this effect has been estimated in [1]. They have demonstrated this phenomenon using the gauge–invariant and covariant perturbation theory of Ellis and Bruni [2]. This theory has also been used to construct explicit solutions of the Ellis–Bruni perturbation equations describing gravitational waves propagating in FLRW universes [3] (see also [4]) *from a point of view which differs significantly from that of* [1]. The histories of the wave fronts of the latter waves are particularly simple families of null hypersurfaces which arise naturally in the FLRW space–times. In this paper we demonstrate how test electromagnetic waves can be constructed from these explicit gravitational waves and how these electromagnetic waves can be viewed as arising from the interaction of the gravitational waves with a test magnetic field, thereby supporting the observation of Marklund, Dunsby and Brodin.

We utilize a test magnetic field which, by its nature, does not perturb the isotropic cosmology but is nevertheless a violation of isotropy. The gravitational waves we use do perturb the isotropic cosmology and this perturbation also breaks the isotropic symmetry since the gravitational waves are uni–directional and thus at each point of the isotropic cosmological model their histories have a unique null propagation direction. The behavior of magnetic fields in cosmological models has been extensively and carefully studied from diverse physical viewpoints in [5]–[10].

The gravitational waves which we utilize in this paper correspond to the simplest (from a geometrical point of view) type of gravitational radiation which can propagate in isotropic cosmologies. The histories of their wave fronts are naturally occurring null hypersurfaces in the isotropic cosmological models and their propagation direction in these space–times is null, geodesic and shear–free. It is thus of some interest to examine the Marklund, Dunsby and Brodin observation in terms of them. As a measure of the strength of the gravitational waves used in this paper, and of the electromagnetic waves which result from their interaction with a test magnetic field, we find that the gravitational field of the gravitational waves (the perturbed Weyl tensor) is proportional to R^{-2} (see eq.(2.34)) where $R(t)$ is the scale factor of the isotropic universe, while the electromagnetic field (see eq. (3.17)) is also proportional to R^{-2}. The coefficients of R^{-2} in both cases involve an arbitrary analytic function whose appearance is a characteristic of electromagnetic radiation (which is shear–free [11] in the optical sense that the null propagation direction in space–time, given by the gradient of the function ϕ in (2.28) below, is shear–free) and of shear–free (in the optical sense) gravitational radiation [12].

The interaction of gravitational waves and electromagnetic fields has led to proposals for a mechanism to detect gravitational waves [13]–[15]. Our approach is influenced by the fundamental paper by Szekeres [16] on the interaction of gravitational waves with matter and with electromagnetic waves. With the inclusion of matter many modelling possibilities open up. For example, a recent model for the generation of gravitational waves from matter and electromagnetic waves can be found in [17].

This paper is organized in the following way: in section 2 the relevant Ellis–Bruni perturbation equations (for tensor perturbations) are listed and the explicit solutions derived in [3] are summarized. Analogous test electromagnetic waves having the same wave fronts as these gravitational waves will be required and they are described in section 3. In section 4 the construction of test electromagnetic waves of the type considered in section 3 are derived from the gravitational waves of section 2. In addition in this section these electromagnetic waves are shown to be capable of being interpreted as resulting from the interaction of the gravitational waves with an explicit test magnetic field. The paper ends with a brief discussion in section 5.

2 Gravity Waves in FLRW Universes

We make use of a four dimensional space–time manifold with a metric tensor having components g_{ab} in a local coordinate system $\{x^a\}$. This manifold contains a preferred congruence of time–like world–lines which are the integral curves of a vector field having components u^a in the coordinate system $\{x^a\}$ and which satisfies $u_a\,u^a = -1$. The electric and magnetic parts of the Weyl tensor C_{abcd} are defined respectively by

$$E_{ab} = C_{apbq}\,u^p\,u^q\ , \qquad H_{ab} = C^*_{apbq}\,u^p\,u^q\ . \tag{2.1}$$

These are equivalent to C_{abcd} and the star superscript denotes the dual $C^*_{apbq} = \frac{1}{2}\eta_{ap}{}^{rs}\,C_{rsbq}$ with $\eta_{abcd} = \sqrt{-g}\,\epsilon_{abcd}$, $g = \det(g_{ab})$ and ϵ_{abcd} the Levi–Civita permutation symbol. We note that for the Weyl tensor the left and right duals are equal. The energy–momentum–stress tensor $T^{ab} = T^{ba}$ describing the matter distribution can be decomposed with the respect to the vector field u^a to read [18]

$$T^{ab} = \mu\,u^a\,u^b + p\,h^{ab} + q^a\,u^b + q^b\,u^a + \pi^{ab}\ , \tag{2.2}$$

with μ the density of matter measured by the observer with 4–velocity u^a, $h^{ab} = g^{ab} + u^a\,u^b$ the projection tensor, p the isotropic pressure, q^a the energy flow measured by the observer with 4–velocity u^a and $\pi^{ab} = \pi^{ba}$ the anisotropic stress. Here

$$q^a\,u_a = 0\ , \qquad \pi^{ab}\,u_b = 0\ , \qquad \pi^a{}_a = 0\ . \tag{2.3}$$

The covariant derivative $u_{a;b}$ is decomposed into

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{1}{3}\theta\,h_{ab} - \dot{u}_a\,u_b\ , \tag{2.4}$$

with the dot in general indicating covariant differentiation in the direction of u^a (and thus in particular $\dot{u}^a = u^a{}_{;b}\,u^b$). Here

$$\omega_{ab} = u_{[a;b]} + \dot{u}_{[a}\,u_{b]}\ , \tag{2.5}$$

is the vorticity tensor (with $\omega_{ab} = -\omega_{ba}$, $\omega_{ab}\,u^b = 0$ and square brackets denote skew symmetrization),

$$\sigma_{ab} = u_{(a;b)} + \dot{u}_{(a}\,u_{b)} - \frac{1}{3}\theta\,h_{ab}\ , \tag{2.6}$$

is the shear tensor (with $\sigma_{ab} = \sigma_{ba}$, $\sigma^a{}_a = 0$, $\sigma_{ab}\,u^b = 0$ and round brackets denote symmetrization) and

$$\theta = u^a{}_{;a}\ , \tag{2.7}$$

is the expansion or contraction scalar.

In the isotropic FLRW space–times with u^a the 4–velocity of matter we have $q^a = 0$ and $\pi^{ab} = 0$ and thus (2.2) specialises to the perfect fluid form

$$T^{ab} = \mu\,u^a\,u^b + p\,h^{ab}\ , \tag{2.8}$$

with, in addition, $h^b_a\,p_{,b} = 0$ and $h^b_a\,\mu_{,b} = 0$. We also have in this case $\dot{u}^a = 0$, $\omega_{ab} = 0$, $\sigma_{ab} = 0$ and $h^b_a\,\theta_{,b} = 0$ which has the effect of simplifying (2.4) to

$$u_{a;b} = \frac{1}{3}\theta\,h_{ab}\ . \tag{2.9}$$

In this case θ satisfies the simplified Raychaudhuri equation

$$\dot{\theta} + \frac{1}{3}\theta^2 = -\frac{1}{2}(\mu + 3\,p)\ , \tag{2.10}$$

and the equations $T^{ab}{}_{;b} = 0$ reduce in this case to the single equation

$$\dot{\mu} + \theta\,(\mu + p) = 0\ . \tag{2.11}$$

The space–times are now necessarily conformally flat and thus $E_{ab} = 0$ and $H_{ab} = 0$. The metric tensor g_{ab} is given via the line–element in the Robertson–Walker form

$$ds^2 = R^2(t)\,\frac{[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]}{\left(1 + \frac{k}{4}\,r^2\right)^2} - dt^2 \ , \tag{2.12}$$

with $R(t)$ the scale factor, $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ and $k(= 0,\ \pm 1)$ is the Gaussian curvature of the $t =$ constant space–like hypersurfaces. In these coordinates $u^a\partial/\partial x^a = \partial/\partial t$ and x^α $(\alpha = 1, 2, 3)$ are constant on each integral curve of the vector field $\partial/\partial t$. In addition $\theta = \theta(t),\ \mu = \mu(t)$ and $p = p(t)$ are obtained from the scale factor $R(t)$ as $\theta = 3\,\dot{R}/R$ with $\mu,\ p$ given by (2.11) along with an equation of state while $R(t)$ is derived from the Friedmann equation (whose time derivative in this case coincides with (2.10); see, for example [18]).

In the Ellis–Bruni [2] theory perturbations of the FLRW models in particular are described by the basic gauge invariant variables $E_{ab},\ H_{ab},\ \sigma_{ab},\ \omega_{ab},\ \dot{u}^a$, $h^b_a\,\theta_{,b},\ h^b_a\,\mu_{,b},\ h^b_a\,p_{,b},\ q^a$ and π^{ab}, which are taken to be small of first order. The differential equations determining these variables are obtained by retaining first order small terms in the Ricci identities, the Bianchi identities and the equations $T^{ab}{}_{;b} = 0$ with T^{ab} given by (2.2). There are two exceptions to this procedure: the propagation equation for θ along the integral curves of u^a (Raychaudhuri's equation), which follows from the Ricci identities, and the propagation equation for μ along the integral curves of u^a, which follows from $T^{ab}{}_{;b} = 0$. These equations are converted into differential equations for the gauge invariant variables listed above by projecting their gradients orthogonal to u^a (see [2]). For perturbations that exclusively describe gravitational waves propagating through FLRW universes we only require the variables $E_{ab},\ H_{ab},\ \sigma_{ab},\ \pi^{ab}$ (the so–called "tensor" perturbations), with the remaining gauge invariant variables vanishing. The linear equations determining these first order quantities are [3]

$$\dot{\sigma}_{ab} + \frac{2}{3}\,\theta\,\sigma_{ab} - \frac{1}{2}\,\pi_{ab} + E_{ab} = 0 \ , \tag{2.13}$$

and

$$H_{ab} = -\sigma_{(a}{}^{g;c}\,\eta_{b)fgc}\,u^f \ , \tag{2.14}$$

from the Ricci identities and

$$E^{bd}{}_{;d} = -\frac{1}{2}\,\pi^{bd}{}_{;d} \ , \tag{2.15}$$

$$H^{bd}{}_{;d} = 0 \ , \tag{2.16}$$

$$\dot{E}^{bt} + \theta\, E^{bt} = -u_r\, H^{(b}_{s;d}\, \eta^{t)rsd} - \frac{1}{2}\, \dot{\pi}^{bt} - \frac{1}{6}\, \theta\, \pi^{bt} \ , \qquad (2.17)$$

$$\dot{H}^{bt} + \theta\, H^{bt} = u_r\, E^{(b}_{s;d}\, \eta^{t)rsd} - \frac{1}{2}\, \eta^{(b}{}_{rad}\, \pi^{t)a;d}\, u_r \ , \qquad (2.18)$$

from the Bianchi identities. To obtain simple solutions of these equations we look for solutions which have an arbitrary dependence on a scalar function $\phi(x^a)$ by writing

$$\sigma_{ab} = s_{ab}\, F(\phi) \ , \qquad \pi_{ab} = \Pi_{ab}\, F(\phi) \ . \qquad (2.19)$$

We will then substitute these into (2.13) and (2.14) to obtain expressions for E_{ab} and H_{ab} which depend on F and its derivative with respect to ϕ. These, along with (2.19), are then substituted into (2.15)–(2.18) to arrive at differential equations for s_{ab} and Π_{ab}. We note that

$$s_{ab} = s_{ba},\ \ s_{ab}\, u^b = 0,\ \ g^{ab}\, s_{ab} = s^a{}_a = 0 \ , \qquad (2.20)$$

with similar equations holding for Π_{ab}. When the substitutions indicated above are carried out the consistency of the resulting equations requires [3]

$$g^{ab}\, \phi_{,a}\, \phi_{,b} = 0 \ , \qquad (2.21)$$

(the comma denoting partial differentiation with respect to x^a) so that the hypersurfaces $\phi(x^a)$ = constant are null, and the following consistent equations (also consistent with (2.20)) must hold:

$$s^{ab}\, \phi_{,b} = 0 \ , \qquad \Pi^{ab}\, \phi_{,b} = 0 \ , \qquad (2.22)$$

along with

$$s^{ab}{}_{;b} = 0 \ , \qquad \Pi^{ab}{}_{;b} = 0 \ , \qquad (2.23)$$

the propagation equation for s^{ab} along the generators of the null hypersurfaces $\phi(x^a)$ = constant,

$$s^{ab;c}\, \phi_{,c} + \left(\frac{1}{2}\phi^{,d}{}_{;d} - \frac{1}{3}\theta\, \dot{\phi}\right)\, s^{ab} = -\frac{1}{2}\dot{\phi}\, \Pi^{ab} \ , \qquad (2.24)$$

and the wave equation,

$$s^{ab;d}{}_{;d} - \frac{2}{3}\theta\, \dot{s}^{ab} + \left(-\frac{1}{3}\theta^2 + \frac{3}{2}p - \frac{1}{6}\mu\right)\, s^{ab} = -\dot{\Pi}^{ab} - \frac{2}{3}\theta\, \Pi^{ab} \ . \qquad (2.25)$$

The null hypersurfaces $\phi(x^a)$ = constant are the histories in the FLRW space–times of the wave fronts of the gravitational waves described by these perturbations (see [3], [4]). To exhibit explicit solutions we first choose these null hypersurfaces. To obtain some naturally occuring null hypersurfaces which will lead to surveyable solutions we start by writing the Robertson–Walker line–element (2.12) in the form [19]

$$ds^2 = R^2(t)\{dx^2 + p_0^{-2} f^2(dy^2 + dz^2)\} - dt^2 \ , \tag{2.26}$$

with $p_0 = 1 + (K/4)(y^2 + z^2),\ K = \text{constant},\ f = f(x)$. The following cases are allowed: (i) if $k = +1$ then $K = +1$ and $f(x) = \sin x$; (ii) if $k = 0$ then $K = 0, +1$ with $f(x) = 1$ when $K = 0$ and $f(x) = x$ when $K = +1$; (iii) if $k = -1$ then $K = 0,\ \pm 1$ with $f(x) = \frac{1}{2}\,e^x$ when $K = 0$, $f(x) = \sinh x$ when $K = +1$ and $f(x) = \cosh x$ when $K = -1$. The details of these special cases are given in [3]. The following equations are satisfied in the cases (i)–(iii):

$$f'' + k\,f = 0 \ , \qquad (f')^2 + k\,f^2 = K \ , \tag{2.27}$$

with the prime denoting differentiation with respect to x. A convenient family of null hypersurfaces is given by

$$\phi(x^a) = x - T(t) = \text{constant} \ , \tag{2.28}$$

with $dT/dt = R^{-1}$.

Using the null hypersurfaces (2.28) and the equations (2.23)–(2.25) for s^{ab}, Π^{ab} subject to (2.20) (and similar conditions on Π^{ab}) and (2.22) yields solutions in the form

$$s^{ab} = \bar{s}\,m^a\,m^b + s\,\bar{m}^a\,\bar{m}^b \ , \tag{2.29}$$

$$\Pi^{ab} = \bar{\Pi}\,m^a\,m^b + \Pi\,\bar{m}^a\,\bar{m}^b \ , \tag{2.30}$$

with m^a given by the 1–form $m_a\,dx^a = R\,p_0^{-1} f\,d\zeta/\sqrt{2}$ with $\zeta = y + iz$ (and thus $m_a\,m^a = 0 = \bar{m}_a\,\bar{m}^a$ and $m_a\,\bar{m}^a = +1$) and the bar denoting complex conjugation. In addition

$$\bar{s} = -\frac{p_0^2}{R\,f}\,\mathcal{G}(\zeta, x, t) \ , \tag{2.31}$$

$$\bar{\Pi} = -\frac{2\,p_0^2}{R^2 f}\,(D\mathcal{G} + \dot{R}\,\mathcal{G}) \ . \tag{2.32}$$

Here $\mathcal{G}$ is a complex analytic function of its argument and $D = \partial/\partial x + R\,\partial/\partial t$. Also $\mathcal{G}$ satisfies

$$D^2\mathcal{G} + k\,\mathcal{G} = 0 \ , \tag{2.33}$$

with $k = 0, \pm 1$ and this can easily be solved with each case involving two arbitrary complex analytic functions, of ζ and $x - T = \phi$, of integration. The corresponding perturbed Weyl tensor is given by

$$E^{ab} + iH^{ab} = -\frac{2\,p_0^2}{R^2 f}\,\frac{\partial}{\partial x}(\mathcal{G}\,F(\phi))\,m^a\,m^b \ , \tag{2.34}$$

which is a Petrov type N Weyl tensor with degenerate principal null direction $\phi_{,a}$, confirming the interpretation of the perturbations of the FLRW space–times described here as being due to gravitational waves having the null hypersurfaces ϕ = constant as the histories of their wave fronts. Further properties of these waves are discussed in [3].

3 Electromagnetic Waves in FLRW Universes

Electromagnetic test fields on the space–time described at the beginning of section 2 are encoded in a skew–symmetric tensor with components $F_{ab} = -F_{ba}$ having electric and magnetic parts defined by [18]

$$E_a = F_{ab}\,u^b\ , \qquad H_a = F^*_{ab}\,u^b\ , \tag{3.1}$$

with $F^*_{ab} = \frac{1}{2}\eta_{ab}{}^{rs}\,F_{rs}$. The vectors E_a, H_a are equivalent to a knowledge of F_{ab} with

$$F_{ab} = u_a\,E_b - u_b\,E_a - \eta_{abcd}\,u^c\,H^d\ . \tag{3.2}$$

If these are source–free electromagnetic test fields on the FLRW space–times then they must satisfy Maxwell's equations in the form [18]

$$E^a{}_{;a} = 0\ , \qquad H^a{}_{;a} = 0\ , \tag{3.3}$$

and

$$\dot{E}^a + \frac{2}{3}\theta\,E^a = -\eta^{abed}\,u_b\,H_{e;d}\ , \tag{3.4}$$

$$\dot{H}^a + \frac{2}{3}\theta\,H^a = \eta^{abed}\,u_b\,E_{e;d}\ . \tag{3.5}$$

To obtain solutions analogous to the gravitational waves described in section 2 we first introduce a 4–potential σ^a with [20]

$$\sigma^a\,u_a = 0\ , \qquad \sigma^a{}_{;a} = 0\ , \tag{3.6}$$

and

$$F_{ab} = \sigma_{b;a} - \sigma_{a;b}\ . \tag{3.7}$$

Using the Ricci identities along with the conformal flatness of the FLRW space–times we have

$$\sigma_{a;dc} - \sigma_{a;cd} = \frac{2}{3}\mu\,g_{a[c}\,\sigma_{d]} + (\mu + p)\,u_a\,u_{[c}\,\sigma_{d]}\ . \tag{3.8}$$

This helps to establish that (3.3) are now satisfied automatically. In addition (3.4) reduces to the wave equation

$$\sigma^{a;d}{}_{;d} = \frac{1}{2}(\mu - p)\,\sigma^a \ , \tag{3.9}$$

and (3.5) is automatically satisfied. The equations to be satisfied by σ^a are (3.6) and (3.9). Following (2.19) we look for solutions of the form

$$\sigma^a = s^a\,F(\phi) \ , \tag{3.10}$$

with F an arbitrary function of $\phi(x^a)$. With $s^a \neq 0$ we arrive again at (2.21) along with

$$s^a{}_{;a} = 0 \ , \qquad s^a\,\phi_{,a} = 0 \ . \tag{3.11}$$

We also obtain the propagation equation for s^a along the generators of the null hypersurfaces $\phi(x^a) =$ constant,

$$s^{a;b}\,\phi_{,b} + \frac{1}{2}\phi_{,d}{}^{;d}\,s^a = 0 \ , \tag{3.12}$$

and the wave equation,

$$s^{a;d}{}_{;d} = \frac{1}{2}(\mu - p)\,s^a \ . \tag{3.13}$$

Solving these with $\phi(x^a)$ given by (2.28) we find

$$s^a = \bar{s}_0\,m^a + s_0\,\bar{m}^a \ , \tag{3.14}$$

with m^a given following (2.29) and

$$\bar{s}_0 = \frac{p_0}{R\,f}\,\mathcal{F}(\zeta, x, t) \ , \tag{3.15}$$

with $\mathcal{F}$ a complex analytic function required to satisfy

$$D\mathcal{F} = 0 \ , \tag{3.16}$$

with the operator D defined following (2.32). Thus $\mathcal{F} = \mathcal{F}(\zeta,\ x - T)$ and is otherwise arbitrary. The corresponding electric and magnetic fields (3.1) are given via

$$E^a + iH^a = \frac{2\,p_0}{R^2 f}\,\frac{\partial}{\partial x}(\mathcal{F}\,F)\,m^a \ , \tag{3.17}$$

analogous to (2.34). The radiative nature of this electromagnetic field, with propagation direction $\phi_{,a}$, is evident from (3.17).

4 Electromagnetic Waves from Gravity Waves

We now demonstrate how electromagnetic waves of the type described in section 3 can be constructed from the gravitational waves given in section 2. The construction is not unique (see the discussion in the next section) but it can be given the interpretation of arising from the coupling of the gravitational waves of section 2 with a test magnetic field in agreement with [1].

The gravitational waves of section 2 are obtained from a complex analytic function $\mathcal{G}(\zeta, x, t)$ which satisfies (2.33). We see from the argument in section 3 that the electromagnetic waves given there are obtained from a complex analytic function $\mathcal{F}(\zeta, x, t)$ which satisfies (3.16). Given $\mathcal{G}$ satisfying (2.33) and $f(x)$ satisfying (2.27) we define

$$\mathcal{F}(\zeta, x, t) = f\, D\mathcal{G} - f'\, \mathcal{G}\ . \tag{4.1}$$

Clearly this function satisfies (3.16) and so $\mathcal{F} = \mathcal{F}(\zeta, x-T)$. With this choice of $\mathcal{F}$ electromagnetic waves in section 3 corresponding to the gravitational waves of section 2 are described by the 4–potential (3.10) with s^a given by (3.14) and (3.15).

A source–free test magnetic field on the FLRW space–times must satisfy (3.3)–(3.5) with $H_a \neq 0$ and $E_a = 0$. A simple example of such a field is given by

$$H_a = \frac{\lambda_a}{R\, f^2}\ , \qquad \lambda_a = h^b_a\, \phi_{,b}\ . \tag{4.2}$$

This field can be derived from a 4–potential in the manner of (3.7) with (3.6). In this case the potential is given by the 1–form

$$_H\sigma_a\, dx^a = \frac{y\, dz - z\, dy}{2\, p_0}\ , \tag{4.3}$$

or equivalently by the covariant vector

$$_H\sigma^a = \frac{i(\zeta\, \bar{m}^a - \bar{\zeta}\, m^a)}{2\sqrt{2}\, R\, f}\ , \tag{4.4}$$

with m^a given following (2.30). We note in passing that any source–free magnetic test field on the FLRW space–times is given by $H_a = R^{-1} q_{,a}$ with $q_{,a} u^a = 0$ and $g^{ab} q_{,a;b} = 0$ and (4.2) corresponds to $q = q(x)$ given by $dq/dx = f^{-2}$.

Observing that the function $\mathcal{F}$ in (4.1) is linear in the arbitrary analytic function $\mathcal{G}$ and its derivative $D\mathcal{G}$ we first note from (2.29)–(2.32) that

$$s^{ab} = -\frac{p_0^2}{R\, f}\, \mathcal{G}\, m^a\, m^b - \frac{p_0^2}{R\, f}\, \bar{\mathcal{G}}\, \bar{m}^a\, \bar{m}^b\ , \tag{4.5}$$

and

$$\Pi^{ab}-\frac{2}{3}\,\theta\,s^{ab}=-\frac{2\,p_0^2}{R^2 f}\,D\mathcal{G}\,m^a\,m^b-\frac{2\,p_0^2}{R^2 f}\,D\bar{\mathcal{G}}\,\bar{m}^a\,\bar{m}^b\ . \tag{4.6}$$

Hence using (4.4) we arrive at

$$-\,2\,p_0^{-1}R\,f\,f'\,s^{ab}{}_H\sigma_b-p_0^{-1}R^2 f^2\left(\Pi^{ab}-\frac{2}{3}\theta\,s^{ab}\right){}_H\sigma_b=\frac{p_0}{R\,f}\left(\mathcal{J}\,m^a+\bar{\mathcal{J}}\,\bar{m}^a\right)\ , \tag{4.7}$$

with

$$\mathcal{J}=\frac{i\zeta}{\sqrt{2}}\,\mathcal{F}\ , \tag{4.8}$$

and $\mathcal{F}$ is given by (4.1). The right hand side of (4.7) is the 4–potential of a radiative Maxwell test field on the FLRW space–times of the type described in section 3. The left hand side of (4.7) provides a Marklund–Dunsby–Brodin physical interpretation of the origin of this electromagnetic field as a coupling of the gravitational waves described by s^{ab} and Π^{ab} with the magnetic test field described by the 4–potential ${}_H\sigma^a$, due to the appearance of the products $s^{ab}{}_H\sigma_b$ and $\Pi^{ab}{}_H\sigma_b$.

5 Discussion

We make two observations regarding the electromagnetic waves constructed in section 4 from the gravitational waves in section 2. The first observation concerns the lack of uniqueness of the resulting electromagnetic waves. To see this we note that if $\mathcal{G}$ and $\mathcal{H}$ are two complex analytic functions satisfying (2.33) then

$$\mathcal{F}(\zeta,x,t)=k\,\mathcal{H}\,\mathcal{G}+D\mathcal{H}\,D\mathcal{G}\ , \tag{5.1}$$

satisfies (3.16). In view of (2.27) a choice of $\mathcal{H}$ is simply $\mathcal{H}=f(x)$. In this case

$$\mathcal{F}(\zeta,x,t)=k\,f\,\mathcal{G}+f'D\mathcal{G}\ . \tag{5.2}$$

It is easy to see from (3.17) that these waves are distinct from those obtained using $\mathcal{F}$ given in (4.1). In addition we note that there are no waves of this type for the case (ii) with $k=K=0$ given following (2.26), in contradistinction to the example (4.1). On the other hand since (5.1) is a linear combination of $\mathcal{G}$ and $D\mathcal{G}$ these electromagnetic waves also lend themselves to the interpretation of arising from the coupling of the gravitational waves of section 2 to the test magnetic field given in section 4.

The significance of utilizing a test magnetic field rather than a test electric field in (4.2) is clarified by the following observation: A source–free test electric field given by (3.1), (3.6) and (3.7) with $H_a=0$ has a potential

1–form of the form ${}_E\sigma_a\,dx^a = a_1\,dx + a_2\,dy + a_3\,dz$ with a_α ($\alpha = 1, 2, 3$) functions of x, y, z, t. Maxwell's equations (3.3)–(3.5) with $H_a = 0$ imply that ${}_E\sigma^a = R^{-1}l(x, y, z)\,u^a$, for some function l, up to a gauge transformation. The scalar products of this 4–potential with s^{ab} and Π^{ab} are therefore zero.

Finally we remark that astrophysical gravitational waves are generally low frequency and in universes containing ionized matter the corresponding electromagnetic waves propagate poorly. In our treatment no mention has been made of the frequency of the resulting electromagnetic waves with 4–potential (4.7). This would be introduced via a Fourier analysis of the arbitrary analytic function $\mathcal{G}$. To further study this electromagnetic radiation and follow the dissipative consequences of its interaction with the matter and gravitational radiation would require calculating its perturbative effect on the cosmological model. In this paper this electromagnetic radiation is weak but is still considered a test field.

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